hexagon:2610.00023v2

A FINITE POLYHEDRAL CONSTRUCTION OF THE RANK-TWO QUANTUM GREEDY BASIS

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Abstract. We give a finite polyhedral construction of the quantum greedy basis for coefficient-free rank-two quantum cluster algebras. The construction specializes coefficientwise to the classical polyhedral construction at $v = 1$. Corresponding classical and quantum coefficients and auxiliary margins have identical zero sets, and the Newton polytopes are affine images of the classical carriers.

1. Introduction

Li and Pan's polyhedral construction for coefficient-free rank-two cluster algebras assigns weights by taking the maximum of two directional convolutions. Subtracting those convolutions gives two nonnegative margins, and the convex hull of the nonzero weights gives a polytope [6, Section 3.2, equations (15)–(17)]. We use this construction as the starting point for a finite quantum recursion. Our boundary conventions for the auxiliary margins are stated in Section 3; they differ from the printed Li–Pan rule for some degenerate labels.

For each $h \in \mathbb{Z}^2$, we first compute the classical carrier and the direction choices at its nonskeleton lattice points. We then replace the ordinary binomial kernels by symmetric Gaussian kernels and use these fixed choices to construct $\rho_h^{(v)}$ in the based quantum torus. Theorem 4.2 proves that the recursion terminates and defines a bar-invariant Laurent element. Theorem 4.3 compares the updates themselves and gives

$$\mathrm{sp}_1(\rho_h^{(v)}) = \rho_h.$$

This specialization is proved directly from the classical and quantum recursions, before identifying either table with greedy coefficients.

The greedy basis was constructed in the commutative setting by Lee, Li, and Zelevinsky [4], following the canonical bases of Sherman and Zelevinsky in finite and affine rank-two cluster algebras [8]. Lee, Li, Rupel, and Zelevinsky constructed its quantum analogue [2, 3]. Our identification theorem is the exact equality

$$\rho_h^{(v)} = G_{(-h_1,-h_2+c[-h_1]_+)} \qquad (h \in \mathbb{Z}^2).$$

By the label bijection and the Lee–Li–Rupel–Zelevinsky theorem, the constructed elements belong to the quantum cluster algebra and form its existing quantum greedy basis.

We also prove equality of the classical and quantum coefficient supports and of the corresponding margin zero sets. The Newton-polytope formula then follows from the Laurent-exponent map. These are statements about vanishing: quantum greedy coefficients need not be nonnegative.

The identification must justify omitting predecessors and using the classical choice of direction in the quantum recursion. Classical support saturation [5, Proposition 3.4 and Corollary 3.5], the quantum support bound, and cluster-monomial positivity give coefficient

2020 *Mathematics Subject Classification.* 13F60, 16T20, 52B20.

Key words and phrases. quantum cluster algebra, greedy basis, Newton polytope.

zero transfer. Quantum reflection transfers this result to inverse convolutions. A joint induction then identifies the weight table and both margin tables.

Throughout, b, c are positive integers, $[x]_+ = \max\{x, 0\}$, and $\mathbb{Z}_{\geq 0}$ denotes the nonnegative integers. After the conventions in Section 2, Sections 3–5 give the constructions, specialization and identification. Section 6 gives the consequences, and Section 7 contains examples and further questions.

AI disclosure. The main results of this paper were generated using GPT-6-Astra and the Danus system. The author supplied relevant references, checked the manuscript, and revised its exposition.

2. Quantum tori and quantum greedy elements

2.1. **Quantum tori and rank-two seeds.** Let v be an indeterminate. Let $L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and let $\Lambda : L \times L \to \mathbb{Z}$ be the skew-symmetric bilinear form whose matrix in the ordered basis (e_1, e_2) is

$$\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus $\Lambda(e_1, e_2) = -1$ and $\Lambda(e_2, e_1) = 1$.

Definition 2.1. The based quantum torus $\mathcal{T}_v$ is the free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{X^a : a \in L\}$ and multiplication

$$X^a X^{a'} = v^{\Lambda(a,a')} X^{a+a'}. \tag{1}$$

Its skew field of fractions is denoted by $\mathcal{F}$.

Set $X_i = X^{e_i}$. Equation (1) gives

$$X_2 X_1 = v^2 X_1 X_2, \qquad X^{(r,t)} = v^{rt} X_1^r X_2^t. \tag{2}$$

In particular,

$$\begin{aligned} X_1 X^{(r,t)} &= v^{-t} X^{(r+1,t)}, \quad X^{(r,t)} X_1 = v^t X^{(r+1,t)}, \\ X_2 X^{(r,t)} &= v^r X^{(r,t+1)}, \quad X^{(r,t)} X_2 = v^{-r} X^{(r,t+1)}. \end{aligned} \tag{3}$$

Definition 2.2. The bar map is the $\mathbb{Z}$-linear antiautomorphism of $\mathcal{T}_v$ determined by $\overline{v} = v^{-1}$ and $\overline{X^a} = X^a$. Thus $\overline{fg} = \overline{g}\,\overline{f}$.

Definition 2.3. A toric frame in $\mathcal{F}$ is a map $M : \mathbb{Z}^2 \to \mathcal{F}^\times$ of the form $M(a) = \varphi(X^{\eta(a)})$, where η is a lattice isomorphism and φ is a $\mathbb{Q}(v)$-algebra automorphism of $\mathcal{F}$ (so it fixes v); in particular it satisfies

$$M(a)M(a') = v^{\Lambda_M(a,a')} M(a + a')$$

for a skew form Λ_M. We use the toric-frame definition and ground-field convention of [1, Sections 4.2–4.3], with $q^{1/2} = v$. Let $B = (b_{ij})$ be a 2×2 integer exchange matrix. The pair (Λ_M, B) is compatible when $B^T \Lambda_M$ is diagonal with positive diagonal entries. A quantum seed is a compatible pair (M, B).

Let $M_{\mathrm{init}}(a) = X^a$ for $a \in \mathbb{Z}^2$. The initial quantum seed is (M_{init}, B), where

$$B = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}, \qquad B^T \Lambda = \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix}, \tag{4}$$

where $b, c \in \mathbb{Z}_{>0}$.

Convention 2.4. For $k \in \{1,2\}$, mutation replaces B by $B' = (b'_{ij})$, where

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+, & i,j \neq k, \end{cases}$$

Coefficient-free Berenstein–Zelevinsky mutation fixes $M(e_j)$ for $j \neq k$ and gives

$$M'(e_k) = M\left(-e_k + \sum_i [b_{ik}]_+ e_i\right) + M\left(-e_k + \sum_i [-b_{ik}]_+ e_i\right), \tag{5}$$

The rule is the coefficient-free specialization of Definitions 4.3 and 4.8 and Proposition 4.9 of [1]; Proposition 4.7 of the same source ensures that (M', B') is again a quantum seed, hence compatible.

For example, mutation in direction 1 gives

$$X'_1 = X^{(-1,c)} + X^{(-1,0)}.$$

Since $\Lambda(-e_1 + ce_2, e_1) = c$, multiplication on the right by X_1 produces

$$X'_1 X_1 = v^c X_2^c + 1. \tag{6}$$

Definition 2.5. The coefficient-free rank-two quantum cluster algebra $\mathcal{A}_v(b,c)$ is the $\mathbb{Z}[v^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by all cluster variables obtained by iterated mutation from the seed (4). A quantum cluster monomial is a based monomial in two variables from one cluster with nonnegative exponents.

Starting from X_1, X_2, index the cluster variables in both directions by

$$X_{m+1}X_{m-1} = 1 + v^{r_m} X_m^{r_m}, \qquad r_m = \begin{cases} b, & m \text{ odd}, \\ c, & m \text{ even}, \end{cases} \quad m \in \mathbb{Z}. \tag{7}$$

Here $X_3 = X'_1$ from (6), and X_0 is obtained by mutating the second initial variable. The forward and backward recursions are the corresponding seed mutations. Consecutive variables satisfy $X_{m+1}X_m = v^2 X_m X_{m+1}$ and generate a based quantum torus. For the ordered cluster (X_m, X_{m+1}), write $\Sigma_m = (M_m, B_m)$, where

$$M_m(a,d) = v^{ad} X_m^a X_{m+1}^d, \qquad B_m = \begin{cases} \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}, & m \text{ odd}, \\ \begin{pmatrix} 0 & -c \\ b & 0 \end{pmatrix}, & m \text{ even}. \end{cases}$$

In particular, $M_1 = M_{\text{init}}$ and $\Sigma_1 = (M_{\text{init}}, B)$. Each ordered frame has skew form Λ, and $B_m^T \Lambda$ is $\operatorname{diag}(c,b)$ or $\operatorname{diag}(b,c)$, respectively. In fixed mutation slots, the matrices instead alternate between B and $-B$: the first mutated cluster is (X_3, X_2), and exchanging its two slots gives the ordered belt cluster (X_2, X_3) and matrix B_2 above. Mutation is involutive, so adjacent repetitions in a mutation word cancel. Every reduced word in two directions alternates, and the two-sided belt therefore covers all rank-two clusters.

Evaluation at $v = 1$ is the ring homomorphism

$$\varepsilon : \mathbb{Z}[v^{\pm 1}] \longrightarrow \mathbb{Z}, \qquad v \longmapsto 1.$$

On a finite based-torus sum it induces

$$\operatorname{sp}_1\left(\sum_a f_a(v) X^a\right) = \sum_a f_a(1) x^a \tag{8}$$

into the commutative Laurent ring $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$. The monomials X^a are based monomials, with the phase fixed by (2).

2.2. **Gaussian kernels and quantum greedy elements.**

Definition 2.6. For $d = (d_1, d_2) \in \mathbb{Z}^2$, a finite element of $\mathcal{T}_v$ is pointed at d if it has the form

$$F = \sum_{p,q \geq 0} e_d^v(p,q) X^{(-d_1+bp, -d_2+cq)}, \qquad e_d^v(0,0) = 1. \tag{9}$$

The pair (p, q) is the pointed coordinate of the displayed term.

