

Continuity of solutions to abstract linear control systems

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Abstract

It has long been known that the solutions to abstract linear control systems are continuous in time for controls in L^p with $1 \leq p < \infty$. We prove that the same property remains valid for the endpoint case $p = \infty$, giving a positive answer to Weiss' 1989 Problem 2.4.

The proof relies on a direct semigroup argument based on Phillips' lemma, does not require the input map to have an integral representation (which is not always the case for $p = \infty$), and actually entails that such systems are all of the zero-class (which fails for $1 \leq p < \infty$).

1 Introduction

1.1 Context

Let X be a Banach space (the *state space*) describing the possible values for the state $x(t) \in X$ and let U be a Banach space (the *input space*) describing the possible values for the control $u(t) \in U$. Fix $p \in [1, \infty]$. We consider controls $u \in L^p(\mathbb{R}_+; U)$, where $\mathbb{R}_+ := [0, \infty)$.

The following definition has been popularized by [18]; see also [16, 17].

Definition 1.1. An abstract linear control system with state space X and input space U is a pair $(\mathbb{T}, \Phi)$ of families of operators such that

- $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ is a strongly continuous semigroup of bounded linear operators on X ;
- $\Phi = (\Phi_t)_{t \geq 0}$ is a family of bounded linear operators from $L^p(\mathbb{R}_+; U)$ to X , called input maps, such that, for all $t, \tau \geq 0$ and $u, v \in L^p(\mathbb{R}_+; U)$,

$$\Phi_{\tau+t}(u \diamond_{\tau} v) = \mathbb{T}_t \Phi_{\tau} u + \Phi_t v, \quad (1.1)$$

where $u \diamond_{\tau} v \in L^p(\mathbb{R}_+; U)$ denotes the τ -concatenation of u and v defined as:

$$(u \diamond_{\tau} v)(s) := \begin{cases} u(s) & \text{for } s \in [0, \tau), \\ v(s - \tau) & \text{for } s \geq \tau. \end{cases} \quad (1.2)$$

For a given initial data $x^\circ \in X$ and control $u \in L^p(\mathbb{R}_+; U)$, one thinks of $x(t) := \mathbb{T}_t x^\circ + \Phi_t u$ as the solution to a linear control system with initial data x° and control u . One is interested in knowing if such solutions are continuous in time. Basic semigroup theory automatically yields the continuity of the uncontrolled part (see e.g. [12, Chapter 1, Corollary 2.3]).

Proposition 1.2. For any $x^\circ \in X$, $t \mapsto \mathbb{T}_t x^\circ$ is continuous on $\mathbb{R}_+$.

When $p \in [1, \infty)$, an elementary argument entails that the controlled part is continuous too (see [18, Proposition 2.3]). The argument uses that $p < \infty$ twice: first it uses that $\|u\|_{L^p((0,t);U)} \rightarrow 0$ as $t \rightarrow 0$, and second it uses that translations are continuous in $L^p(\mathbb{R}_+; U)$.

Proposition 1.3. Assume that $p \in [1, \infty)$. For all $u \in L^p(\mathbb{R}_+; U)$, $t \mapsto \Phi_t u$ is continuous on $\mathbb{R}_+$.

The endpoint case $p = \infty$ was left open as Problem 2.4 in [18]. Recent research papers [10, p. 23] or [5, Section 6] still mention this case as open in full generality. It is nevertheless known that continuity does hold for various classes of systems (see e.g. [8] or [14, Section 4.4]).

1.2 Zero-class systems

From the composition relation (1.1) with $t = \tau = 0$, one obtains that $\Phi_0 = 0$. For $t \geq 0$, define

$$\kappa(t) := \|\Phi_t\|_{\mathcal{L}(L^p(\mathbb{R}_+; U), X)}. \quad (1.3)$$

Taking $u = 0$ in (1.1), one also obtains that, for all $t, \tau \geq 0$,

$$\kappa(t) \leq \kappa(t + \tau). \quad (1.4)$$

One can then wonder whether $\kappa(t) \rightarrow 0$ as $t \rightarrow 0$. This leads to the following definition, introduced in [20] in the context of observation operators (see also [6]).

Definition 1.4. *An abstract linear control system $(\mathbb{T}, \Phi)$ is said to be of the zero-class when*

$$\lim_{t \rightarrow 0} \kappa(t) = 0. \quad (1.5)$$

Many papers underline the importance of this notion, sufficient conditions for systems to be of the zero-class, and consequences thereof (see e.g. [1]). In particular, in [5, Proposition 2.5], the authors prove that the solutions to zero-class systems for $p = \infty$ and admitting an integral representation are continuous in time.

Not all systems are of the zero-class. For instance, with $p = 1$, take $X = U = \mathbb{R}$, $\mathbb{T}_t = \text{Id}$ and $\Phi_t u := \int_0^t u(s) ds$. Then, for any $t > 0$, $\kappa(t) = 1$. See Section 3.1 for a general $p \in [1, \infty)$.

