

# Continuity of solutions to abstract linear control systems

Contributed by: Frédéric Marbach\*

October 3, 2026

## Abstract

It has long been known that the solutions to abstract linear control systems are continuous in time for controls in  $L^p$  with  $1 \leq p < \infty$ . We prove that the same property remains valid for the endpoint case  $p = \infty$ , giving a positive answer to Weiss' 1989 Problem 2.4.

The proof relies on a direct semigroup argument based on Phillips' lemma, does not require the input map to have an integral representation (which is not always the case for  $p = \infty$ ), and actually entails that such systems are all of the zero-class (which fails for  $1 \leq p < \infty$ ).

## 1 Introduction

### 1.1 Context

Let  $X$  be a Banach space (the *state space*) describing the possible values for the state  $x(t) \in X$  and let  $U$  be a Banach space (the *input space*) describing the possible values for the control  $u(t) \in U$ . Fix  $p \in [1, \infty]$ . We consider controls  $u \in L^p(\mathbb{R}_+; U)$ , where  $\mathbb{R}_+ := [0, \infty)$ .

The following definition has been popularized by [18]; see also [15, 17].

**Definition 1.1.** An abstract linear control system with state space  $X$  and input space  $U$  is a pair  $(\mathbb{T}, \Phi)$  of families of operators such that

- $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  is a strongly continuous semigroup of bounded linear operators on  $X$ ;
- $\Phi = (\Phi_t)_{t \geq 0}$  is a family of bounded linear operators from  $L^p(\mathbb{R}_+; U)$  to  $X$ , called input maps, such that, for all  $t, \tau \geq 0$  and  $u, v \in L^p(\mathbb{R}_+; U)$ ,

$$\Phi_{\tau+t}(u \underset{\tau}{\diamond} v) = \mathbb{T}_t \Phi_{\tau} u + \Phi_t v, \quad (1.1)$$

where  $u \underset{\tau}{\diamond} v \in L^p(\mathbb{R}_+; U)$  denotes the  $\tau$ -concatenation of  $u$  and  $v$  defined as:

$$(u \underset{\tau}{\diamond} v)(s) := \begin{cases} u(s) & \text{for } s \in [0, \tau), \\ v(s - \tau) & \text{for } s \geq \tau. \end{cases} \quad (1.2)$$

For a given initial data  $x^\circ \in X$  and control  $u \in L^p(\mathbb{R}_+; U)$ , one thinks of  $x(t) := \mathbb{T}_t x^\circ + \Phi_t u$  as the solution to a linear control system with initial data  $x^\circ$  and control  $u$ . One is interested in knowing if such solutions are continuous in time. Basic semigroup theory automatically yields the continuity of the uncontrolled part (see e.g. [11, Chapter 1, Corollary 2.3]).

**Proposition 1.2.** For any  $x^\circ \in X$ ,  $t \mapsto \mathbb{T}_t x^\circ$  is continuous on  $\mathbb{R}_+$ .

When  $p \in [1, \infty)$ , an elementary argument entails that the controlled part is continuous too (see [18, Proposition 2.3]). The argument uses that  $p < \infty$  twice: first it uses that  $\|u\|_{L^p((0,t);U)} \rightarrow 0$  as  $t \rightarrow 0$ , and second it uses that translations are continuous in  $L^p(\mathbb{R}_+; U)$ .

**Proposition 1.3.** Assume that  $p \in [1, \infty)$ . For all  $u \in L^p(\mathbb{R}_+; U)$ ,  $t \mapsto \Phi_t u$  is continuous on  $\mathbb{R}_+$ .

The endpoint case  $p = \infty$  was left open as Problem 2.4 in [18]. Recent research papers [9, p. 23] or [4, Section 6] still mention this case as open in full generality. It is nevertheless known that continuity does hold for various classes of systems (see e.g. [7] or [13, Section 4.4]).

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\*DMA, École normale supérieure, Université PSL, CNRS, 75005 Paris, France

## 1.2 Zero-class systems

From the composition relation (1.1) with  $t = \tau = 0$ , one obtains that  $\Phi_0 = 0$ . For  $t \geq 0$ , define

$$\kappa(t) := \|\Phi_t\|_{\mathcal{L}(L^p(\mathbb{R}_+; U), X)}. \quad (1.3)$$

Taking  $u = 0$  in (1.1), one also obtains that, for all  $t, \tau \geq 0$ ,

$$\kappa(t) \leq \kappa(t + \tau). \quad (1.4)$$

One can then wonder whether  $\kappa(t) \rightarrow 0$  as  $t \rightarrow 0$ . This leads to the following definition, introduced in [20] in the context of observation operators (see also [5]).