For $n \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$, and $r \in \mathbb{Z}_{>0}$, define in $\mathbb{Q}(v)$

$$[n]_{v^r} = \frac{v^{rn} - v^{-rn}}{v^r - v^{-r}}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{v^r} = \prod_{j=1}^{k} \frac{[n-j+1]_{v^r}}{[j]_{v^r}}, \tag{10}$$

where the empty product is 1. For $k < 0$, set $\begin{bmatrix} n \\ k \end{bmatrix}_{v^r} = 0$. For nonnegative upper index these are bar-invariant Laurent polynomials. For negative upper index, the identity

$$\begin{bmatrix} -n \\ k \end{bmatrix}_{v^r} = (-1)^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{v^r} \qquad (n > 0,\ k \geq 0) \tag{11}$$

follows by replacing each $[-n-j+1]_{v^r}$ with $-[n+j-1]_{v^r}$ in the product and reversing the numerator factors. It includes $k = 0$ and shows that the products belong to $\mathbb{Z}[v^{\pm 1}]$ and are bar invariant for all integral upper indices. We use the zero-extended kernel

$$K_r(n,k) = \begin{cases} \begin{bmatrix} n \\ k \end{bmatrix}_{v^r}, & 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Let $n \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{>0}$, and let t be a central formal variable. The quantum binomial identity in $\mathbb{Z}[v^{\pm 1}][t]$ is

$$\sum_{k=0}^{n} K_r(n,k) t^k = \prod_{j=1}^{n} \left(1 + v^{r(n+1-2j)} t\right). \tag{13}$$

Its inverse in $\mathbb{Z}[v^{\pm 1}][[t]]$ is

$$\left(\sum_{j=0}^{n} K_r(n,j) t^j \right)^{-1} = \sum_{j \geq 0} (-1)^j \begin{bmatrix} n+j-1 \\ j \end{bmatrix}_{v^r} t^j. \tag{14}$$

The $j = 0$ coefficient is one. When $n = 0$, every positive coefficient on the right is interpreted as zero, so the inverse is one. Multiplication of (13) and (14) verifies the identity coefficientwise.

Theorem 2.7 (Lee–Li–Rupel–Zelevinsky recurrence and uniqueness)**.** *For every $d = (d_1, d_2) \in \mathbb{Z}^2$ there is a unique pointed element G_d whose coefficients satisfy, for $(p,q) \neq (0,0)$,*

$$e_d^v(p,q) = \begin{cases} \displaystyle\sum_{k=1}^{p} (-1)^{k-1} e_d^v(p-k,q) \begin{bmatrix} [d_2 - cq]_+ + k - 1 \\ k \end{bmatrix}_{v^b}, & cd_1 q \leq bd_2 p, \\ \displaystyle\sum_{\ell=1}^{q} (-1)^{\ell-1} e_d^v(p,q-\ell) \begin{bmatrix} [d_1 - bp]_+ + \ell - 1 \\ \ell \end{bmatrix}_{v^c}, & cd_1 q \geq bd_2 p. \end{cases} \tag{15}$$

On the equality wall both displayed values agree. The element G_d belongs to $\mathcal{A}_v(b,c)$.

This is Definition 1.6 and Theorem 1.7 of [3]. The proof is completed in their Section 5 by identifying the upper and lower quasi-greedy elements (Theorem 5.8). Membership is Corollary 1.8.

The first branch advances p with scale v^b; the second advances q with scale v^c.

Remark 2.8 (Directional scale check). In arXiv:1405.2414v1 (10 May 2014), page 6, the first divisibility display in equation (2.1) contains v^c where the p-direction divisibility, the rank-two exchange of parameters, and the authors' subsequent formulas require v^b, as in (15). This observation concerns only that display in that preprint version; the carrier-scale correspondence is checked in the proof of Theorem 5.9.

The Lee–Li–Rupel–Zelevinsky pointed support region is defined by the following six cases. All sets below are understood as subsets of $\mathbb{R}^2_{\geq 0}$.

(i) if $d_1, d_2 \leq 0$, it is $\{(0,0)\}$;
(ii) if $d_1 \leq 0 < d_2$, it is $\{(p,0) : 0 \leq p \leq d_2\}$;
(iii) if $d_2 \leq 0 < d_1$, it is $\{(0,q) : 0 \leq q \leq d_1\}$;
(iv) if $0 < bd_2 \leq d_1$, it is $\{(p,q) : p \leq d_2,\ q \leq d_1 - bp\}$;
(v) if $0 < cd_1 \leq d_2$, it is $\{(p,q) : q \leq d_1,\ p \leq d_2 - cq\}$;
(vi) if $0 < d_1 < bd_2$ and $0 < d_2 < cd_1$, it is the union

$$\left\{(p,q) : 0 \leq p < \frac{d_1}{b},\ q + \left(b - \frac{bd_2}{cd_1}\right) p < d_1\right\} \tag{16}$$

$$\cup \left\{(p,q) : 0 \leq q < \frac{d_2}{c},\ p + \left(c - \frac{cd_1}{bd_2}\right) q < d_2\right\} \cup \{(0,d_1),(d_2,0)\}. \tag{17}$$

This is Lee–Li–Rupel–Zelevinsky Definition 3.1; Proposition 3.7 of [3] gives the pointed support containment for G_d.

Theorem 2.9 (Lee–Li–Rupel–Zelevinsky basis theorem)**.** *For fixed positive b, c, the family $\{G_d : d \in \mathbb{Z}^2\}$ is an $\mathbb{Z}[v^{\pm 1}]$-basis of $\mathcal{A}_v(b,c)$. It is bar invariant, independent of the initial cluster, contains all quantum cluster monomials, and specializes at $v = 1$ to the classical greedy basis.*

The cited assertions are Theorem 1.7 of [4] after specialization and Theorem 1.9 of [3] in the quantum setting.

We also use the reflection symmetry of Lee–Li–Rupel–Zelevinsky Proposition 4.5 and the cluster-monomial sector of Corollary 4.6, with their precise coefficient translations given in Section 5.

3. The classical polyhedral construction

We use the rank-two polyhedral recursion of [6, Section 3.2, equations (15)–(17)], with the two boundary normalizations specified below. Li–Pan's Proposition 3.1 states the label correspondence for their construction. For the normalized recursion here, well-definedness, nonnegativity and finiteness are proved in this section; the classical table identification and degenerate calculations in Section 5 then prove $\rho_h = g_{(-h_1,-h_2+c[-h_1]_+)}$ in Corollary 5.4. This identification does not follow merely by applying their proposition to changed boundary data. We use their notation $co_{(u,w)}$, m_1, m_2, N_h, and ρ_h; the superscript (v) distinguishes quantum weights, margins, and functions. Here ρ_h denotes the classical element defined by our normalized recursion. We do not identify it with a mechanical application of all the printed boundary rules. We keep the coordinate w because v is the quantum parameter:

their (u,v) is our (u,w). Their exchange matrix is $\left(\begin{smallmatrix} 0 & b \\ -c & 0 \end{smallmatrix}\right) = -B$. For comparison with their coefficient-free specialization,

$$\widehat{y}_1\big|_{y=1} = x_2^{-c}, \qquad \widehat{y}_2\big|_{y=1} = x_1^{b}, \qquad \widehat{y}_1^{u}\widehat{y}_2^{w}x^{h}\big|_{y=1} = x_1^{h_1+bw}x_2^{h_2-cu}.$$

We work only with this coefficient-free specialization. The recursion is defined on the full strip before applying a finite cutoff.

Definition 3.1. For integers n, j, let

$$\widetilde{\binom{n}{j}} = \begin{cases} \binom{n}{j}, & 0 \le j \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus empty sums and positive-index kernels of order zero contribute nothing.

Our convention gives $\widetilde{\binom{0}{0}} = 1$. In contrast, the modified binomial coefficient printed at the beginning of Section 3 of [6] is zero whenever its upper index is nonpositive. For positive lower indices the kernels agree; the difference at $(n,j) = (0,0)$ concerns the initialization of a zero-length skeleton. We use the value one there, rather than the zero endpoint weight that the printed rule would give. Thus not all skeleton weights agree entrywise with that rule.

We first define the classical table on $\{0,\dots,[-h_1]_+\} \times \mathbb{Z}_{\ge 0}$. The cutoff lemma below will allow us to restrict it to the finite rectangle

$$\left\{ (u,w) \in \mathbb{Z}^2 : \begin{array}{c} 0 \le u \le [-h_1]_+, \\ 0 \le w \le [-h_2 + c[-h_1]_+]_+ + [\left\lceil \dfrac{[-h_1]_+}{b} \right\rceil - 1]_+ \end{array} \right\}. \tag{18}$$

Definition 3.2 (Semi-infinite classical table)**.** The table is defined on $\{0,\dots,[-h_1]_+\}\times\mathbb{Z}_{\ge 0}$, row by row in increasing w and within each row in decreasing u. No finite support bound is assumed here. On the bottom edge initialize $co_{(u,0)} = \widetilde{\binom{[-h_1]_+}{u}}$, and on the right edge initialize $co_{([-h_1]_+,w)} = \widetilde{\binom{[-h_2+c[-h_1]_+]_+}{w}}$ for every $w \ge 0$. At the common corner $([-h_1]_+, 0)$ set $co_{(u,w)} = m_1 = m_2 = 1$. At every other bottom point set $m_1 = 0$, $m_2 = co_{(u,w)}$; at every other right-edge point set $m_1 = co_{(u,w)}$, $m_2 = 0$. The common corner is initialized only once, including when a skeleton has length zero or the two skeletons coincide at a point.

Independently of the zero-order binomial normalization, we modify the first skeleton margin condition: Li–Pan's equation (15) uses $u = -h_1$, whereas here it is $u = [-h_1]_+$. When $h_1 > 0$, we do not claim entrywise equality with that source's margins. For $h = (1,-1)$, the right edge is $u = 0$ and has length one, so $co_{(0,1)} = \widetilde{\binom{1}{1}} = 1$ and our rule gives $m_1(0,1) = 1$. The printed condition would require $u = -1$, and hence gives $m_1(0,1) = 0$. Thus the weight and label correspondence does not extend to all boundary margins. For positive denominator labels, the full-strip margin dictionary is proved below in Proposition 5.3.

At each remaining point, the horizontal and vertical classical sums are

$$\sum_{j=1}^{[-h_1]_+-u} m_1(u+j,w)\widetilde{\binom{-h_1-bw}{j}}, \tag{19}$$

$$\sum_{j=1}^{w} m_2(u,w-j)\widetilde{\binom{-h_2+cu}{j}}. \tag{20}$$

Set

(21)
$$co_{(u,w)} = \max\left\{ \sum_{j=1}^{[-h_1]_+-u} m_1(u+j,w)\binom{\widetilde{-h_1-bw}}{j}, \sum_{j=1}^{w} m_2(u,w-j)\binom{\widetilde{-h_2+cu}}{j} \right\},$$
$$m_1(u,w) = co_{(u,w)} - \sum_{j=1}^{[-h_1]_+-u} m_1(u+j,w)\binom{\widetilde{-h_1-bw}}{j},$$
$$m_2(u,w) = co_{(u,w)} - \sum_{j=1}^{w} m_2(u,w-j)\binom{\widetilde{-h_2+cu}}{j}.$$

When the sums agree, select the horizontal one.