1.3 Main results

In contrast with the case $p \in [1, \infty)$, we establish that all abstract linear control systems are of the zero-class for $p = \infty$ (even without assuming any integral representation).

Theorem 1.5. *Let X, U be arbitrary Banach spaces and $(\mathbb{T}, \Phi)$ be an abstract linear control system with $p = \infty$. Then the system is of the zero-class, i.e. $\lim_{t \rightarrow 0} \|\Phi_t\|_{\mathcal{L}(L^\infty(\mathbb{R}_+; U), X)} = 0$.*

Arora, Preußler and Schwenninger proved in an independent very recent preprint [2] that every L^∞ -admissible control operator $B \in \mathcal{L}(U, X_{-1})$ is admissible for an Orlicz heart E_F associated with a suitable Young function. Their theorem yields both the zero-class property and continuity of mild solutions. Thus, for systems admitting the usual semigroup-convolution representation, their result provides a stronger admissibility conclusion than the one proved here. The present paper works directly with the abstract input maps and their concatenation identity, without assuming the existence of a control operator or an integral representation. Its only additional functional-analytic ingredient is Phillips' lemma. In particular, the argument does not require sun-dual observation operators or Orlicz-space duality.

For $p \in [1, \infty)$, all abstract linear control systems admit an integral representation (see [18, Theorem 3.9]). It is known that this is not the case for $p = \infty$ (see [19, Section 3, especially Remark 3.7]). In Section 3.2, we give a variant of the classical invariant-mean construction behind the failure of integral representation at $p = \infty$. For this system, $X = U = \mathbb{R}$ and $\mathbb{T}_t = \text{Id}$, and we show that it is not E_F -admissible for any finite-valued Young function F . Thus the results of [2] do not apply directly to all systems covered by Proposition 1.5 and Proposition 1.6, and the Orlicz improvement obtained there does not extend to the full class of abstract input maps.

As in [5, Proposition 2.5], Proposition 1.5 entails the following extension of Proposition 1.3 to $p = \infty$ (even without assuming any integral representation; see Section 2.4).

Corollary 1.6. *Let X, U be arbitrary Banach spaces and $(\mathbb{T}, \Phi)$ be an abstract linear control system with $p = \infty$. For all $u \in L^\infty(\mathbb{R}_+; U)$, $t \mapsto \Phi_t u$ is continuous on $\mathbb{R}_+$.*

2 Proofs

2.1 Elementary consequences of the composition property

Lemma 2.1. *For all $s, t, \tau \geq 0$ and $u, v \in L^p(\mathbb{R}_+; U)$,*

$$\Phi_\tau(u \diamond_\tau v) = \Phi_\tau u, \quad (2.1)$$

$$\Phi_{s+t}(0 \diamond_s v) = \Phi_t v, \quad (2.2)$$

$$\Phi_{\tau+t}(u \diamond_\tau 0) = \mathbb{T}_t \Phi_\tau u. \quad (2.3)$$

In words: $\Phi_\tau u$ only depends on $u|_{[0, \tau)}$; an initial period of zero input has no effect; once the input is switched off, the state evolves freely.

Proof. These are (1.1) with $t = 0$ (recall $\Phi_0 = 0$), with $u = 0$, and with $v = 0$ respectively. $\square$

2.2 Phillips' lemma

We use the following classical form of Phillips' lemma (see [13, Corollary 3.4] or [11, Section 6.1]).

Lemma 2.2. *Let $(\mu_n)_{n \geq 0}$ be a sequence in $(\ell^\infty(\mathbb{N}))^*$ such that, for every $a \in \ell^\infty(\mathbb{N})$, $\mu_n(a) \rightarrow 0$. If $(e_k)_{k \geq 0}$ denote the canonical vectors of $\ell^\infty(\mathbb{N})$, then*

$$\sum_{k=0}^{\infty} |\mu_n(e_k)| \rightarrow 0. \quad (2.4)$$

In particular $\mu_n(e_n) \rightarrow 0$.

We will use it through the following consequence. By definition of a strongly continuous semigroup, for each fixed $x \in X$, $(\mathbb{T}_t - \text{Id})x \rightarrow 0$ as $t \rightarrow 0$, but this convergence is in general not uniform on bounded sets. The lemma below states that it is nevertheless uniform along the images of the canonical vectors under any bounded operator from $\ell^\infty(\mathbb{N})$.