**Definition 1.4.** *An abstract linear control system  $(\mathbb{T}, \Phi)$  is said to be of the zero-class when*

$$\lim_{t \rightarrow 0} \kappa(t) = 0. \quad (1.5)$$

Many papers underline the importance of this notion, sufficient conditions for systems to be of the zero-class, and consequences thereof (see e.g. [1]). In particular, in [4, Proposition 2.5], the authors prove that the solutions to zero-class systems for  $p = \infty$  and admitting an integral representation are continuous in time.

Not all systems are of the zero-class. For instance, with  $p = 1$ , take  $X = U = \mathbb{R}$ ,  $\mathbb{T}_t = \text{Id}$  and  $\Phi_t u := \int_0^t u(s) ds$ . Then, for any  $t > 0$ ,  $\kappa(t) = 1$ . See Section 3.1 for a general  $p \in [1, \infty)$ .

## 1.3 Main results

In contrast with the case  $p \in [1, \infty)$ , we establish that all abstract linear control systems are of the zero-class for  $p = \infty$  (even without assuming any integral representation).

**Theorem 1.5.** *Let  $X, U$  be arbitrary Banach spaces and  $(\mathbb{T}, \Phi)$  be an abstract linear control system with  $p = \infty$ . Then the system is of the zero-class, i.e.  $\lim_{t \rightarrow 0} \|\Phi_t\|_{\mathcal{L}(L^\infty(\mathbb{R}_+; U), X)} = 0$ .*

Arora, Preußler and Schwenninger proved in an independent very recent preprint [2] that every  $L^\infty$ -admissible control operator  $B \in \mathcal{L}(U, X_{-1})$  is admissible for an Orlicz heart  $E_F$  associated with a suitable Young function (see Section 3.2). Their theorem yields both the zero-class property and continuity of mild solutions. Thus, for systems admitting the usual semigroup-convolution representation, their result provides a stronger admissibility conclusion than the one proved here.

Our paper works directly with the abstract input maps and its concatenation identity, without assuming the existence of a control operator or an integral representation. Our only additional functional-analytic ingredient is Phillips' lemma. In particular, the argument does not require sun-dual observation operators or Orlicz-space duality.

For  $p \in [1, \infty)$ , all abstract linear control systems admit an integral representation (see [18, Theorem 3.9]). It is not the case for  $p = \infty$  (see [19, Section 3 and Remark 3.7]). In Section 3.2, we give a variant of the classical invariant-mean construction behind the failure of integral representation at  $p = \infty$ . For this system,  $X = U = \mathbb{R}$  and  $\mathbb{T}_t = \text{Id}$ , and we show that it is not  $E_F$ -admissible for any finite-valued Young function  $F$ . Thus the results of [2] do not apply directly to all systems covered by Theorem 1.5 and Corollary 1.6, and the Orlicz improvement obtained there does not extend to the full class of abstract input maps.

As in [4, Proposition 2.5], Theorem 1.5 entails the following extension of Proposition 1.3 to  $p = \infty$  (even without assuming any integral representation; see Section 2.4).

**Corollary 1.6.** *Let  $X, U$  be arbitrary Banach spaces and  $(\mathbb{T}, \Phi)$  be an abstract linear control system with  $p = \infty$ . For all  $u \in L^\infty(\mathbb{R}_+; U)$ ,  $t \mapsto \Phi_t u$  is continuous on  $\mathbb{R}_+$ .*

## 2 Proofs

### 2.1 Elementary consequences of the composition property

**Lemma 2.1.** *For all  $s, t, \tau \geq 0$  and  $u, v \in L^p(\mathbb{R}_+; U)$ ,*

$$\Phi_\tau(u \diamond_\tau v) = \Phi_\tau u, \quad (2.1)$$

$$\Phi_{s+t}(0 \diamond_s v) = \Phi_t v, \quad (2.2)$$

$$\Phi_{\tau+t}(u \diamond_\tau 0) = \mathbb{T}_t \Phi_\tau u. \quad (2.3)$$

In words:  $\Phi_\tau u$  only depends on  $u|_{[0, \tau)}$ ; an initial period of zero input has no effect; once the input is switched off, the state evolves freely.

*Proof.* These are (1.1) with  $t = 0$  (recall  $\Phi_0 = 0$ ), with  $u = 0$ , and with  $v = 0$  respectively.  $\square$

### 2.2 Phillips' lemma

We use the following classical form of Phillips' lemma (see [12, Corollary 3.4] or [10, Section 6.1]).