Lemma 3.3 (Well-definedness and nonnegativity)**.** *The semi-infinite table is uniquely defined and all its entries are nonnegative integers. At every point of the strip,*

$$0 \le m_1(u,w) \le co_{(u,w)}, \qquad 0 \le m_2(u,w) \le co_{(u,w)}. \tag{22}$$

Proof. Each row has $[-h_1]_+ + 1$ points and each sum is finite. Every summand uses an earlier row or an earlier point of the current row. The initialized binomial coefficients and margins are nonnegative integers. Inductively, nonnegative earlier margins and kernels make both classical sums nonnegative. Taking their maximum and subtracting either nonnegative proposal gives an integer margin between zero and $co_{(u,w)}$. On each ordinary skeleton point the margins are zero and $co_{(u,w)}$, and at the common corner all three entries are one. These checks also apply when the strip has width zero or a skeleton length vanishes. The induction proves the bounds together with existence and uniqueness. □

Lemma 3.4 (Finite classical cutoff)**.** *For the semi-infinite table, all coefficients and both margins vanish when*

$$w > [-h_2 + c[-h_1]_+]_+ + [\left\lceil \frac{[-h_1]_+}{b} \right\rceil - 1]_+.$$

Hence the support lies in the rectangle (18)*.*

Proof. Both proposals are nonnegative by Lemma 3.3. If $[-h_1]_+ > 0$ and $w \ge \lceil [-h_1]_+/b \rceil$, then $-h_1 - bw \le 0$. Every positive-index horizontal kernel is zero, so the horizontal proposal vanishes. At a noninitialized point in such a row, the coefficient equals the vertical proposal and the vertical margin is zero. A nonzero vertical margin can therefore occur only in rows

$$0 \le w \le [\left\lceil \frac{[-h_1]_+}{b} \right\rceil - 1]_+,$$

apart from the right edge, where it is zero away from the common corner. If $[-h_1]_+ = 0$, the whole strip is the right edge and only its common corner can have a nonzero vertical margin; this is in row zero.

For $0 \le u \le [-h_1]_+$ we have $-h_2 + cu \le [-h_2 + c[-h_1]_+]_+$. A nonzero term of the vertical convolution must therefore satisfy

$$w - j \le [\left\lceil \frac{[-h_1]_+}{b} \right\rceil - 1]_+, \qquad j \le [-h_2 + c[-h_1]_+]_+.$$

Adding these inequalities gives the bound in (18); above it, the vertical proposal therefore vanishes. If $[-h_1]_+ > 0$, then $\lceil [-h_1]_+/b \rceil \ge 1$, and an integer w above this bound also satisfies $w \ge \lceil [-h_1]_+/b \rceil$. Thus all positive-index horizontal kernels vanish as well. The

right-edge coefficient is zero by its binomial support, so both proposals, the coefficient and both margins vanish throughout such a row. If $[-h_1]_+ = 0$, every entry is on the initialized right edge and its binomial support gives the same conclusion directly. These arguments include segments and the point case. □

The cutoff gives the finite classical element

$$\rho_h = \sum_{\substack{0\le u\le [-h_1]_+ \\ w\ge 0}} co_{(u,w)} x_1^{h_1+bw} x_2^{h_2-cu}. \tag{23}$$

Definition 3.5 (Support-generated carrier)**.** Let

$$\begin{aligned} \operatorname{Supp}_{(u,w)} \rho_h &= \{(u,w) \in \mathbb{Z}^2_{\ge 0} : 0 \le u \le [-h_1]_+,\ co_{(u,w)} \ne 0\}, \\ N_h &= \operatorname{conv}\big(\operatorname{Supp}_{(u,w)} \rho_h\big). \end{aligned} \tag{24}$$

We call $\operatorname{Supp}_{(u,w)} \rho_h$ the support, N_h the convex carrier, and $(N_h \cap \mathbb{Z}^2)$ the lattice carrier. A carrier point means a lattice point of N_h, which need not belong to the support.

Corollary 3.6 (Vanishing classical margins)**.** *At a point of the classical strip with $co_{(u,w)} = 0$, both margins vanish. In particular, this holds at every strip point outside N_h.*

Proof. Use (22) and the inclusion of the coefficient support in N_h. □

Outside the strip, we extend the classical coefficient and margin arrays by zero. With this convention, deleting predecessors outside N_h does not change either classical sum.

Definition 3.7. For $h = (h_1, h_2) \in \mathbb{Z}^2$, put

$$d = \big(d_1, d_2\big) = \big(-h_1, -h_2 + c[-h_1]_+\big). \tag{25}$$

Its inverse is

$$h_1 = -d_1, \qquad h_2 = c[d_1]_+ - d_2, \tag{26}$$

so this correspondence is a bijection of $\mathbb{Z}^2$. The two skeleton lengths in the polyhedral construction are $[-h_1]_+$ and $[-h_2 + c[-h_1]_+]_+$ as in Li–Pan.

The degenerate carriers follow directly from the classical table. If $[-h_1]_+ = 0$, the strip is the initialized right edge, whose weights are nonzero exactly at $(0, w)$ with $0 \le w \le [-h_2 + c[-h_1]_+]_+$. If $[-h_2 + c[-h_1]_+]_+ = 0$, then $-h_2 + c[-h_1]_+ \le 0$, and hence $-h_2 + cu \le 0$ throughout the strip. Every positive-index vertical kernel is zero. For each $w > 0$ the right-edge weight and both margins are zero; induction from right to left makes the horizontal proposal, weight and both margins zero at every remaining point of that row. The support is therefore exactly the bottom points $(u, 0)$ with $0 \le u \le [-h_1]_+$, whose binomial weights are positive. Taking convex hulls gives the corresponding vertical or horizontal segment. When both lengths vanish, only the common corner remains.

Definition 3.8 (Coordinate maps)**.** For every h, define pointed coordinates by

$$(p, q) = (w, [-h_1]_+ - u), \qquad (u, w) = ([-h_1]_+ - q, p). \tag{27}$$

The Laurent-exponent map is

$$(h_1 + bw, h_2 - cu) = (-d_1 + bp, -d_2 + cq). \tag{28}$$

When $[-h_1]_+ = 0$, the surviving segment maps to $(p, 0)$, $0 \le p \le [-h_2 + c[-h_1]_+]_+$; when $[-h_2 + c[-h_1]_+]_+ = 0$, it maps to $(0, q)$, $0 \le q \le [-h_1]_+$; when both vanish, it maps

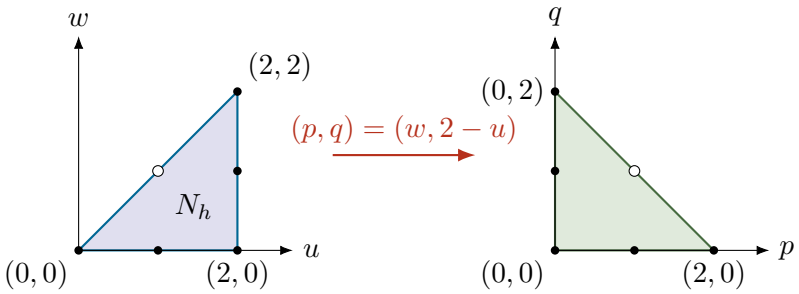


FIGURE 1. The carrier for $(b, c) = (2, 2)$ and $h = (-2, 2)$ and its image in pointed coordinates. The vertices $(0, 0), (2, 0), (2, 2)$ map respectively to $(0, 2), (0, 0), (2, 0)$. Filled points have nonzero weights; the open point $(1, 1)$ has weight zero in both coordinate systems. This is the specific affine example of Section 7.2.

to $(0, 0)$. The linear part of the Laurent-exponent map has determinant $bc > 0$, so this map is injective on $\mathbb{R}^2$ and on the lattice. For a finite sum $F = \sum_{u,w} f_{u,w} X^{(h_1+bw, h_2-cu)}$, define

$$\operatorname{Supp}_{(u,w)} F = \{(u, w) : f_{u,w} \neq 0\},$$
$$\operatorname{Supp}_{\text{Laurent}} F = \{(h_1 + bw, h_2 - cu) : (u, w) \in \operatorname{Supp}_{(u,w)} F\},$$

and $\operatorname{Newt}(F) = \operatorname{conv} \operatorname{Supp}_{\text{Laurent}} F$.

4. The quantum polyhedral construction

The quantum procedure uses the precomputed $(N_h \cap \mathbb{Z}^2)$, order, and classical branch decisions. In particular, it never takes a maximum of two Laurent polynomials.

Algorithm 4.1 (Finite quantum local elimination)**.** The input is (b, c, h) together with the classical carrier and the comparison of the sums (19) and (20). The state consists of three $\mathbb{Z}[v^{\pm 1}]$-valued tables $co^{(v)}_{(u,w)}, m^{(v)}_1, m^{(v)}_2$ on $(N_h \cap \mathbb{Z}^2)$.

Initialization. First initialize the common corner $(u, w) = ([-h_1]_+, 0)$ with $co^{(v)}_{(u,w)} = m^{(v)}_1(u, w) = m^{(v)}_2(u, w) = 1$. At bottom carrier points $(u, 0) \in N_h \cap \mathbb{Z}^2$ with $u < [-h_1]_+$, set $co^{(v)}_{(u,0)} = K_c([-h_1]_+, u)$, $m^{(v)}_1(u, 0) = 0$, and $m^{(v)}_2(u, 0) = co^{(v)}_{(u,0)}$. At right-edge carrier points $([-h_1]_+, w) \in N_h \cap \mathbb{Z}^2$ with $w > 0$, set $co^{(v)}_{([-h_1]_+,w)} = K_b([-h_2 + c[-h_1]_+]_+, w)$, $m^{(v)}_1([-h_1]_+, w) = co^{(v)}_{([-h_1]_+,w)}$, and $m^{(v)}_2([-h_1]_+, w) = 0$. These three cases are disjoint. When $[-h_1]_+ = 0$, there are no other bottom points; when the other skeleton length is zero, there are no right-edge carrier points with $w > 0$. A point carrier uses only the common-corner initialization.

Recursion. Visit all remaining points in increasing w and then decreasing u. The horizontal and vertical quantum sums are, respectively,

$$\sum_{\substack{1 \le j \le [-h_1]_+ - u \\ (u+j,w) \in N_h \cap \mathbb{Z}^2}} m^{(v)}_1(u + j, w) K_c(-h_1 - bw, j), \tag{29}$$

$$\sum_{\substack{1 \le j \le w \\ (u,w-j) \in N_h \cap \mathbb{Z}^2}} m^{(v)}_2(u, w - j) K_b(-h_2 + cu, j). \tag{30}$$

If the classical sum (19) is at least (20), assign (29) to $co^{(v)}_{(u,w)}$; otherwise assign (30). Then set

$$
\begin{aligned}
m_1^{(v)}(u,w) &= co^{(v)}_{(u,w)} - \sum_{\substack{1\le j\le [-h_1]_+ - u \\ (u+j,w)\in N_h\cap\mathbb{Z}^2}} m_1^{(v)}(u+j,w)K_c(-h_1-bw,j),\\
m_2^{(v)}(u,w) &= co^{(v)}_{(u,w)} - \sum_{\substack{1\le j\le w \\ (u,w-j)\in N_h\cap\mathbb{Z}^2}} m_2^{(v)}(u,w-j)K_b(-h_2+cu,j).
\end{aligned}
\tag{31}
$$

An out-of-domain predecessor is skipped, an empty sum is zero, and the kernel is zero extended by (12).