Lemma 2.3. *Let $S \in \mathcal{L}(\ell^\infty(\mathbb{N}), X)$ and $(t_n)_{n \geq 0}$ be nonnegative times with $t_n \rightarrow 0$. Then*

$$\|(\mathbb{T}_{t_n} - \text{Id})S e_n\|_X \rightarrow 0. \quad (2.5)$$

Proof. By the Hahn–Banach theorem, for each n there exists $x_n^* \in X^*$ with $\|x_n^*\|_{X^*} \leq 1$ such that $x_n^*((\mathbb{T}_{t_n} - \text{Id})S e_n) = \|(\mathbb{T}_{t_n} - \text{Id})S e_n\|_X$. Define $\mu_n \in (\ell^\infty(\mathbb{N}))^*$ by $\mu_n(\alpha) := x_n^*((\mathbb{T}_{t_n} - \text{Id})S \alpha)$. For each fixed $\alpha \in \ell^\infty(\mathbb{N})$, $|\mu_n(\alpha)| \leq \|(\mathbb{T}_{t_n} - \text{Id})S \alpha\|_X \rightarrow 0$ by the definition of a strongly continuous semigroup. Hence Proposition 2.2 yields $\mu_n(e_n) \rightarrow 0$, which is the claim. $\square$

2.3 Proof of the zero-class property

By (1.4), κ is nonnegative and nondecreasing, so the limit

$$\ell := \lim_{t \rightarrow 0^+} \kappa(t) = \inf_{t > 0} \kappa(t) \quad (2.6)$$

exists in $[0, \infty)$. Proving Proposition 1.5 amounts to proving that $\ell = 0$.

2.3.1 Idea of the proof

Let $0 < h \ll 1$ and u be a control on $[0, h]$ of L^∞ norm at most 1 such that $x := \Phi_h u$ is nearly extremal, i.e. $\|x\|_X \approx \kappa(h) \approx \ell$. Playing u twice in a row leads, at time $2h$, to the state $\mathbb{T}_h x + x$.

- On the one hand, this state is reached in time $2h$ with a control of L^∞ norm at most 1, so its norm is at most $\kappa(2h) \approx \ell$.
- On the other hand, if the semigroup barely moves x during the time h , this state is close to $2x$, whose norm is approximately 2ℓ .

Hence $2\ell \leq \ell$, i.e. $\ell = 0$.

The only delicate point is to guarantee that $(\mathbb{T}_h - \text{Id})x$ is small. Strong continuity gives this for a fixed x as $h \rightarrow 0$, but here x depends on h . The required uniformity will be provided by Proposition 2.3, once the nearly extremal controls are packed into a single bounded operator on $\ell^\infty(\mathbb{N})$.

2.3.2 Proof

Proof of Proposition 1.5. Let $K := \sup_{0 \leq t \leq 1} \|\mathbb{T}_t\|_{\mathcal{L}(X)} < \infty$ by [12, Chapter 1, Theorem 2.2].

Step 1: A doubling inequality. Let $h > 0$ and $u \in L^\infty(\mathbb{R}_+; U)$ with $\|u\|_{L^\infty} \leq 1$. Set $x := \Phi_h u$ and $w := u \diamond_h u$, which consists of two consecutive copies of $u|_{[0, h]}$. Then $\|w\|_{L^\infty} \leq 1$ and the composition property (1.1) with $\tau = t = h$ gives

$$\Phi_{2h} w = \mathbb{T}_h \Phi_h u + \Phi_h u = \mathbb{T}_h x + x. \quad (2.7)$$

Writing $2x = \Phi_{2h} w - (\mathbb{T}_h - \text{Id})x$, we obtain

$$2\|x\|_X \leq \kappa(2h) + \|(\mathbb{T}_h - \text{Id})x\|_X. \quad (2.8)$$

Consequently, it suffices to construct times $h_n \rightarrow 0$ and controls u_n with $\|u_n\|_{L^\infty} \leq 1$ such that the states $x_n := \Phi_{h_n} u_n$ satisfy

$$\|x_n\|_X \rightarrow \ell \quad \text{and} \quad \|(\mathbb{T}_{h_n} - \text{Id})x_n\|_X \rightarrow 0. \quad (2.9)$$

Indeed, applying (2.8) with $h := h_n$ and $u := u_n$, and letting $n \rightarrow \infty$ (recall that $\kappa(2h_n) \rightarrow \ell$ since $2h_n \rightarrow 0$), then yields $2\ell \leq \ell$, hence $\ell = 0$.

Step 2: Nearly extremal controls with short time scales. We construct h_n , u_n and x_n inductively for $n \geq 0$, starting from $h_0 := 1/2$. Given h_n , by definition of $\kappa(h_n)$ as an operator norm, there exists $u_n \in L^\infty(\mathbb{R}_+; U)$ with $\|u_n\|_{L^\infty} \leq 1$ and $\|\Phi_{h_n} u_n\|_X \geq \kappa(h_n) - 2^{-n}$. By (2.1), replacing u_n by $u_n|_{[0, h_n]}$ does not change $\Phi_{h_n} u_n$, so we can moreover assume that $u_n = 0$ on $[h_n, \infty)$. We set $x_n := \Phi_{h_n} u_n$, so that

$$\kappa(h_n) - 2^{-n} \leq \|x_n\|_X \leq \kappa(h_n). \quad (2.10)$$

Then, by strong continuity of the semigroup at the fixed vector x_n , we choose $h_{n+1} \in (0, h_n/2]$ such that

$$\|\mathbb{T}_t x_n - x_n\|_X \leq 2^{-n} \quad \text{for all } t \in [0, 2h_{n+1}]. \quad (2.11)$$

In words: the state x_n is essentially frozen by the semigroup during the time needed to play all the subsequent controls (see (2.12) below).