**Lemma 2.2.** *Let  $(\mu_n)_{n \geq 0}$  be a sequence in  $(\ell^\infty(\mathbb{N}))^*$  such that, for every  $a \in \ell^\infty(\mathbb{N})$ ,  $\mu_n(a) \rightarrow 0$ . If  $(e_k)_{k \geq 0}$  denote the canonical vectors of  $\ell^\infty(\mathbb{N})$ , then*

$$\sum_{k=0}^{\infty} |\mu_n(e_k)| \rightarrow 0. \quad (2.4)$$

In particular  $\mu_n(e_n) \rightarrow 0$ .

We will use it through the following consequence. By definition of a strongly continuous semigroup, for each fixed  $x \in X$ ,  $(\mathbb{T}_t - \text{Id})x \rightarrow 0$  as  $t \rightarrow 0$ , but this convergence is in general not uniform on bounded sets. The lemma below states that it is nevertheless uniform along the images of the canonical vectors under any bounded operator from  $\ell^\infty(\mathbb{N})$ .

**Lemma 2.3.** *Let  $S \in \mathcal{L}(\ell^\infty(\mathbb{N}), X)$  and  $(t_n)_{n \geq 0}$  be nonnegative times with  $t_n \rightarrow 0$ . Then*

$$\|(\mathbb{T}_{t_n} - \text{Id})Se_n\|_X \rightarrow 0. \quad (2.5)$$

*Proof.* By the Hahn–Banach theorem, for each  $n$  there exists  $x_n^* \in X^*$  with  $\|x_n^*\|_{X^*} \leq 1$  such that  $x_n^*((\mathbb{T}_{t_n} - \text{Id})Se_n) = \|(\mathbb{T}_{t_n} - \text{Id})Se_n\|_X$ . Define  $\mu_n \in (\ell^\infty(\mathbb{N}))^*$  by  $\mu_n(\alpha) := x_n^*((\mathbb{T}_{t_n} - \text{Id})S\alpha)$ . For each fixed  $\alpha \in \ell^\infty(\mathbb{N})$ ,  $|\mu_n(\alpha)| \leq \|(\mathbb{T}_{t_n} - \text{Id})S\alpha\|_X \rightarrow 0$  by the definition of a strongly continuous semigroup. Hence Lemma 2.2 yields  $\mu_n(e_n) \rightarrow 0$ , which is the claim.  $\square$

### 2.3 Proof of the zero-class property

By (1.4),  $\kappa$  is nonnegative and nondecreasing, so the limit

$$\ell := \lim_{t \rightarrow 0^+} \kappa(t) = \inf_{t > 0} \kappa(t) \quad (2.6)$$

exists in  $[0, \infty)$ . Proving Theorem 1.5 amounts to proving that  $\ell = 0$ .

#### Idea of the proof

Let  $0 < h \ll 1$  and  $u$  be a control on  $[0, h]$  of  $L^\infty$  norm at most 1 such that  $x := \Phi_h u$  is nearly extremal, i.e.  $\|x\|_X \approx \kappa(h) \approx \ell$ . Playing  $u$  twice in a row leads, at time  $2h$ , to the state  $\mathbb{T}_h x + x$ .

- On the one hand, this state is reached in time  $2h$  with a control of  $L^\infty$  norm at most 1, so its norm is at most  $\kappa(2h) \approx \ell$ .
- On the other hand, if the semigroup barely moves  $x$  during the time  $h$ , this state is close to  $2x$ , whose norm is approximately  $2\ell$ .

Hence  $2\ell \leq \ell$ , i.e.  $\ell = 0$ .

The only delicate point is to guarantee that  $(\mathbb{T}_h - \text{Id})x$  is small. Strong continuity gives this for a fixed  $x$  as  $h \rightarrow 0$ , but here  $x$  depends on  $h$ . The required uniformity will be provided by Lemma 2.3, once the nearly extremal controls are packed into a single bounded operator on  $\ell^\infty(\mathbb{N})$ .

## Detailed proof

*Proof of Theorem 1.5.* Let  $K := \sup_{0 \leq t \leq 1} \|\mathbb{T}_t\|_{\mathcal{L}(X)} < \infty$  by [11, Chapter 1, Theorem 2.2].