Output.

$$
\rho_h^{(v)} = \sum_{(u,w)\in(N_h\cap\mathbb{Z}^2)} co^{(v)}_{(u,w)} X^{(h_1+bw,h_2-cu)}. \tag{32}
$$

For comparisons, extend $co^{(v)}_{(u,w)}, m_1^{(v)}, m_2^{(v)}$ by zero outside $(N_h\cap\mathbb{Z}^2)$; the algorithm does not read these entries. Pointed coefficient arrays are zero at negative indices. The classical arrays retain the full-strip definition and zero extension of Section 3; their vanishing outside the carrier follows from Corollary 3.6. The carrier is fixed: the quantum construction changes its weights and the associated Laurent element. The equality of classical and quantum nonzero positions is proved in Section 6.

Theorem 4.2 (Finite quantum polyhedral construction)**.** *For every $b,c\in\mathbb{Z}_{>0}$ and $h\in\mathbb{Z}^2$, Algorithm 4.1 is well defined. It assigns every point of $(N_h\cap\mathbb{Z}^2)$ exactly once, terminates, and produces an element of $\mathcal{T}_v$ with coefficients in $\mathbb{Z}[v^{\pm1}]$. Every coefficient and each of the two residual tables is bar invariant.*

Proof. Finiteness of $(N_h\cap\mathbb{Z}^2)$ follows from Lemma 3.4. Rank a point by

$$
\mathrm{rk}(u,w) = (w, [-h_1]_+ - u)
$$

in lexicographic order. A predecessor $(u+j,w)$ in (29) has the same first rank coordinate and a smaller second coordinate. A predecessor $(u,w-j)$ in (30) has a smaller first coordinate. Thus every predecessor has already been initialized or computed. Initialization takes priority, and each remaining point is processed once; finiteness of the carrier gives termination.

The kernels and initial data lie in $\mathbb{Z}[v^{\pm1}]$ and are bar invariant. Finite addition, multiplication and subtraction preserve these properties in the coefficient ring, and the classical selector is fixed by bar. Induction over the same rank gives both properties for the three tables. Since bar fixes based monomials, it also fixes $\rho_h^{(v)}\in\mathcal{T}_v$. □

Membership in $\mathcal{A}_v(b,c)$ is proved in Section 5.

4.1. Stepwise classical specialization.

Theorem 4.3 (Algorithmic classical specialization)**.** *At every visited point,*

$$
\begin{aligned}
\varepsilon(co^{(v)}_{(u,w)}) &= co_{(u,w)},\\
\varepsilon(m_1^{(v)}(u,w)) &= m_1(u,w),\\
\varepsilon(m_2^{(v)}(u,w)) &= m_2(u,w).
\end{aligned}
\tag{33}
$$

Consequently

$$
\mathrm{sp}_1(\rho_h^{(v)}) = \rho_h. \tag{34}
$$

Proof. For $n \geq 0$, apply ε coefficientwise in the Laurent polynomial ring to (13). This gives

$$\sum_{j=0}^{n} \varepsilon(K_r(n,j))t^j = (1+t)^n.$$

Comparing coefficients of t yields

$$\varepsilon(K_r(n,j)) = \widetilde{\binom{n}{j}} \tag{35}$$

for $0 \leq j \leq n$. For negative upper indices or out-of-range lower indices, both sides are zero by the kernel definitions; for $n = j = 0$, both sides are one. Thus the identity holds for all integer n, j. This evaluates Laurent polynomials, not the Gaussian quotient at $v = 1$. The bottom, right-edge and common-corner initializations therefore specialize correctly.

By Corollary 3.6 and zero extension outside the strip, every omitted classical predecessor has weight and margins zero. Restricting the classical sums to the quantum predecessor domain therefore leaves them unchanged.

Induct in the rank $(w, [-h_1]_+ - u)$. Assume (33) for every predecessor of (u, w). Applying ε to the finite sums (29)–(30), and using (35), we recover the horizontal and vertical classical sums (19)–(20), respectively. The quantum and classical runs use the same precomputed selector. Hence the selected proposal specializes to $co_{(u,w)}$, and subtracting the two proposals gives the two margin identities. Common-corner priority, empty sums, and the segment and point initializations cover the remaining boundary cases.

Applying (8) to the finite output proves (34), since all classical weights outside the carrier are zero. □

Example 4.4 (One update in wild type)**.** Take $(b, c) = (2, 3)$ and $h = (-2, 2)$. At $(u, w) = (1, 1)$ the horizontal kernel has order $-h_1 - bw = 0$, so both horizontal proposals vanish. The vertical kernel has order $-h_2 + cu = 1$. Its only predecessor is the bottom point $(1, 0)$, where $m_2(1, 0) = 2$ and $m_2^{(v)}(1, 0) = K_3(2, 1)$. Thus the classical vertical proposal is $2\binom{1}{1} = 2$, and the selector chooses it. The quantum vertical proposal is

$$K_3(2,1)K_2(1,1) = v^{-3} + v^3.$$

The update gives

$$co_{(1,1)}^{(v)} = m_1^{(v)}(1,1) = v^{-3} + v^3, \qquad m_2^{(v)}(1,1) = 0.$$

At $v = 1$ these become $co_{(1,1)} = m_1(1, 1) = 2$ and $m_2(1, 1) = 0$. The complete table appears in Section 7.3.

Remark 4.5 (Arithmetic cost and reuse). Assume that the kernel coefficients are available. At each point (u, w) of the rectangle (18), direct classical summation uses at most $[-h_1]_+ - u + w$ predecessor terms, together with one comparison and two subtractions. Storage is proportional to the number of lattice points in that rectangle. The quantum run uses at most

$$\sum_{(u,w) \in N_h \cap \mathbb{Z}^2} ([-h_1]_+ - u + w)$$

predecessor terms and stores three ring elements per carrier point, apart from the kernel tables. Computing the convex hull takes a number of comparisons and orientation tests bounded by a constant times the number of input points times its logarithm. Scanning the rectangle against the hull takes at most one test per rectangle-point/hull-edge pair; segments and points are treated separately. These are unit-cost arithmetic counts and do not bound bit complexity or Laurent coefficient size.

Direct evaluation of the greedy recurrence on the same rectangle has a comparable bound: each point requires at most $w + [-h_1]_+ - u$ predecessor terms. Our construction also requires classical preprocessing and a convex hull, so these counts do not establish a uniform asymptotic speedup. The domain and selector are computed once and can be reused under coefficient-ring specializations.

5. Identification with the quantum greedy basis

For every $d \in \mathbb{Z}^2$, write $e^v_{b,c;d}(p,q)$ for the based pointed coefficient of G_d introduced in Section 2, and put $e^1_{b,c;d}(p,q) = e^v_{b,c;d}(p,q)|_{v=1}$. We omit b, c from the subscripts when they are fixed.

We use $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise.

5.1. **Degenerate cases.**

Lemma 5.1 (Nonpositive coordinates and degenerate carriers)**.** *If $d_1 \le 0$ and $d_2 \le 0$, the nonzero support of each coefficient table is the common point, with coefficient* 1*. If exactly one coordinate is positive, the nonzero supports are the lattice points of the corresponding axis segment, with coefficients $\binom{n}{k}$ and $K_r(n,k)$. The carrier is that point or segment. All coefficients and both margins vanish outside the stated support on the classical strip; quantum arrays are zero-extended outside the lattice carrier. Common-corner priority and coefficient and residual zero transfer hold in every case.*

Proof. Write $d = (d_1, d_2)$ as in (25). The recurrence (15) and its classical specialization give the following complete coefficient arrays:

condition	$e^v_d(p,q)$	$e^1_d(p,q)$
$d_1, d_2 \le 0$	$\delta_{p0}\delta_{q0}$	$\delta_{p0}\delta_{q0}$
$d_1 \le 0 < d_2$	$\delta_{q0} K_b([d_2]_+, p)$	$\delta_{q0}\binom{[d_2]_+}{p}$
$d_2 \le 0 < d_1$	$\delta_{p0} K_c([d_1]_+, q)$	$\delta_{p0}\binom{[d_1]_+}{q}$

For the second row, the first branch of (15) is valid at every $(p,q) \neq (0,0)$, since $cd_1q \le 0 \le bd_2p$. For each fixed $q > 0$, the value at $p = 0$ is the empty sum, hence zero; induction on p then gives $e^v_d(p,q) = 0$ for all $p \ge 0$. For $q = 0$, the same recurrence with $e^v_d(0,0) = 1$ gives the Gaussian row $K_b([d_2]_+, p)$: its generating series multiplied by the inverse kernel in (14) is 1, so it is the product in (13). The third row follows by simultaneously exchanging the coordinates and parameters. In the first row both orders are zero, so all positive-index inverse-kernel coefficients vanish; either applicable branch therefore gives zero at every noninitial point. Specializing gives the stated classical arrays. In carrier coordinates these are respectively a point, $(0,p)$, and $([d_1]_+ - q, 0)$. In the second row, the horizontal margin is the coefficient row and the vertical margin is δ_{p0}; in the third, the horizontal margin is δ_{q0} and the vertical margin is the coefficient row. At the common corner both margins equal one. For off-segment classical entries, if $[d_1]_+ = 0$ the strip is the initialized right edge. If $[d_2]_+ = 0 < [d_1]_+$, then $-h_2 + cu \le -h_2 + c[d_1]_+ = d_2 \le 0$ throughout the strip, so every positive-index vertical kernel is zero. For each $w > 0$ the right-edge coefficient and both margins are zero; induction in decreasing u then gives a zero horizontal proposal, coefficient, and both margins along the entire row. Quantum off-carrier entries are zero by definition. Gaussian rows have the same nonzero indices as ordinary binomial rows, proving all stated zero transfers. These calculations also agree with Lee–Li–Rupel–Zelevinsky Proposition 3.5 and Corollary 3.6. □

5.2. **Inverse convolutions and reflection.** Assume $d = (d_1, d_2) \in \mathbb{Z}^2_{>0}$, with h related by (25). The specialized coefficients $e^1_d(p,q)$ are nonnegative integers by Lee–Li–Zelevinsky Theorem 1.7. One has $e^1_d(0,0) = 1$ and, away from $(0,0)$,

$$e^1_d(p,q) = \begin{cases} \displaystyle\sum_{k=1}^{p} (-1)^{k-1} e^1_d(p-k,q) \binom{[d_2 - cq]_+ + k - 1}{k}, & cd_1 q \le bd_2 p, \\ \displaystyle\sum_{j=1}^{q} (-1)^{j-1} e^1_d(p,q-j) \binom{[d_1 - bp]_+ + j - 1}{j}, & cd_1 q \ge bd_2 p. \end{cases} \tag{36}$$