Since $h_n \leq 2^{-n-1}$, we have $h_n \rightarrow 0$, hence $\kappa(h_n) \rightarrow \ell$ and, by (2.10), $\|x_n\|_X \rightarrow \ell$. This is the first half of (2.9).

Step 3: Packing the controls into one operator. Let $s_n := h_0 + \dots + h_{n-1}$ (with $s_0 = 0$) and $H := \sum_{k \geq 0} h_k$. Since $h_{k+1} \leq h_k/2$, we have $H \leq 2h_0 = 1$ and

$$H - s_{n+1} = \sum_{k > n} h_k \leq 2h_{n+1}. \quad (2.12)$$

The intervals $[s_n, s_{n+1})$ partition $[0, H)$. For $\alpha \in \ell^\infty(\mathbb{N})$, we play the controls one after the other, the n -th one with amplitude α_n :

$$u^\alpha(s) := \begin{cases} \alpha_n u_n(s - s_n) & \text{for } s \in [s_n, s_{n+1}), \ n \geq 0, \\ 0 & \text{for } s \geq H. \end{cases} \quad (2.13)$$

The map $\alpha \mapsto u^\alpha$ is linear and, since $\|u_n\|_{L^\infty} \leq 1$, $\|u^\alpha\|_{L^\infty} \leq \|\alpha\|_{\ell^\infty}$. Hence $S\alpha := \Phi_H u^\alpha$ defines $S \in \mathcal{L}(\ell^\infty(\mathbb{N}), X)$.

Let us compute Se_n . Since $u_n = 0$ on $[h_n, \infty)$, we have $u^{e_n} = 0 \diamond_{s_n} u_n$ and $u_n = u_n \diamond_{h_n} 0$. Thus, by (2.2) and then (2.3), $Se_n = \Phi_{H-s_n} u_n = \mathbb{T}_{H-s_{n+1}} \Phi_{h_n} u_n = \mathbb{T}_{H-s_{n+1}} x_n$. In words, Se_n is the

n -th state x_n , after it has evolved freely while the later controls are played. By (2.11) and (2.12), this evolution is negligible:

$$\|Se_n - x_n\|_X \leq 2^{-n}. \quad (2.14)$$

Step 4: Conclusion. By (2.14) and $\|\mathbb{T}_{h_n} - \text{Id}\|_{\mathcal{L}(X)} \leq K + 1$,

$$\|(\mathbb{T}_{h_n} - \text{Id})x_n\|_X \leq \|(\mathbb{T}_{h_n} - \text{Id})Se_n\|_X + (K + 1)2^{-n}, \quad (2.15)$$

which tends to 0 by Proposition 2.3 with $t_n := h_n$.

This is the second half of (2.9), which concludes the proof by Step 1. $\square$

2.4 Proof of the time continuity

In [5, Proposition 2.5], the authors prove that the solutions to zero-class L^∞ systems are continuous in time. Their proof assumes that the input map has an integral representation. We show that the composition property is sufficient to reach the conclusion.

Proof of Proposition 1.6. By Proposition 1.5, one has (1.5), i.e. $\lim_{t \rightarrow 0} \kappa(t) = 0$, where $\kappa(t) = \|\Phi_t\|$. Fix $u \in L^\infty(\mathbb{R}_+; U)$. We want to prove that $t \mapsto \Phi_t u$ is continuous on $\mathbb{R}_+ = [0, \infty)$.

- At $t_0 = 0$, using (1.5),

$$\|\Phi_t u\|_X \leq \kappa(t)\|u\|_{L^\infty} \longrightarrow 0 \quad (2.16)$$

so $t \mapsto \Phi_t u$ is continuous at $t_0 = 0$.

- We now fix $t_0 > 0$. Let $\varepsilon > 0$. Using (1.5), choose $0 < \delta < t_0$ small enough such that

$$2\kappa(2\delta)\|u\|_{L^\infty} \leq \frac{1}{2}\varepsilon. \quad (2.17)$$

Let $t_1 := t_0 - \delta$ and $u_1(s) := u(t_1 + s)$. By the composition property (1.1), for $t \geq t_1$,

$$\Phi_t u = \mathbb{T}_{t-t_1} \Phi_{t_1} u + \Phi_{t-t_1} u_1. \quad (2.18)$$

In particular,

$$\Phi_{t_0} u = \mathbb{T}_\delta \Phi_{t_1} u + \Phi_\delta u_1. \quad (2.19)$$

Thus, for $t \in [t_0 - \delta, t_0 + \delta]$, subtracting both identities,

$$\|\Phi_t u - \Phi_{t_0} u\|_X \leq \|(\mathbb{T}_{t-t_1} - \mathbb{T}_\delta) \Phi_{t_1} u\|_X + \|\Phi_{t-t_1} u_1\|_X + \|\Phi_\delta u_1\|_X. \quad (2.20)$$