*Step 1: A doubling inequality.* Let  $h > 0$  and  $u \in L^\infty(\mathbb{R}_+; U)$  with  $\|u\|_{L^\infty} \leq 1$ . Set  $x := \Phi_h u$  and  $w := u \diamond_h u$ , which consists of two consecutive copies of  $u|_{[0,h]}$ . Then  $\|w\|_{L^\infty} \leq 1$  and the composition property (1.1) with  $\tau = t = h$  gives

$$\Phi_{2h} w = \mathbb{T}_h \Phi_h u + \Phi_h u = \mathbb{T}_h x + x. \quad (2.7)$$

Writing  $2x = \Phi_{2h} w - (\mathbb{T}_h - \text{Id})x$ , we obtain

$$2\|x\|_X \leq \kappa(2h) + \|(\mathbb{T}_h - \text{Id})x\|_X. \quad (2.8)$$

Consequently, it suffices to construct times  $h_n \rightarrow 0$  and controls  $u_n$  with  $\|u_n\|_{L^\infty} \leq 1$  such that the states  $x_n := \Phi_{h_n} u_n$  satisfy

$$\|x_n\|_X \rightarrow \ell \quad \text{and} \quad \|(\mathbb{T}_{h_n} - \text{Id})x_n\|_X \rightarrow 0. \quad (2.9)$$

Indeed, applying (2.8) with  $h := h_n$  and  $u := u_n$ , and letting  $n \rightarrow \infty$  (recall that  $\kappa(2h_n) \rightarrow \ell$  since  $2h_n \rightarrow 0$ ), then yields  $2\ell \leq \ell$ , hence  $\ell = 0$ .

*Step 2: Nearly extremal controls with short time scales.* We construct  $h_n$ ,  $u_n$  and  $x_n$  inductively for  $n \geq 0$ , starting from  $h_0 := 1/2$ . Given  $h_n$ , by definition of  $\kappa(h_n)$  as an operator norm, there exists  $u_n \in L^\infty(\mathbb{R}_+; U)$  with  $\|u_n\|_{L^\infty} \leq 1$  and  $\|\Phi_{h_n} u_n\|_X \geq \kappa(h_n) - 2^{-n}$ . By (2.1), replacing  $u_n$  by  $u_n|_{[0,h_n]}$  does not change  $\Phi_{h_n} u_n$ , so we can moreover assume that  $u_n = 0$  on  $[h_n, \infty)$ . We set  $x_n := \Phi_{h_n} u_n$ , so that

$$\kappa(h_n) - 2^{-n} \leq \|x_n\|_X \leq \kappa(h_n). \quad (2.10)$$

Then, by strong continuity of the semigroup at the fixed vector  $x_n$ , we choose  $h_{n+1} \in (0, h_n/2]$  such that

$$\|\mathbb{T}_t x_n - x_n\|_X \leq 2^{-n} \quad \text{for all } t \in [0, 2h_{n+1}]. \quad (2.11)$$

In words: the state  $x_n$  is essentially frozen by the semigroup during the time needed to play all the subsequent controls (see (2.12) below).

Since  $h_n \leq 2^{-n-1}$ , we have  $h_n \rightarrow 0$ , hence  $\kappa(h_n) \rightarrow \ell$  and, by (2.10),  $\|x_n\|_X \rightarrow \ell$ . This is the first half of (2.9).

*Step 3: Packing the controls into one operator.* Let  $s_n := h_0 + \dots + h_{n-1}$  (with  $s_0 = 0$ ) and  $H := \sum_{k \geq 0} h_k$ . Since  $h_{k+1} \leq h_k/2$ , we have  $H \leq 2h_0 = 1$  and

$$H - s_{n+1} = \sum_{k > n} h_k \leq 2h_{n+1}. \quad (2.12)$$

The intervals  $[s_n, s_{n+1})$  partition  $[0, H)$ . For  $\alpha \in \ell^\infty(\mathbb{N})$ , we play the controls one after the other, the  $n$ -th one with amplitude  $\alpha_n$ :

$$u^\alpha(s) := \begin{cases} \alpha_n u_n(s - s_n) & \text{for } s \in [s_n, s_{n+1}), \ n \geq 0, \\ 0 & \text{for } s \geq H. \end{cases} \quad (2.13)$$

The map  $\alpha \mapsto u^\alpha$  is linear and, since  $\|u_n\|_{L^\infty} \leq 1$ ,  $\|u^\alpha\|_{L^\infty} \leq \|\alpha\|_{\ell^\infty}$ . Hence  $S\alpha := \Phi_H u^\alpha$  defines  $S \in \mathcal{L}(\ell^\infty(\mathbb{N}), X)$ .