Both values agree on the wall; an order-zero inverse has all its positive-degree coefficients zero. Let t be a central formal variable; all series below lie in $\mathbb{Z}[v^{\pm 1}][[t]]$. We write $[t^j]F$ for the coefficient of t^j in a formal series F. Define the two inverse-convolution margins by

$$\xi_p(q) = [t^q] \left(\sum_{j=0}^{[d_1 - bp]_+} K_c([d_1 - bp]_+, j) t^j \right)^{-1} \sum_{r \ge 0} e^v_d(p,r) t^r, \tag{37}$$

$$\eta_p(q) = [t^p] \left(\sum_{j=0}^{[d_2 - cq]_+} K_b([d_2 - cq]_+, j) t^j \right)^{-1} \sum_{r \ge 0} e^v_d(r,q) t^r. \tag{38}$$

The kernel polynomials have constant term one, so each coefficient is a finite expression in $\mathbb{Z}[v^{\pm 1}]$. Inverting the defining relations gives

$$e^v_d(p,q) = \xi_p(q) + \sum_{j=1}^{q} K_c([d_1 - bp]_+, j) \xi_p(q-j), \tag{39}$$

$$e^v_d(p,q) = \eta_p(q) + \sum_{j=1}^{p} K_b([d_2 - cq]_+, j) \eta_{p-j}(q). \tag{40}$$

We use the simultaneous exchange

$$(b, c, d_1, d_2, p, q) \longleftrightarrow (c, b, d_2, d_1, q, p), \qquad \xi \longleftrightarrow \eta. \tag{41}$$

On coefficient arrays this sends $e^v_{b,c;d_1,d_2}(p,q)$ to $e^v_{c,b;d_2,d_1}(q,p)$; on the quantum torus it swaps X_1, X_2 and sends v to v^{-1}. The greedy recurrence is preserved, and bar invariance removes this parameter inversion from coefficient comparisons. The q-direction inverse with order $[d_1 - bp]_+$ and scale v^c becomes the p-direction inverse with order $[d_2 - cq]_+$ and scale v^b.

Lemma 5.2 (Inverse coefficients under reflection)**.** *For $d_1, d_2 > 0$, $0 \le p \le d_2$, and $d_1 - bp > 0$,*

$$\xi_p(q) = e^v_{bd_2 - d_1, d_2}(d_2 - p, q) \qquad (q \ge 0). \tag{42}$$

After exchanging (b, c, d_1, d_2, p, q) with (c, b, d_2, d_1, q, p), the analogous formula is $\eta_p(q) = e^v_{d_1, cd_1 - d_2}(p, d_1 - q)$ when $d_2 - cq > 0$ and $0 \le q \le d_1$. These formulas also hold at $v = 1$.

Proof. We use the coefficient calculation of Lee–Li–Rupel–Zelevinsky Lemma 2.2, equation (2.2), and Proposition 4.5 [3]. The semilinear reflection σ_2 sends v to v^{-1}, X_1 to X_3, and fixes X_2. It preserves multiplication order, $\sigma_2(fg) = \sigma_2(f)\sigma_2(g)$, as in [3, Lemma 2.1]. The coefficients $e^v_d(p,q)$ are bar invariant. For any integer r, the formal quantum binomial expansion is

$$X_3^r = \sum_{\ell \ge 0} \begin{bmatrix} r \\ \ell \end{bmatrix}_{v^c} v^{-rc\ell} X_1^{-r} X_2^{c\ell}.$$

For negative r, this expansion is taken in the skew Laurent-series ring

$$\widehat{\mathcal{T}} = \mathbb{Z}[v^{\pm 1}][X_1^{\pm 1}]((X_2;\tau)), \qquad \tau(X_1) = v^2 X_1, \quad \tau(v) = v,$$

whose elements are series $\sum_{j\geq j_0} a_j X_2^j$ with $a_j \in \mathbb{Z}[v^{\pm 1}][X_1^{\pm 1}]$ and multiplication $X_2 a = \tau(a) X_2$. The displayed expansion lies in its X_2^c-power subseries, up to the fixed Laurent shift. Since exponents are bounded below, each coefficient of a product is a finite sum. The initial torus embeds in this ring, and the reflected greedy element is a finite Laurent element by Lee–Li–Rupel–Zelevinsky symmetry; thus its expansion may be compared coefficient by coefficient. This calculation only expands the images of the finite torus expressions under consideration; it does not require an extension of σ_2 to an automorphism of $\widehat{\mathcal{T}}$. Hence

$$\begin{aligned}
&\sigma_2\left(\sum_q e_d^v(p,q) X^{(-d_1+bp,-d_2+cq)}\right) \\
&\quad = \sum_{q,\ell} e_d^v(p,q) \begin{bmatrix} -d_1+bp \\ \ell \end{bmatrix}_{v^c} v^{(d_1-bp)(-d_2+c(q+\ell))} X_1^{d_1-bp} X_2^{-d_2+c(q+\ell)} \\
&\quad = \sum_{q,\ell} e_d^v(p,q) \begin{bmatrix} -d_1+bp \\ \ell \end{bmatrix}_{v^c} X^{(d_1-bp,-d_2+c(q+\ell))}.
\end{aligned}$$

The phase is the based-monomial normalization. The identity $d_1 - bp = -(bd_2 - d_1) + b(d_2 - p)$ identifies the reflected denominator and pointed coordinate. Thus the transformed row has generating series

$$\sum_{q\geq 0} e^v_{(bd_2-d_1,d_2)}(d_2-p,q) t^q = \left(\sum_{\ell\geq 0} \begin{bmatrix} -(d_1-bp) \\ \ell \end{bmatrix}_{v^c} t^\ell\right) \sum_{q\geq 0} e_d^v(p,q) t^q.$$

Here we used $\sigma_2(G_{(d_1,d_2)}) = G_{(bd_2-d_1,d_2)}$, which follows from Lee–Li–Rupel–Zelevinsky Proposition 4.5 and the equality of upper and lower quasi-greedy elements in their Theorem 5.8. The generalized quantum binomial identity gives

$$\sum_{\ell\geq 0} \begin{bmatrix} -(d_1-bp) \\ \ell \end{bmatrix}_{v^c} t^\ell = \prod_{j=1}^{d_1-bp} (1 + v^{c(d_1-bp+1-2j)} t)^{-1}.$$

This is the inverse in (37), proving the assertion coefficient by coefficient. The exchanged calculation also sends v to v^{-1}; bar invariance of the coefficients removes that inversion. □

5.3. **Identification of the classical table.**

Proposition 5.3 (Classical table dictionary on the entire strip)**.** *For a positive denominator label $d = (d_1, d_2)$, every $p \geq 0$ and $0 \leq q \leq d_1$, with $(u, w) = (d_1 - q, p)$, satisfy*

$$co_{(u,w)} = e_d^1(p,q), \qquad m_1(u,w) = \xi_p(q)|_{v=1}, \qquad m_2(u,w) = \eta_p(q)|_{v=1}. \tag{43}$$

Proof. We first prove nonnegativity of both classical inverse margins on the whole quadrant, using only greedy coefficients. If $d_1 - bp \leq 0$, then $\xi_p(q)|_{v=1} = e_d^1(p,q) \geq 0$. If $d_1 - bp > 0$ and $p \leq d_2$, Lemma 5.2 at $v = 1$ identifies $\xi_p(q)|_{v=1}$ with a coefficient of a classical greedy element, which is nonnegative by Lee–Li–Zelevinsky Theorem 1.7, including nonpositive reflected labels. If $p > d_2$, Lee–Li–Zelevinsky Proposition 4.1 [4] makes the entire classical row zero, so its inverse series is zero. The quantum support bound is Lee–Li–Rupel–Zelevinsky Definition 3.1/Proposition 3.7 [3]. The exchange (41) proves $\eta_p(q)|_{v=1} \geq 0$ as well; for $q > d_1$ the entire column is zero. The remaining terms in the direct convolutions

(39) and (40) at $v = 1$ are nonnegative, since both the inverse margins and the ordinary binomial coefficients are nonnegative. Hence, for every $(p, q) \in \mathbb{Z}^2_{\geq 0}$,

$$0 \leq \xi_p(q)|_{v=1} \leq e^1_d(p, q), \qquad 0 \leq \eta_p(q)|_{v=1} \leq e^1_d(p, q). \tag{44}$$

In particular, throughout the same quadrant,

$$e^1_d(p, q) = 0 \implies \xi_p(q)|_{v=1} = \eta_p(q)|_{v=1} = 0. \tag{45}$$

These conclusions include $q > d_1$ and $p > d_2$; they concern the greedy coefficients and inverse margins, while the table dictionary (43) is stated only on the classical strip.

Next, multiplication by the inverse binomial series in (36) gives

$$cd_1q \geq bd_2p \implies \xi_p(q)|_{v=1} = 0, \qquad cd_1q \leq bd_2p \implies \eta_p(q)|_{v=1} = 0 \quad ((p, q) \neq (0, 0)).$$

Thus at least one margin is zero at every noninitial point, and both are zero on the equality wall. This holds throughout the quadrant, including outside the carrier.

Induct on the three identities in increasing p and then q. On the axes, the recurrence gives $e^1_d(0, q) = \binom{d_1}{q}$ and $e^1_d(p, 0) = \binom{d_2}{p}$, zero extended. The inverse series give $\xi_0(q)|_{v=1} = \delta_{q0}$ and $\eta_p(0)|_{v=1} = \delta_{p0}$; also $\xi_p(0)|_{v=1} = e^1_d(p, 0)$ and $\eta_0(q)|_{v=1} = e^1_d(0, q)$. These are the two axis initializations, including $e^1_d(0, 0) = \xi_0(0)|_{v=1} = \eta_0(0)|_{v=1} = 1$. At a nonaxis point assume all earlier entries agree. The horizontal predecessor is $(p, q - j)$ and the vertical predecessor is $(p - j, q)$. The direct convolution identities at $v = 1$ express the horizontal and vertical sums as

$$\sum_{j=1}^{q} \binom{[d_1 - bp]_+}{j} \xi_p(q - j)|_{v=1} = e^1_d(p, q) - \xi_p(q)|_{v=1},$$

$$\sum_{j=1}^{p} \binom{[d_2 - cq]_+}{j} \eta_{p-j}(q)|_{v=1} = e^1_d(p, q) - \eta_p(q)|_{v=1},$$

respectively. Replacing a nonpositive order by zero does not change these positive-index kernels. Nonnegativity of both margins bounds both proposals above by $e^1_d(p, q)$, and a zero margin makes one proposal equal to $e^1_d(p, q)$. Their maximum is consequently $e^1_d(p, q)$; subtracting each proposal proves the other two induction assertions. A tie, whether on or off the slope wall, is exactly $\xi_p(q)|_{v=1} = \eta_p(q)|_{v=1} = 0$ and needs no separate argument.