Using (1.4), we have $\kappa(t - t_1) \leq \kappa(2\delta)$ and $\kappa(\delta) \leq \kappa(2\delta)$. Thus, since $\|u_1\|_{L^\infty} \leq \|u\|_{L^\infty}$,

$$\|\Phi_{t-t_1} u_1\|_X + \|\Phi_\delta u_1\|_X \leq 2\kappa(2\delta)\|u\|_{L^\infty} \leq \frac{1}{2}\varepsilon. \quad (2.21)$$

Moreover, by Proposition 1.2 with $x^\circ = \Phi_{t_1} u$, the map $s \mapsto \mathbb{T}_s(\Phi_{t_1} u)$ is continuous on $\mathbb{R}_+$, and in particular at $s = \delta$. Hence there exists $\delta' > 0$ such that, if $|t - t_0| = |(t - t_1) - \delta| \leq \delta'$,

$$\|(\mathbb{T}_{t-t_1} - \mathbb{T}_\delta) \Phi_{t_1} u\|_X \leq \frac{1}{2}\varepsilon. \quad (2.22)$$

This concludes the proof of the continuity at t_0 . $\square$

3 Sharpness

3.1 Failure of the zero-class property for finite p

Proposition 1.5 establishes that any abstract linear control system with $p = \infty$ is zero-class. In contrast, we illustrate here that, for every $1 \leq p < \infty$, there exists an abstract linear control system with scalar inputs which is not zero-class.

The example given below is classical; see e.g. [7, Example 3.2].

Take $X = L^p(\mathbb{R}_+; \mathbb{R})$, $U = \mathbb{R}$ and let $\mathbb{T}$ be the right-shift semigroup on X , defined by

$$(\mathbb{T}_t f)(s) := \begin{cases} 0, & 0 \leq s < t, \\ f(s - t), & s \geq t. \end{cases} \quad (3.1)$$

This is a semigroup of isometries. Its strong continuity follows from the continuity of translations in $L^p(\mathbb{R})$, after extending f by zero to the negative half-line. This continuity fails for $p = \infty$.

Define the input maps $\Phi_t : L^p(\mathbb{R}_+; U) \rightarrow X$ by

$$(\Phi_t u)(s) := \begin{cases} u(t - s), & 0 \leq s < t, \\ 0, & s \geq t. \end{cases} \quad (3.2)$$

These maps are linear and bounded, since

$$\|\Phi_t u\|_X^p = \int_0^t |u(t - s)|^p ds = \int_0^t |u(s)|^p ds \leq \|u\|_{L^p(\mathbb{R}_+; U)}^p. \quad (3.3)$$

Moreover, for $t, \tau \geq 0$ and $w := u \diamond_{\tau} v$, one has almost everywhere

$$(\Phi_{\tau+t} w)(s) = \begin{cases} v(t - s), & 0 \leq s < t, \\ u(\tau + t - s), & t \leq s < \tau + t, \\ 0, & s \geq \tau + t. \end{cases} \quad (3.4)$$

The right-hand side equals $(\mathbb{T}_t \Phi_{\tau} u)(s) + (\Phi_t v)(s)$, proving the concatenation identity. Thus $(\mathbb{T}, \Phi)$ is an abstract linear control system.

However, for every $t > 0$, the input $u_t := t^{-\frac{1}{p}} \mathbf{1}_{[0,t]}$ satisfies $\|u_t\|_{L^p(\mathbb{R}_+; U)} = \|\Phi_t u_t\|_X = 1$. Consequently, $\kappa(t) = \|\Phi_t\| = 1$ for all $t > 0$, so the system is not zero-class.

3.2 Systems without integral representation for $p = \infty$

Integral representations. Let A be the generator of $\mathbb{T}$ and X_{-1} be the completion of X for the norm $x \mapsto \|(\beta - A)^{-1}x\|_X$, where β is a sufficiently large real number in the resolvent set of A . The semigroup $\mathbb{T}$ extends to a strongly continuous semigroup on X_{-1} , still denoted by $\mathbb{T}$. We say that $(\mathbb{T}, \Phi)$ admits an integral representation when there exists $B \in \mathcal{L}(U, X_{-1})$ such that

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} B u(s) ds \quad \text{for all } t \geq 0 \text{ and } u \in L^p(\mathbb{R}_+; U), \quad (3.5)$$

where the integral is computed in X_{-1} . For $p \in [1, \infty)$, every abstract linear control system admits an integral representation [18, Theorem 3.9]. For $p = \infty$, an operator B is still determined by the responses to constant inputs. Indeed, using the continuity established in Proposition 1.6, the constant-input argument in the proof of [18, Theorem 3.9] yields

$$Bv = \lim_{h \downarrow 0} \frac{\Phi_h \mathbf{v}}{h} \quad \text{in } X_{-1}, \quad \mathbf{v}(s) := v \quad (v \in U), \quad (3.6)$$

and $B \in \mathcal{L}(U, X_{-1})$. The concatenation identity then gives (3.5) for *interval-step inputs*, namely inputs that are constant on each interval of a finite partition of the time interval under consideration. By continuity, the identity also holds for their limits in the essential-supremum norm on that

interval. However, interval-step inputs are not dense in L^∞ , even for scalar inputs; they should not be confused with measurable simple functions, which are uniformly dense in scalar L^∞ .