Let us compute  $Se_n$ . Since  $u_n = 0$  on  $[h_n, \infty)$ , we have  $u^{e_n} = 0 \diamond_{s_n} u_n$  and  $u_n = u_n \diamond_{h_n} 0$ . Thus, by (2.2) and then (2.3),  $Se_n = \Phi_{H-s_n} u_n = \mathbb{T}_{H-s_{n+1}} \Phi_{h_n} u_n = \mathbb{T}_{H-s_{n+1}} x_n$ . In words,  $Se_n$  is the  $n$ -th state  $x_n$ , after it has evolved freely while the later controls are played. By (2.11) and (2.12), this evolution is negligible:

$$\|Se_n - x_n\|_X \leq 2^{-n}. \quad (2.14)$$

*Step 4: Conclusion.* By (2.14) and  $\|\mathbb{T}_{h_n} - \text{Id}\|_{\mathcal{L}(X)} \leq K + 1$ ,

$$\|(\mathbb{T}_{h_n} - \text{Id})x_n\|_X \leq \|(\mathbb{T}_{h_n} - \text{Id})Se_n\|_X + (K + 1)2^{-n}, \quad (2.15)$$

which tends to 0 by Lemma 2.3 with  $t_n := h_n$ .

This is the second half of (2.9), which concludes the proof by Step 1.  $\square$

## 2.4 Proof of the time continuity

In [4, Proposition 2.5], the authors prove that the solutions to zero-class  $L^\infty$  systems are continuous in time. Their proof assumes that the input map has an integral representation. We show that the composition property is sufficient to reach the conclusion.

*Proof of Corollary 1.6.* By Theorem 1.5, one has (1.5), i.e.  $\lim_{t \rightarrow 0} \kappa(t) = 0$ , where  $\kappa(t) = \|\Phi_t\|$ . Fix  $u \in L^\infty(\mathbb{R}_+; U)$ . We want to prove that  $t \mapsto \Phi_t u$  is continuous on  $\mathbb{R}_+ = [0, \infty)$ .

- At  $t_0 = 0$ , using (1.5),

$$\|\Phi_t u\|_X \leq \kappa(t) \|u\|_{L^\infty} \longrightarrow 0 \quad (2.16)$$

so  $t \mapsto \Phi_t u$  is continuous at  $t_0 = 0$ .

- We now fix  $t_0 > 0$ . Let  $\varepsilon > 0$ . Using (1.5), choose  $0 < \delta < t_0$  small enough such that

$$2\kappa(2\delta) \|u\|_{L^\infty} \leq \frac{1}{2}\varepsilon. \quad (2.17)$$

Let  $t_1 := t_0 - \delta$  and  $u_1(s) := u(t_1 + s)$ . By the composition property (1.1), for  $t \geq t_1$ ,

$$\Phi_t u = \mathbb{T}_{t-t_1} \Phi_{t_1} u + \Phi_{t-t_1} u_1. \quad (2.18)$$

In particular,

$$\Phi_{t_0} u = \mathbb{T}_\delta \Phi_{t_1} u + \Phi_\delta u_1. \quad (2.19)$$

Thus, for  $t \in [t_0 - \delta, t_0 + \delta]$ , subtracting both identities,

$$\|\Phi_t u - \Phi_{t_0} u\|_X \leq \|(\mathbb{T}_{t-t_1} - \mathbb{T}_\delta) \Phi_{t_1} u\|_X + \|\Phi_{t-t_1} u_1\|_X + \|\Phi_\delta u_1\|_X. \quad (2.20)$$

Using (1.4), we have  $\kappa(t - t_1) \leq \kappa(2\delta)$  and  $\kappa(\delta) \leq \kappa(2\delta)$ . Thus, since  $\|u_1\|_{L^\infty} \leq \|u\|_{L^\infty}$ ,

$$\|\Phi_{t-t_1} u_1\|_X + \|\Phi_\delta u_1\|_X \leq 2\kappa(2\delta) \|u\|_{L^\infty} \leq \frac{1}{2}\varepsilon. \quad (2.21)$$

Moreover, by Proposition 1.2 with  $x^\circ = \Phi_{t_1} u$ , the map  $s \mapsto \mathbb{T}_s(\Phi_{t_1} u)$  is continuous on  $\mathbb{R}_+$ , and in particular at  $s = \delta$ . Hence there exists  $\delta' > 0$  such that, if  $|t - t_0| = |(t - t_1) - \delta| \leq \delta'$ ,

$$\|(\mathbb{T}_{t-t_1} - \mathbb{T}_\delta) \Phi_{t_1} u\|_X \leq \frac{1}{2}\varepsilon. \quad (2.22)$$

This concludes the proof of the continuity at  $t_0$ .  $\square$

## 3 Examples and counterexamples

### 3.1 Failure of the zero-class property for finite $p$

Theorem 1.5 establishes that any abstract linear control system with  $p = \infty$  is zero-class. In contrast, we illustrate here that, for every  $1 \leq p < \infty$ , there exists an abstract linear control system with scalar inputs which is not zero-class.