The induction holds on the full strip, including outside the classical support. If $e^1_d(p, q) = 0$, nonnegative margins and predecessor sums force both margins and proposals to vanish. The cutoff in Lemma 3.4 then permits restriction to the carrier; Lemma 5.1 covers segments and points. □

Corollary 5.4 (Classical element identification)**.** *For every $h \in \mathbb{Z}^2$ and d given by (25), $\rho_h = g_d$, where g_d is the classical greedy element.*

Proof. For positive denominator labels, Proposition 5.3 identifies the coefficients on the strip, and the classical support bound gives zero for $q > d_1$. Equation (28) identifies the monomials. Lemma 5.1 gives the same equality for the remaining labels from their axis arrays. □

5.4. **Coefficient and residual zero transfer.** The Lee–Li–Rupel–Zelevinsky recurrence says directly that

$$cd_1q \geq bd_2p \Longrightarrow \xi_p(q) = 0, \qquad cd_1q \leq bd_2p \Longrightarrow \eta_p(q) = 0 \tag{46}$$

away from the initialized leading point. The classical selector compares the proposals rather than the sides of the slope wall, so its quantum validity requires the zero-transfer results below.

Lemma 5.5 (Positivity of quantum cluster monomials)**.** *Every quantum cluster monomial in $\mathcal{A}_v(b,c)$ has an initial based Laurent expansion with coefficients in $\mathbb{Z}_{\geq 0}[v^{\pm 1}]$.*

Proof. Take the characteristic-zero ground field in Rupel's Theorem 1 and Remark 2 [7] to be $k = \mathbb{Q}(v)$. These results give noncommutative Laurent polynomials with nonnegative integer coefficients for both coordinates of alternating iterates of

$$F_r(x,y) = (xyx^{-1}, (1+y^r)x^{-1}), \qquad r \in \mathbb{Z}_{>0},$$

for arbitrary positive parameters. We evaluate in $\mathcal{F}$, which contains $k = \mathbb{Q}(v)$.

Let U, V be consecutive cluster variables with $VU = v^2UV$, and write $X_{U,V}^{(a,d)} = v^{ad}U^aV^d$ for the corresponding based monomials. For the exchange exponent r, set $W = (1 + v^rV^r)U^{-1}$. Since $U^{-1}V = v^2VU^{-1}$, we have

$$WV = (1+v^rV^r)U^{-1}V = v^2V(1+v^rV^r)U^{-1} = v^2VW.$$

Moreover, $V^rU^{-1} = v^{-2r}U^{-1}V^r$, so

$$W = U^{-1} + v^{-r}U^{-1}V^r = X_{U,V}^{(-1,0)} + X_{U,V}^{(-1,r)}.$$

Thus W is the based-torus mutation of U, and (V,W) satisfies the same commutation relation. Substituting $x = v^{-1}U$ and $y = vV$ gives

$$xyx^{-1} = v^{-1}V, \qquad (1+y^r)x^{-1} = vW, \qquad F_r(v^{-1}U, vV) = (v^{-1}V, vW).$$

Starting with $(U,V) = (X_1, X_2)$ and alternating $r = c, b, c, \ldots$ therefore produces the forward cluster belt. The tuple iteration reverses the order of composition of the corresponding substitution automorphisms. Reversing an alternating word still gives an alternating word, with at most an exchange of its initial parameter; Rupel's theorem applies to either choice. The backward belt follows from the same calculation with initial pair (X_2, X_1), parameter v^{-1}, and exchange exponents $b, c, b, \ldots$.

Rupel's equality is an equality of noncommutative rational functions. The iterated expression is defined at $(x,y) = (v^{-1}X_1, vX_2)$ in $\mathcal{F}$: each mutation is a sum of two distinct based monomials and hence is nonzero, so all inverses in the iteration exist. Its Laurent-polynomial representative inverts only the nonzero initial variables. Both representatives are therefore defined at this tuple, and their evaluations agree. The same reasoning applies to the swapped initial tuple for the backward belt. Every noncommutative Laurent word normal-orders to a power of v times a based monomial. Rupel's positive Laurent expansions therefore give coefficients in $\mathbb{Z}_{\geq 0}[v^{\pm 1}]$ for all cluster variables in both directions. A quantum cluster monomial is a power of v times a product of cluster variables, and (1) preserves coefficientwise nonnegativity under multiplication. □

Lemma 5.6 (Classical imaginary support, Lee–Li–Zelevinsky)**.** *For $d_1, d_2 > 0$ with $(cd_1^2 - bcd_1d_2 + bd_2^2) \leq 0$, the classical pointed support is the set of nonnegative integer points in the Lee–Li–Rupel–Zelevinsky region (16)–(17).*

Proof. This is Corollary 3.5 of [5]. In that paper's notation,

$$(a_1, a_2, s_1, s_2) = (d_1, d_2, q, p), \qquad Q(a_1, a_2) = (cd_1^2 - bcd_1d_2 + bd_2^2).$$

Their region $P[a_1, a_2]$ is bounded by the broken line

$$(0,0),\ (d_2,0),\ (d_1/b, d_2/c),\ (0,d_1),\ (0,0),$$

with the two finite axis segments included and the other boundary segments excluded. The line from $(0,d_1)$ to $(d_1/b, d_2/c)$ has equation $q = d_1 - bp + bd_2p/(cd_1)$; the line from $(d_2,0)$ to $(d_1/b, d_2/c)$ has equation $p = d_2 - cq + cd_1q/(bd_2)$. Their region is therefore exactly the union in (16)–(17), with the same open boundaries and closed axis endpoints. The

hypothesis $Q \leq 0$ includes affine null roots. When $cd_1^2 - bcd_1d_2 + bd_2^2 = 0$, the intermediate vertex lies on $d_1p + d_2q = d_1d_2$; the support is

$$\{(p,q) \in \mathbb{Z}^2_{\geq 0} : d_1p + d_2q < d_1d_2\} \cup \{(d_2, 0), (0, d_1)\}.$$

For the affine boundary in [5, Proposition 3.4], we also check the selected horizontal and vertical edges using that source's maximal Dyck path conventions (Section 2 and Definition 3.1) and Lemma 3.6. The full support theorem is Corollary 3.5. Here $cd_1^2 - bcd_1d_2 + bd_2^2 \leq 0$ implies $d_1 < bd_2$ and $d_2 < cd_1$: failure of either inequality makes $cd_1(d_1 - bd_2) + bd_2{}^2$ or $cd_1{}^2 + bd_2(d_2 - cd_1)$ strictly positive. Let (p,q) be a lattice point in the region $P[d_1, d_2]$ described above, with $p, q > 0$. This region lies in the closed triangle $d_1p + d_2q \leq d_1d_2$: its axis vertices satisfy the inequality, and the remaining vertex satisfies $d_1{}^2/b + d_2{}^2/c \leq d_1d_2$ because $cd_1^2 - bcd_1d_2 + bd_2^2 \leq 0$. Consequently $d_1p + d_2q \leq d_1d_2 < d_1d_2 + d_1 + d_2$, which verifies the strict hypothesis of [5, Lemma 3.6]; also $p \leq d_2$ and $q \leq d_1$, so the extremal pair of size $(q; p)$ is defined. Use auxiliary coordinates (s,t) and put

$$\mathcal{Q}_{p,q} = \{(s,t) \in \mathbb{Z}^2 : 1 \leq s \leq p,\ 1 \leq t \leq q\}.$$

Define two open half-planes in $\mathbb{R}^2$ by

$$\mathcal{H}_1 = \{(s,t) \in \mathbb{R}^2 : t < d_1 - bp + 1 + \tfrac{bd_2 - d_1}{d_2}(s-1)\},$$
$$\mathcal{H}_2 = \{(s,t) \in \mathbb{R}^2 : t > 1 + \tfrac{d_1(cq - d_2)}{cd_1 - d_2} + \tfrac{d_1}{cd_1 - d_2}(s-1)\}.$$

By the source's Lemma 3.6, compatibility of the extremal pair follows from $\mathcal{Q}_{p,q} \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$. Condition (3.5) there is

$$d_1 \geq bp, \qquad (cd_1 - d_2)(d_1 - bp) > d_1(cq - d_2).$$

The difference of the boundary slopes is

$$\frac{bd_2 - d_1}{d_2} - \frac{d_1}{cd_1 - d_2} = \frac{-(cd_1^2 - bcd_1d_2 + bd_2^2)}{d_2(cd_1 - d_2)}.$$

When $cd_1^2 - bcd_1d_2 + bd_2^2 = 0$ the slopes agree, and the strict inequality above puts the boundary of $\mathcal{H}_1$ strictly above that of $\mathcal{H}_2$ at $s = 1$, hence everywhere. Therefore $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathbb{R}^2$, which contains $\mathcal{Q}_{p,q}$. The alternative condition (3.6), namely $d_2 \geq cq$ and $(bd_2 - d_1)(d_2 - cq) > d_2(bp - d_1)$, is handled by exchanging the two coordinates and parameters. These alternatives describe the two leaves of the source's region; on either axis the extremal pair is automatically compatible. Thus no strict support boundary is changed at $cd_1^2 - bcd_1d_2 + bd_2^2 = 0$.

The source proves saturation using the extremal pair consisting of the first q horizontal and last p vertical edges (Definition 3.1 and Proposition 3.4). □

For positive integers b, c, the denominator labels of cluster monomials are exactly

$$\mathbb{Z}^2 \setminus \{(d_1, d_2) \in \mathbb{Z}^2_{>0} : (cd_1^2 - bcd_1d_2 + bd_2^2) \leq 0\};$$

see [4, Remark 1.9] and the paragraph preceding [3, Corollary 4.6]. Corollary 4.6 of the latter source identifies the quasi-greedy elements at each such label with the corresponding quantum cluster monomial. Combining this with [3, Theorem 5.8], which identifies the quasi-greedy and quantum greedy elements, gives the same identification for G_d. Our pointed label is the signed denominator vector: every exponent is coordinatewise at least $(-d_1, -d_2)$, and the coefficient-one term at $(p,q) = (0,0)$ attains both minima.

Lemma 5.7 (Coefficient zero transfer)**.** *For every $d \in \mathbb{Z}^2$ and $(p,q) \in \mathbb{Z}^2_{\geq 0}$,*

$$e_d^1(p,q) = 0 \quad \Longrightarrow \quad e_d^v(p,q) = 0.$$

Proof. If a coordinate of d is nonpositive, Lemma 5.1 compares the complete ordinary and Gaussian axis arrays, including the point case. Suppose $d = (d_1, d_2)$ has both coordinates positive. For $(cd_1^2 - bcd_1d_2 + bd_2^2) \le 0$, Lemma 5.6 gives the exact classical support, while Lee–Li–Rupel–Zelevinsky Proposition 3.7 gives quantum support contained in that same region. A classical coefficient zero is therefore a quantum coefficient zero.