The failure of the full representation theorem at $p = \infty$ was already recorded by Weiss [19, Remark 3.7], answering the representation question in [18, Problem 3.10] through the invariant-mean construction in [19, Section 3]. We give below a scalar, positive variant of this construction and record its failure of Orlicz-heart admissibility.¹ The results of [2] concern systems given by (3.5) on the whole of $L^\infty(\mathbb{R}_+; U)$.

Orlicz admissibility. Following the convention of [2], a Young function is a finite-valued, convex, continuous, nondecreasing $F: [0, \infty) \rightarrow [0, \infty)$ such that $F(r)/r \rightarrow 0$ as $r \rightarrow 0$ and $F(r)/r \rightarrow \infty$ as $r \rightarrow \infty$. The obstruction below uses only that F is finite-valued, nondecreasing and satisfies $F(0) = 0$. For $t > 0$, the associated Luxemburg norm is

$$\|u\|_{F,t} := \inf \left\{ \lambda > 0 : \int_0^t F \left(\frac{\|u(s)\|_U}{\lambda} \right) ds \leq 1 \right\}. \quad (3.7)$$

For $F(r) = r^q$ with $q \in (1, \infty)$, this is the norm of $L^q((0, t); U)$. Extending the usual terminology to abstract input maps, we say that $(\mathbb{T}, \Phi)$ is E_F -admissible when, for every $t > 0$, there exists $C_t \geq 0$ such that

$$\|\Phi_t u\|_X \leq C_t \|u\|_{F,t} \quad \text{for all } u \in L^\infty(\mathbb{R}_+; U). \quad (3.8)$$

By (2.1), this amounts to Φ_t extending by continuity to the Orlicz heart $E_F((0, t); U)$, the closure of bounded strongly measurable functions (equivalently, of measurable simple functions) for $\|\cdot\|_{F,t}$. By [2, Proposition 2.1], every system of the form (3.5) with $p = \infty$ is E_F -admissible for some Young function F depending on the system; no stability assumption on $\mathbb{T}$ is needed for this admissibility conclusion.

Such an estimate is stronger than the zero-class property. Indeed, at a fixed time $T > 0$, if $\|v\|_{L^\infty} \leq 1$ and $v = 0$ outside a measurable set $E \subset (0, T)$, then, for every $\lambda > 0$,

$$\int_0^T F \left(\frac{\|v(s)\|_U}{\lambda} \right) ds \leq |E| F(1/\lambda). \quad (3.9)$$

Consequently, (3.8) makes $\Phi_T v$ small, uniformly over such inputs, as $|E| \rightarrow 0$. In particular, delaying an input of duration $h \leq T$ to the last h units of this fixed time interval and using (2.2) gives

$$\kappa(h) \leq C_T \|\mathbf{1}_{(T-h, T)}\|_{F,T} \longrightarrow 0 \quad (h \downarrow 0), \quad (3.10)$$

where the norm on the right is the scalar Luxemburg norm. Conversely, the zero-class property, together with (2.2) and (2.3), makes the output small for inputs supported on any single short interval: for $0 \leq a \leq b \leq T$,

$$\|\Phi_T(\mathbf{1}_{[a,b]} u)\|_X \leq M_T \kappa(b-a) \|u\|_{L^\infty}, \quad M_T := \sup_{0 \leq r \leq T} \|\mathbb{T}_r\|_{\mathcal{L}(X)}. \quad (3.11)$$

It does not imply the corresponding smallness for arbitrary measurable supports of small measure, as the following example shows.

An exotic invariant mean. Let L_{per}^∞ denote the space of 1-periodic elements of $L^\infty(\mathbb{R}; \mathbb{R})$.

Lemma 3.1. *There exists a linear map $m: L_{\text{per}}^\infty \rightarrow \mathbb{R}$ such that*

- (a) $m(f) \geq 0$ whenever $f \geq 0$ almost everywhere, and $m(1) = 1$;
- (b) $m(f(\cdot - a)) = m(f)$ for all $f \in L_{\text{per}}^\infty$ and $a \in \mathbb{R}$;
- (c) $m(f) \neq \int_0^1 f(s) ds$ for some $f \in L_{\text{per}}^\infty$.

¹A related bilateral example based on a Banach limit at $-\infty$ appears in [3, Appendix A3, p. 276]; see also [19, Remark 3.8]. That operator vanishes on inputs which are zero before time 0, so it does not by itself give the present half-line example.