The example given below is classical; see e.g. [6, Example 3.2].

**Proposition 3.1.** *Let  $p \in [1, \infty)$ . There exist Banach spaces  $X$  and  $U$ , and an abstract linear control system  $(\mathbb{T}, \Phi)$  such that  $\kappa(t) = 1$  for all  $t > 0$ .*

*Proof.* Take  $X = L^p(\mathbb{R}_+; \mathbb{R})$ ,  $U = \mathbb{R}$  and let  $\mathbb{T}$  be the right-shift semigroup on  $X$ , defined by

$$(\mathbb{T}_t f)(s) := \begin{cases} 0, & 0 \leq s < t, \\ f(s - t), & s \geq t. \end{cases} \quad (3.1)$$

This is a semigroup of isometries. Its strong continuity follows from the continuity of translations in  $L^p(\mathbb{R})$ , after extending  $f$  by zero to the negative half-line. This continuity fails for  $p = \infty$ .

Define the input maps  $\Phi_t : L^p(\mathbb{R}_+; U) \rightarrow X$  by

$$(\Phi_t u)(s) := \begin{cases} u(t-s), & 0 \leq s < t, \\ 0, & s \geq t. \end{cases} \quad (3.2)$$

These maps are linear and bounded, since

$$\|\Phi_t u\|_X^p = \int_0^t |u(t-s)|^p ds = \int_0^t |u(s)|^p ds \leq \|u\|_{L^p(\mathbb{R}_+; U)}^p. \quad (3.3)$$

Moreover, for  $t, \tau \geq 0$  and  $w := u \diamond_{\tau} v$ , one has almost everywhere

$$(\Phi_{\tau+t} w)(s) = \begin{cases} v(t-s), & 0 \leq s < t, \\ u(\tau+t-s), & t \leq s < \tau+t, \\ 0, & s \geq \tau+t. \end{cases} \quad (3.4)$$

The right-hand side equals  $(\mathbb{T}_t \Phi_{\tau} u)(s) + (\Phi_t v)(s)$ , proving the concatenation identity. Thus  $(\mathbb{T}, \Phi)$  is an abstract linear control system.

However, for every  $t > 0$ , the input  $u_t := t^{-\frac{1}{p}} \mathbf{1}_{[0,t)}$  satisfies  $\|u_t\|_{L^p(\mathbb{R}_+; U)} = \|\Phi_t u_t\|_X = 1$ . Consequently,  $\kappa(t) = \|\Phi_t\| = 1$  for all  $t > 0$ , so the system is not zero-class.  $\square$

### 3.2 Systems without integral representation for $p = \infty$

**Context.** Let  $X_{-1}$  be the extrapolation space associated with the generator  $A$  of  $\mathbb{T}$ , to which  $\mathbb{T}$  extends as a strongly continuous semigroup (see e.g. [16, Section 2.10]). We say that  $(\mathbb{T}, \Phi)$  *admits an integral representation* when there exists  $B \in \mathcal{L}(U, X_{-1})$  such that

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} B u(s) ds \quad \text{for all } t \geq 0 \text{ and } u \in L^p(\mathbb{R}_+; U). \quad (3.5)$$

For  $p \in [1, \infty)$ , every abstract linear control system admits an integral representation [18, Theorem 3.9]. For  $p = \infty$ , this fails, as shown by Weiss with a construction based on invariant means [19, Section 3 and Remark 3.7].