For $(cd_1^2 - bcd_1d_2 + bd_2^2) > 0$, the classification and quantum identification just recalled show that G_d is a quantum cluster monomial. Lemma 5.5 gives $e_d^v(p,q) \in \mathbb{Z}_{\ge 0}[v^{\pm 1}]$, and Lee–Li–Rupel–Zelevinsky Theorem 1.9(e) gives $e_d^v(p,q)|_{v=1} = e_d^1(p,q)$. A Laurent polynomial with nonnegative coefficients and value zero at $v = 1$ is identically zero. □

Corollary 5.8 (Zero transfer for positive denominator labels)**.** *For every $b, c > 0$ and every positive denominator label (d_1, d_2), the classical and quantum inverse-convolution margins satisfy, at every $(p,q) \in \mathbb{Z}^2_{\ge 0}$,*

$$\xi_p(q)|_{v=1} = 0 \Longrightarrow \xi_p(q) = 0, \qquad \eta_p(q)|_{v=1} = 0 \Longrightarrow \eta_p(q) = 0. \tag{47}$$

Moreover, coefficient zeros transfer in the same direction:

$$e_d^1(p,q) = 0 \Longrightarrow e_d^v(p,q) = 0. \tag{48}$$

These assertions include points outside $(N_h \cap \mathbb{Z}^2)$ under $(u,w) = (d_1 - q, p)$.

Proof. The coefficient assertion is Lemma 5.7. If $d_1 - bp \le 0$, the inverse kernel has order zero, so $\xi_p(q)|_{v=1} = e_d^1(p,q)$ and $\xi_p(q) = e_d^v(p,q)$. If $d_1 - bp > 0$ and $p \le d_2$, Lemma 5.2 and its classical specialization give

$$\xi_p(q) = e^v_{bd_2 - d_1, d_2}(d_2 - p, q), \qquad \xi_p(q)|_{v=1} = e^1_{bd_2 - d_1, d_2}(d_2 - p, q).$$

Lemma 5.7 applies to the reflected label, including a nonpositive first coordinate, and proves the first implication. For $p > d_2$, the entire classical and quantum row is zero by the support bounds used in Proposition 5.3, so both inverse series are zero. The exchange (41) gives the second implication, including zero kernel order and $q > d_1$. □

5.5. **Identification of the quantum table.**

Theorem 5.9 (Coefficientwise identification)**.** *For all $b, c \in \mathbb{Z}_{>0}$ and $h \in \mathbb{Z}^2$,*

$$\rho_h^{(v)} = G_{(-h_1, -h_2 + c[-h_1]_+)}. \tag{49}$$

Let d be given by (25). If $d_1, d_2 > 0$, then for every carrier point (u,w), with $(p,q) = (w, d_1 - u)$, the three tables satisfy

$$co^{(v)}_{(u,w)} = e_d^v(p,q), \tag{50}$$

$$m_1^{(v)}(u,w) = \xi_p(q), \qquad m_2^{(v)}(u,w) = \eta_p(q). \tag{51}$$

These identities include the initialized skeleton. They also hold for every $(p,q) \in \mathbb{Z}^2_{\ge 0}$ whose coordinate $(d_1 - q, p)$ is outside the carrier, with the quantum tables extended by zero: all three right-hand sides then vanish. The inverse-margin identities are asserted only for positive denominator labels; the other labels have the arrays in Lemma 5.1.

Proof. If a denominator coordinate is nonpositive, Lemma 5.1 gives the identity from the segment or point coefficients. We may therefore assume that both are positive. Then (27) sends the carrier point (u,w) to the Lee–Li–Rupel–Zelevinsky pointed coordinate $(p,q) = (w, d_1 - u)$, and (28) gives the identical based monomial. If ordered monomials are used, the coefficient is multiplied by the explicit phase $v^{(-d_1+bp)(-d_2+cq)}$ from (2).

Under this coordinate map, the horizontal carrier proposal (29) is the q-direction convolution (39), with order $[d_1 - bp]_+$ and scale v^c. The vertical carrier proposal (30) is

the p-direction convolution (40), with order $[d_2 - cq]_+$ and scale v^b. These are the two scales in (15).

Consider a predecessor omitted by the algorithm. For every $(p, q) \in \mathbb{Z}^2_{\geq 0}$ whose carrier coordinate $(d_1 - q, p)$ is outside $(N_h \cap \mathbb{Z}^2)$, one has $e^1_d(p, q) = 0$. Within the classical strip this follows from the definition of the true-support carrier; for $q > d_1$ it follows from the classical support bound used in Proposition 5.3. The full-quadrant implication (45) then gives $\xi_p(q)|_{v=1} = \eta_p(q)|_{v=1} = 0$. Since Corollary 5.8 applies on the whole quadrant, it follows that

$$e^v_d(p, q) = \xi_p(q) = \eta_p(q) = 0 \quad \text{whenever } (d_1 - q, p) \notin (N_h \cap \mathbb{Z}^2).$$

Thus omitted predecessors contribute zero to both quantum convolutions.

We induct in the lexicographic order $(w, d_1 - u) = (p, q)$, maintaining (50)–(51). On the bottom axis, $e^v_d(0, q) = K_c(d_1, q) = K_c(d_1, d_1 - q)$, $\xi_0(q) = \delta_{q0}$, and $\eta_0(q) = e^v_d(0, q)$. On the right axis, $e^v_d(p, 0) = K_b(d_2, p)$, $\xi_p(0) = e^v_d(p, 0)$, and $\eta_p(0) = \delta_{p0}$. These agree with the skeleton initialization, including the common corner, where all three entries equal one.

At a nonskeleton point, the induction hypothesis identifies the horizontal predecessor residuals with $\xi_p(q - j)$ and the vertical predecessor residuals with $\eta_{p-j}(q)$. Adding the omitted zero terms, the two direct convolution identities show that the horizontal and vertical quantum sums equal $e^v_d(p, q) - \xi_p(q)$ and $e^v_d(p, q) - \eta_p(q)$, respectively. If the classical selector chooses the horizontal classical sum, then Proposition 5.3 gives $\xi_p(q)|_{v=1} = 0$, and Corollary 5.8 gives $\xi_p(q) = 0$. Consequently the horizontal quantum sum is $e^v_d(p, q)$, the value assigned to $co^{(v)}_{(u,w)}$. If it chooses the vertical classical sum, the same argument with $\eta_p(q)|_{v=1}$ and $\eta_p(q)$ shows that the vertical quantum sum is $e^v_d(p, q)$. Subtracting each proposal from this assigned value proves both residual identities in (51) at the same step. At a classical tie, both classical margins vanish, hence $\xi_p(q) = \eta_p(q) = 0$ and both proposals equal $e^v_d(p, q)$. The induction and the preceding zero statement prove the three identities on and outside the carrier. The coefficient equality and (28) give (49). Since $G_d \in \mathcal{A}_v(b, c)$, the constructed element belongs to $\mathcal{A}_v(b, c)$. □

Corollary 5.10 (Quantum polyhedral basis)**.** *The family* $\{\rho^{(v)}_h : h \in \mathbb{Z}^2\}$ *is an* $\mathbb{Z}[v^{\pm 1}]$*-basis of* $\mathcal{A}_v(b, c)$.

Proof. The label map is bijective by (26). By Theorem 5.9, the displayed family is therefore exactly $\{G_d : d \in \mathbb{Z}^2\}$. The assertion follows from the Lee–Li–Rupel–Zelevinsky basis theorem. □

Together with Theorem 4.3 and Corollary 5.4, this gives a bijection between the quantum and classical basis elements, $\rho^{(v)}_h \mapsto \rho_h$, indexed by $h \in \mathbb{Z}^2$.

6. Support and further properties

Theorem 6.1 (Exact coefficient support)**.** *For every* $h \in \mathbb{Z}^2$,

$$\operatorname{Supp}_{(u,w)} \rho^{(v)}_h = \operatorname{Supp}_{(u,w)} \rho_h. \tag{52}$$

Proof. Outside $N_h \cap \mathbb{Z}^2$, the classical weights vanish by the definition of N_h from the classical support, and the quantum weights vanish by the zero-extension convention. It therefore suffices to consider points in $N_h \cap \mathbb{Z}^2$. If $co_{(u,w)} > 0$, then (33) says that the corresponding quantum coefficient has nonzero value at $v = 1$; hence it is not the zero Laurent polynomial.

For the reverse inclusion, suppose $co_{(u,w)} = 0$. For a positive denominator label, Proposition 5.3 gives $e^1_d(p, q) = 0$, Corollary 5.8 gives $e^v_d(p, q) = 0$, and Theorem 5.9

identifies this coefficient with $co^{(v)}_{(u,w)}$. For a nonpositive denominator coordinate, Lemma 5.1 gives the same conclusion from the explicit segment or point arrays. This proves (52). □

Corollary 6.2 (Laurent support and Newton polytope)**.** *For every $h \in \mathbb{Z}^2$,*

$$\begin{aligned} \operatorname{Supp}_{\text{Laurent}} \rho_h^{(v)} &= \{(h_1 + bw, h_2 - cu) : (u,w) \in \operatorname{Supp}_{(u,w)} \rho_h\}, \\ \operatorname{Newt}(\rho_h^{(v)}) &= \{(h_1 + bw, h_2 - cu) : (u,w) \in N_h\}. \end{aligned} \tag{53}$$

Proof. By Theorem 6.1, the coordinate supports agree. The affine exponent map (28) is injective and commutes with convex hulls, so $N_h = \operatorname{conv} \operatorname{Supp}_{(u,w)} \rho_h$ gives (53). □

The affine example in Section 7.2 has a zero coefficient at a nonskeleton lattice point on the carrier's sloping boundary.

Theorem 6.3 (Residual/margin zero-set correspondence)**.** *For every carrier point,*

$$m_1^{(v)}(u,w) = 0 \iff m_1(u,w) = 0, \qquad m_2^{(v)}(u,w) = 0 \iff m_2(u,w) = 0. \tag{54}$$

Proof. For nonpositive denominator coordinates the assertions follow from Lemma 5.1. For positive denominator labels, the implications from classical zero margins to quantum zero residuals follow from Proposition 5.3, Corollary 5.8, and (51). Conversely, a zero Laurent polynomial specializes to zero. Theorem 4.3 therefore gives $m_1^{(v)} = 0 \Rightarrow m_1 = 0$ and $m_2^{(v)} = 0 \Rightarrow m_2 = 0$, proving (54).