Identifying L_{per}^∞ with $L^\infty(\mathbb{R}/\mathbb{Z})$, this is the classical existence of a translation-invariant mean on L^∞ of the circle which differs from the Haar integral; see [15, 4]. The corresponding construction of a nonzero invariant functional annihilating regulated functions is recalled in [19, Lemma 3.3], whose proof invokes [15, Examples 2.2 and Theorem 3.4]. Since m is defined on equivalence classes, it vanishes on null sets, and (a) entails $|m(f)| \leq \|f\|_{L^\infty}$. Thus m is norm-continuous. It is not weak-* continuous for the duality with L^1 : an L^1 density invariant under all translations would be constant, and normalization would make m the Haar integral, contrary to (c).

A scalar realization.

Proposition 3.2. *There exists an abstract linear control system $(\mathbb{T}, \Phi)$ with $p = \infty$, $X = U = \mathbb{R}$ and $\mathbb{T}_t = \text{Id}$ for all $t \geq 0$, such that:*

- (i) $\kappa(t) = t$ for every $t \geq 0$ and, for every $u \in L^\infty(\mathbb{R}_+; \mathbb{R})$, $t \mapsto \Phi_t u$ is Lipschitz with constant $\|u\|_{L^\infty}$;
- (ii) there is no $g \in L^1(0, 1)$ such that $\Phi_1 u = \int_0^1 g(s)u(s)ds$ for all $u \in L^\infty(\mathbb{R}_+; \mathbb{R})$; in particular, $(\mathbb{T}, \Phi)$ admits no integral representation;
- (iii) $(\mathbb{T}, \Phi)$ is not E_F -admissible, for any finite-valued Young function F .

Proof. Step 1: A translation-invariant positive functional on functions with bounded support. Let $f \in L^\infty(\mathbb{R}; \mathbb{R})$ vanish almost everywhere outside a bounded interval, and choose a representative that is zero outside this interval. Its periodization $Pf(r) := \sum_{k \in \mathbb{Z}} f(r+k)$ is a locally finite sum, with a uniformly bounded number of nonzero summands, so $Pf \in L_{\text{per}}^\infty$. This is well defined on equivalence classes: changing f on a null set changes Pf only on a countable union of integer translates of that null set. Set $J(f) := m(Pf)$. Then J is linear and positive, and it is invariant under all translations since $P(f(\cdot - a)) = (Pf)(\cdot - a)$. We claim that

$$J(\mathbf{1}_{[a,b]}) = b - a \quad \text{for all } a \leq b. \quad (3.12)$$

Let $q(t) := J(\mathbf{1}_{[0,t]})$ for $t \geq 0$. By linearity and invariance, $q(s+t) = J(\mathbf{1}_{[0,s]}) + J(\mathbf{1}_{[s,s+t]}) = q(s) + q(t)$. By positivity, q is nondecreasing. Moreover $q(1) = m(1) = 1$ since $P\mathbf{1}_{[0,1]} = 1$. Hence $q(k/n) = k/n$ for all integers $k \geq 0$, $n \geq 1$, and by monotonicity $q(t) = t$ for all $t \geq 0$. Invariance then yields (3.12). By positivity, if f vanishes outside $[a, b]$, then

$$|J(f)| \leq J(|f|) \leq \|f\|_{L^\infty} J(\mathbf{1}_{[a,b]}) = (b-a)\|f\|_{L^\infty}. \quad (3.13)$$

Step 2: The system. For $t \geq 0$ and $u \in L^\infty(\mathbb{R}_+; \mathbb{R})$, set

$$\Phi_t u := J(\mathbf{1}_{[0,t]} u), \quad (3.14)$$

where $\mathbf{1}_{[0,t]} u$ is extended by zero to $\mathbb{R}$. By (3.13), $|\Phi_t u| \leq t\|u\|_{L^\infty}$, with equality for $u = 1$, so $\kappa(t) = t$. For $t, \tau \geq 0$ and $w := u \diamond_\tau v$, one has almost everywhere $\mathbf{1}_{[0,\tau+t]} w = \mathbf{1}_{[0,\tau]} u + (\mathbf{1}_{[0,t]} v)(\cdot - \tau)$. By invariance of J , $\Phi_{\tau+t} w = \Phi_\tau u + \Phi_t v$, which is (1.1) with $\mathbb{T}_t = \text{Id}$. Finally, for $0 \leq s \leq t$, (3.13) gives $|\Phi_t u - \Phi_s u| = |J(\mathbf{1}_{[s,t]} u)| \leq (t-s)\|u\|_{L^\infty}$. This proves (i).

Step 3: Inputs of small support with non-small output. We claim that there exists a measurable set $E \subset [0, 1)$ such that

$$d := J(\mathbf{1}_E) - |E| \neq 0. \quad (3.15)$$

Otherwise, $J(f) = \int_0^1 f(s)ds$ for every simple function f vanishing outside $[0, 1)$. By uniform density of measurable scalar simple functions and (3.13), the same identity would then hold for every $f \in L^\infty$ vanishing outside $[0, 1)$. Since every $f \in L_{\text{per}}^\infty$ satisfies $f = P(\mathbf{1}_{[0,1]} f)$, this would give $m(f) = \int_0^1 f(s)ds$ for all f , contradicting (c).