Following [2], a *Young function* is a finite-valued, convex, continuous, nondecreasing function  $F: [0, \infty) \rightarrow [0, \infty)$  such that  $F(r)/r \rightarrow 0$  as  $r \rightarrow 0$  and  $F(r)/r \rightarrow \infty$  as  $r \rightarrow \infty$ . For  $t > 0$ , the associated Luxemburg norm is

$$\|u\|_{F,t} := \inf \left\{ \lambda > 0 : \int_0^t F \left( \frac{\|u(s)\|_U}{\lambda} \right) ds \leq 1 \right\}. \quad (3.6)$$

We say that  $(\mathbb{T}, \Phi)$  is  $E_F$ -admissible when, for every  $t > 0$ , there exists  $C_t \geq 0$  such that

$$\|\Phi_t u\|_X \leq C_t \|u\|_{F,t} \quad \text{for all } u \in L^\infty(\mathbb{R}_+; U). \quad (3.7)$$

By [2, Proposition 2.1], every system with  $p = \infty$  admitting an integral representation is  $E_F$ -admissible for some Young function  $F$  depending on the system. This is stronger than the zero-class property: by (2.2) and (3.7),  $\kappa(h) \leq C_T \|\mathbf{1}_{(T-h, T)}\|_{F,T} \rightarrow 0$  as  $h \rightarrow 0$ . The following example shows that this strategy cannot cover all the systems of Theorem 1.5.

**An example based on an exotic invariant mean.** Let  $L_{\text{per}}^\infty$  denote the space of 1-periodic elements of  $L^\infty(\mathbb{R}; \mathbb{R})$ . We use the following classical fact; see [14, 3], or [19, Lemma 3.3].

**Lemma 3.2.** *There exists a linear map  $m: L_{\text{per}}^\infty \rightarrow \mathbb{R}$  such that*

- (a)  $m(f) \geq 0$  whenever  $f \geq 0$  almost everywhere, and  $m(1) = 1$ ;
- (b)  $m(f(\cdot - a)) = m(f)$  for all  $f \in L_{\text{per}}^\infty$  and  $a \in \mathbb{R}$ ;
- (c)  $m(f) \neq \int_0^1 f(s) ds$  for some  $f \in L_{\text{per}}^\infty$ .

**Proposition 3.3.** *There exists an abstract linear control system  $(\mathbb{T}, \Phi)$  with  $p = \infty$ ,  $X = U = \mathbb{R}$  and  $\mathbb{T}_t = \text{Id}$  for all  $t \geq 0$ , such that:*

- (i)  $\kappa(t) = t$  for every  $t \geq 0$ ;
- (ii)  $(\mathbb{T}, \Phi)$  admits no integral representation;
- (iii)  $(\mathbb{T}, \Phi)$  is not  $E_F$ -admissible, for any Young function  $F$ .

*Proof. Step 1: A translation-invariant functional.* Let  $f \in L^\infty(\mathbb{R}; \mathbb{R})$  vanish outside a bounded interval. Its periodization  $Pf(r) := \sum_{k \in \mathbb{Z}} f(r+k)$  has a uniformly bounded number of nonzero terms, so  $Pf \in L^\infty_{\text{per}}$ , and  $Pf$  does not depend on the representative of  $f$ . Set  $J(f) := m(Pf)$ . Then  $J$  is linear and positive, and it is translation invariant since  $P(f(\cdot - a)) = (Pf)(\cdot - a)$ . We claim that

$$J(\mathbf{1}_{[a,b]}) = b - a \quad \text{for all } a \leq b. \quad (3.8)$$

Indeed,  $q(t) := J(\mathbf{1}_{[0,t]})$  is additive by linearity and invariance, nonnegative by positivity (hence nondecreasing), and satisfies  $q(1) = m(1) = 1$  because  $P\mathbf{1}_{[0,1]} = 1$ . Thus  $q(k/n) = k/n$  for all integers  $k \geq 0$  and  $n \geq 1$ , so  $q(t) = t$  for all  $t \geq 0$  by monotonicity, and invariance yields (3.8). By positivity, if  $f$  vanishes outside  $[a, b]$ , then

$$|J(f)| \leq \|f\|_{L^\infty} J(\mathbf{1}_{[a,b]}) = (b - a)\|f\|_{L^\infty}. \quad (3.9)$$

*Step 2: The system.* For  $t \geq 0$  and  $u \in L^\infty(\mathbb{R}_+; \mathbb{R})$ , set  $\Phi_t u := J(\mathbf{1}_{[0,t]} u)$ , where  $\mathbf{1}_{[0,t]} u$  is extended by zero to  $\mathbb{R}$ . By (3.9),  $|\Phi_t u| \leq t\|u\|_{L^\infty}$ , with equality for  $u = 1$  by (3.8), so  $\kappa(t) = t$ . This proves (i). For  $t, \tau \geq 0$  and  $w := u \diamond_\tau v$ , one has  $\mathbf{1}_{[0, \tau+t]} w = \mathbf{1}_{[0, \tau]} u + (\mathbf{1}_{[0, t]} v)(\cdot - \tau)$  almost everywhere. By invariance of  $J$ ,  $\Phi_{\tau+t} w = \Phi_\tau u + \Phi_t v$ , which is (1.1) with  $\mathbb{T}_t = \text{Id}$ .