□

Corollary 6.4 (Slope vanishing)**.** *In the nondegenerate chamber $h_1 < 0$ and $h_2 + ch_1 < 0$, let (u,w) be a carrier point and put $(p,q) = (w, [-h_1]_+ - u)$. Away from the common leading corner,*

$$\begin{aligned} c[-h_1]_+ q \geq b[-h_2 + c[-h_1]_+]_+ p &\Longrightarrow m_1^{(v)}(u,w) = 0, \\ c[-h_1]_+ q \leq b[-h_2 + c[-h_1]_+]_+ p &\Longrightarrow m_2^{(v)}(u,w) = 0. \end{aligned} \tag{55}$$

On the slope wall both residuals vanish, except at the common corner $(p,q) = (0,0)$, where both are initialized to one. Off-wall classical ties may also belong to the common zero set.

Proof. The implications (46), transported by the coordinate map (27) and the residual dictionary (51), give the two directions in (55). On the equality wall both weak inequalities apply. At the common corner, however, the skeleton initialization takes priority and sets both margins to one. At an ordinary classical tie, both classical sums equal $co_{(u,w)}$, so both classical margins, and hence both quantum residuals, vanish even when the point is off the slope wall. □

For degenerate carriers, including empty sums and zero kernel order, Lemma 5.1 gives (54); the slope conditions do not apply.

Corollary 6.5 (Inherited properties)**.** *Every $\rho_h^{(v)}$ is bar invariant; the family contains every quantum cluster monomial and is independent of the initial cluster. Its pointed row and column generating functions satisfy the Lee–Li–Rupel–Zelevinsky Gaussian divisibility conditions. Explicitly, let $e_d^v(p,q)$ denote the pointed coefficients of $\rho_h^{(v)} = G_d$. For a central*

variable t,

$$\prod_{j=1}^{d_2-cq}\left(1+v^{b(d_2-cq+1-2j)}t\right) \;\Big|\; \sum_{p\geq 0} e_d^v(p,q)t^p, \qquad 0\leq q<d_2/c, \tag{56}$$

$$\prod_{j=1}^{d_1-bp}\left(1+v^{c(d_1-bp+1-2j)}t\right) \;\Big|\; \sum_{q\geq 0} e_d^v(p,q)t^q, \qquad 0\leq p<d_1/b. \tag{57}$$

Here divisibility is in $\mathbb{Z}[v^{\pm 1}][t]$, the free indices p, q are nonnegative integers, and a condition with an empty index range is vacuous.

Proof. Bar invariance is Theorem 4.2. By Theorem 5.9, the family is the quantum greedy basis; its independence of the initial cluster follows from [3, Theorem 1.9(b)], and its inclusion of all quantum cluster monomials follows from [3, Theorem 1.9(c)]. Since $\rho_h^{(v)} = G_d \in \mathcal{A}_v(b,c)$, the pointed row and column divisibility conditions follow from [3, Lemma 2.2 and equation (2.1)]. The displayed factors are the kernel polynomials in (13), with order $d_2 - cq$ and scale v^b in the p-direction, and order $d_1 - bp$ and scale v^c in the q-direction, as checked in Section 2. Each factor has a unit leading coefficient in $\mathbb{Z}[v^{\pm 1}]$, so divisibility over $\mathbb{Q}(v)[t]$ also gives divisibility in $\mathbb{Z}[v^{\pm 1}][t]$ by polynomial division. □

7. Examples and further questions

All coefficients below refer to based monomials, with pointed coordinates (p, q).

7.1. **A finite-type example.** Take $(b, c) = (1, 1)$ and $d = (2, 2)$, so $h = (-2, 0)$. The carrier is the triangle with vertices $(u, w) = (0, 0), (2, 0), (2, 2)$. The bottom and right skeleton coefficients are

$$K_1(2,0), K_1(2,1), K_1(2,2) = 1, v^{-1} + v, 1.$$

At the nonskeleton support point $(u, w) = (1, 1)$ on the sloping boundary, both classical proposals equal 2. The horizontal-first rule is selected, while the two quantum proposals agree and equal $v^{-1} + v$. Thus

$$\begin{aligned}\rho_{(-2,0)}^{(v)} = {} & X^{(-2,-2)} + (v^{-1}+v)X^{(-2,-1)} + X^{(-2,0)} \\ & + (v^{-1}+v)X^{(-1,-2)} + (v^{-1}+v)X^{(-1,-1)} + X^{(0,-2)}.\end{aligned}$$

At $v = 1$ the coefficient list is $1, 2, 1, 2, 2, 1$. The true support has six points, and its Newton polytope is the triangle with vertices $(-2, -2), (-2, 0), (0, -2)$.

7.2. **An affine example with a carrier zero.** Take $(b, c) = (2, 2)$ and $d = (2, 2)$, hence $h = (-2, 2)$. Again N_h is the triangle with vertices $(0, 0), (2, 0), (2, 2)$ in carrier coordinates. At $(u, w) = (1, 1)$, however, both kernel orders are zero:

$$-h_1 - bw = 0, \qquad -h_2 + cu = 0.$$

Both proposals and the output coefficient vanish. Thus this point belongs to the carrier but not to the true support. The output is

$$\begin{aligned}\rho_{(-2,2)}^{(v)} = {} & X^{(-2,-2)} + (v^{-2}+v^2)X^{(-2,0)} + X^{(-2,2)} \\ & + (v^{-2}+v^2)X^{(0,-2)} + X^{(2,-2)}.\end{aligned}$$

Its specialization has coefficients $1, 2, 1, 2, 1$. Its Newton polytope is

$$\operatorname{conv}\{(-2,-2), (-2,2), (2,-2)\}.$$

The pointed sloping edge is open away from its two axis endpoints, in agreement with Lemma 5.6.

7.3. **A wild example.** Take $(b,c)=(2,3)$ and $d=(2,4)$, for which $3\cdot 2^2-2\cdot 3\cdot 2\cdot 4+2\cdot 4^2=-4$ and $h=(-2,2)$. The carrier triangle has vertices $(0,0),(2,0),(2,4)$. The complete classical selector table on its nine lattice points is

(u,w)	$co_{(u,w)}$	horizontal sum	vertical sum	$m_1(u,w)$	$m_2(u,w)$
$(0,0)$	1	–	–	0	1
$(1,0)$	2	–	–	0	2
$(2,0)$	1	–	–	1	1
$(2,1)$	4	–	–	4	0
$(1,1)$	2	0	2	2	0
$(2,2)$	6	–	–	6	0
$(1,2)$	0	0	0	0	0
$(2,3)$	4	–	–	4	0
$(2,4)$	1	–	–	1	0

A dash marks prescribed skeleton data. Example 4.4 computes the vertical choice at $(u,w)=(1,1)$, giving $co^{(v)}_{(1,1)}=v^{-3}+v^3$. At the next nonskeleton point $(1,2)$ on the sloping boundary, the horizontal order is negative and the only possible vertical predecessor margin is already zero. Both quantum sums therefore vanish, and $co^{(v)}_{(1,2)}=0$. The coordinate maps read

$$(1,1)\mapsto(p,q)=(1,1)\mapsto(0,-1),\qquad (1,2)\mapsto(2,1)\mapsto(2,-1),$$

and the latter exponent is absent from the true Laurent support. The complete output is

$$\begin{aligned}\rho^{(v)}_{(-2,2)} &= X^{(-2,-4)}+(v^{-3}+v^3)X^{(-2,-1)}+X^{(-2,2)}\\ &\quad+(v^{-6}+v^{-2}+v^2+v^6)X^{(0,-4)}+(v^{-3}+v^3)X^{(0,-1)}\\ &\quad+(v^{-8}+v^{-4}+2+v^4+v^8)X^{(2,-4)}\\ &\quad+(v^{-6}+v^{-2}+v^2+v^6)X^{(4,-4)}+X^{(6,-4)}.\end{aligned}$$

Evaluation at $v=1$ gives coefficient values $1,2,1,4,2,6,4,1$. The true support consists of the eight displayed exponents, whereas the Newton polytope is the larger triangle with vertices $(-2,-4),(-2,2),(6,-4)$.

7.4. **A negative quantum coefficient.** The example in [2, Section 3], with $(b,c)=(2,3)$ and $d=(3,4)$, gives

$$e^v_d(2,1)=v^2-1+v^{-2},\qquad e^1_d(2,1)=1.$$

This coefficient has a negative term despite its positive specialization. Our construction uses the classical selector, whose validity is proved by zero transfer.

7.5. **Further questions.** For $i=1,2$, left multiplication gives an element $X_i\rho^{(v)}_h$ with a unique finite expansion in the quantum greedy basis. The question is how to determine its coefficients and the labels of its nonzero terms explicitly from the carrier and residual tables. One may also compare these labels with those in the classical greedy-basis expansion of $x_i\rho_h$; these are labels of basis elements, rather than Laurent monomial supports. We leave these questions, as well as extensions to principal coefficients and higher rank, open.

Appendix A. Conventions

The following boundary rules accompany the notation table.

(1) The common bottom-right corner is initialized once, before recursion.

(2) Empty sums are zero. The zero-extended kernels K_r and $\widetilde{\binom{n}{j}}$ vanish at negative order or out-of-range index. The generalized negative-top Gaussian coefficients in the reflection proof are distinct from these zero-extended kernels.

(3) The quantum run does not allocate or read entries outside $(N_h \cap \mathbb{Z}^2)$; the classical preprocessing is defined on the semi-infinite strip and then truncated.

(4) When $[-h_1]_+ = 0$ or $[-h_2 + c[-h_1]_+]_+ = 0$, the carrier is a segment; when both vanish it is a point.

(5) A classical tie uses the horizontal branch.

(6) The affine sloping edge and the two wild sloping leaves retain their strict inequalities; the coordinate-axis endpoints are included as stated.

Object	Convention in this paper	Correspondence
Skew form	$\Lambda(e_1, e_2) = -1$	Gives $X_2X_1 = v^2X_1X_2$ by (1).
Exchange matrix	$B = \left(\begin{smallmatrix} 0 & -b \\ c & 0 \end{smallmatrix}\right)$	$B^T\Lambda = \mathrm{diag}(c, b)$; mutation in direction 1 is given by (6).
Based monomial	$X^{(r,t)} = v^{rt}X_1^rX_2^t$	Bar fixes the based monomial; Ordered coefficients carry the displayed phase.
Denominator label	$d = (d_1, d_2)$	Related to the polyhedral label h by (25).
Pointed coordinate	(p, q)	Carrier coordinate is $(u, w) = ([-h_1]_+ - q, p)$.
Laurent exponent	$(-d_1 + bp, -d_2 + cq)$	Equals $(h_1 + bw, h_2 - cu)$.
First greedy branch	p direction, scale v^b	Corresponds to the carrier vertical sum (30).
Second greedy branch	q direction, scale v^c	Corresponds to the carrier horizontal sum (29).
Equality branch	both weak inequalities	Both quantum residuals vanish at a noninitial tie, so the two proposal values agree.
Classical specialization	$v \mapsto 1$ on based sums	Defined by (8); nonzero specialization implies a nonzero coefficient.

References

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