*Step 3: Inputs with small support and non-small output.* We claim that there exists a measurable set  $E \subset [0, 1]$  such that  $d := J(\mathbf{1}_E) - |E| \neq 0$ . Otherwise,  $J(f) = \int_0^1 f$  for every simple function  $f$  vanishing outside  $[0, 1]$ , hence, by uniform density of simple functions and (3.9), for every  $f \in L^\infty$  vanishing outside  $[0, 1]$ . Since every  $f \in L^\infty_{\text{per}}$  satisfies  $f = P(\mathbf{1}_{[0,1]} f)$ , this would give  $m(f) = \int_0^1 f$  for all  $f$ , contradicting (c).

By regularity of the Lebesgue measure, there exist sets  $I_n \subset [0, 1]$ , each a finite disjoint union of intervals  $[a, b]$ , such that  $A_n := E \triangle I_n$  satisfies  $|A_n| \rightarrow 0$ . Set  $u_n := \mathbf{1}_E - \mathbf{1}_{I_n}$ , so that  $|u_n| = \mathbf{1}_{A_n}$ . By (3.8),  $J(\mathbf{1}_{I_n}) = |I_n|$ , so

$$\Phi_1 u_n = J(\mathbf{1}_E) - |I_n| \longrightarrow J(\mathbf{1}_E) - |E| = d \neq 0. \quad (3.10)$$

*Step 4: Proof of (ii).* Here  $A = 0$  is bounded, so  $X_{-1} = X = \mathbb{R}$  with equivalent norms, and any  $B \in \mathcal{L}(U, X_{-1})$  is the multiplication by some  $b \in \mathbb{R}$ . An integral representation would thus give  $|\Phi_1 u_n| = |b \int_0^1 u_n| \leq |b| |A_n| \rightarrow 0$ , contradicting (3.10).

*Step 5: Proof of (iii).* Let  $F$  be a Young function and  $\lambda > 0$ . Since  $F(0) = 0$ ,  $\int_0^1 F(|u_n|/\lambda) = |A_n| F(1/\lambda) \rightarrow 0$ , so  $\|u_n\|_{F,1} \leq \lambda$  for  $n$  large enough. Hence  $\|u_n\|_{F,1} \rightarrow 0$ , and (3.7) at  $t = 1$  would force  $\Phi_1 u_n \rightarrow 0$ , contradicting (3.10).  $\square$

A different separation of zero-class admissibility from Orlicz-heart admissibility, for continuous inputs, appears in [2, Example 4.2].

## Research provenance

In June 2025, I gave a course at the EUR MINT 2025 Summer School, *Control, Inverse Problems and Spectral Theory*, in Toulouse, France. In this course, I wanted to present several classical methods in control theory based on time-iteration arguments, and to formulate them as reusable black boxes. This required, in particular, the introduction of abstract *nonlinear* control systems (see [8, Section 3.1]), for which the case  $p = \infty$  arises naturally. Continuity in time of the solutions was essential to the time-iteration arguments I intended to present. This led me to stumble upon the difficulty of the open case  $p = \infty$  of [18, Problem 2.4].

Throughout 2025 and early 2026, I made several unsuccessful attempts, both unaided and computer-assisted (up to Gemini 3.1 Pro and GPT-5.2), to settle this question, accumulating personal notes on the problem before eventually giving up.

On September 29, 2026, motivated by the launch of the [hexagonmath.org](https://hexagonmath.org) website, I made a new attempt. Supplied with my old notes, GPT-6 Pro produced in a single attempt the proofs of [Theorem 1.5](#) and [Corollary 1.6](#), notably by identifying Phillips’ lemma as the key ingredient. The following days, I rewrote this text by hand and with the help of Opus 5.5 to give appropriate credit to prior works and to make the proof easier to understand. On October 2, I became aware of the independent preprint [2] posted on September 30 to arXiv, and revised the manuscript accordingly.

GPT-6 Pro and Opus 5.5 formalized in Lean the statements and proofs of all the numbered results of this write-up, including in particular [Theorem 1.5](#) and [Corollary 1.6](#), as well as the examples and intermediate lemmas/propositions. The formalizations were uploaded to the Palomar registry on October 3, at <https://palomar-registry.org/entry?id=PALOMAR-2026-10-03-000001>

*What a time!*

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