By regularity of the Lebesgue measure, there exist sets $I_n \subset [0, 1)$, each a finite disjoint union of intervals $[a, b]$, such that $|E \triangle I_n| \rightarrow 0$. Set $u_n := \mathbf{1}_E - \mathbf{1}_{I_n}$ and $A_n := E \triangle I_n$. Then $|u_n| = \mathbf{1}_{A_n}$ with $|A_n| \rightarrow 0$. By (3.12) and additivity, $J(\mathbf{1}_{I_n}) = |I_n|$, so

$$\Phi_1 u_n = J(\mathbf{1}_E) - |I_n| \longrightarrow J(\mathbf{1}_E) - |E| = d \neq 0. \quad (3.16)$$

Step 4: Proof of (ii). If $\Phi_1 u = \int_0^1 g u$ for some $g \in L^1(0, 1)$ and all u , then $|\Phi_1 u_n| \leq \int_{A_n} |g| \rightarrow 0$ by absolute continuity of the integral, which contradicts (3.16). Moreover, here $A = 0$ is bounded, so $X_{-1} = X = \mathbb{R}$ as vector spaces, with equivalent norms, and any $B \in \mathcal{L}(U, X_{-1})$ is multiplication by some $b \in \mathbb{R}$. Thus (3.5) would give $\Phi_1 u = b \int_0^1 u$, which has just been excluded.

Step 5: Proof of (iii). By the definition above, it suffices to rule out (3.8) at $t = 1$. Let F be a finite-valued Young function and $\lambda > 0$. Since $F(0) = 0$,

$$\int_0^1 F\left(\frac{|u_n(s)|}{\lambda}\right) ds = |A_n| F(1/\lambda) \rightarrow 0, \quad (3.17)$$

so $\|u_n\|_{F,1} \leq \lambda$ for n large enough. As $\lambda > 0$ is arbitrary, $\|u_n\|_{F,1} \rightarrow 0$. An estimate $|\Phi_1 u| \leq C_1 \|u\|_{F,1}$ would then force $\Phi_1 u_n \rightarrow 0$, contradicting (3.16). $\square$

Remark 3.3. *The obstruction is neither the size of Φ on short intervals ($\kappa(t) = t$), nor a lack of compactness (Φ_1 has rank one). It is that the functional $\Phi_1 \in (L^\infty(0, 1))^*$ has no L^1 density. In the proof of [2, Proposition 2.1], the convolution representation is used to identify the adjoint input map restricted to the sun-dual space with an operator taking values in $L^1((0, 1); U^*)$; uniform integrability of the resulting densities is then established. Here the sun-dual space is the whole scalar dual, and this initial density representation already fails. Steps 3 to 5 actually show that Φ_1 is not bounded for any norm N on $L^\infty(0, 1)$ such that $N(u) \leq N(v)$ whenever $|u| \leq |v|$ and such that $N(\mathbf{1}_A) \rightarrow 0$ as $|A| \rightarrow 0$.*

Remark 3.4. *For $p = \infty$, the semigroup together with the operator B recovered from constant inputs does not determine the abstract input maps. In the example above, $\Phi_h(1) = h$, so the operator obtained by testing Φ on constant inputs is $B = 1$. This is also the case for the input maps $\Phi_t^L u := \int_0^t u(s) ds$, which do admit an integral representation. Hence $\Phi - \Phi^L$ is a nonzero family of input maps for $\mathbb{T} = \text{Id}$ which, by (3.12), vanishes on all interval-step inputs and their essential-supremum limits on each finite time interval. The operator recovered from constant inputs for this difference system is $B = 0$.*

Remark 3.5. *A different separation of zero-class admissibility from Orlicz-heart admissibility appears in [2, Example 4.2], using the Cantor measure and a zero-class C -admissible control operator. That example concerns continuous inputs; the present input maps are instead defined on all Lebesgue equivalence classes in L^∞ and satisfy the abstract concatenation identity, but not the convolution representation.*

Research provenance

In June 2025, I gave a course at the EUR MINT 2025 Summer School, *Control, Inverse Problems and Spectral Theory*, in Toulouse, France. In this course, I wanted to present several classical methods in control theory based on time-iteration arguments, and to formulate them as reusable black boxes. This required, in particular, the introduction of abstract *nonlinear* control systems (see [9, Section 3.1]), for which the case $p = \infty$ arises naturally. Continuity in time of the solutions was essential to the time-iteration arguments I intended to present. This led me to stumble upon the difficulty of the open case $p = \infty$ of [18, Problem 2.4].

Throughout 2025 and early 2026, I made several unsuccessful attempts, both unaided and computer-assisted (up to Gemini 3.1 Pro and GPT-5.2), to settle this question, accumulating personal notes on the problem before eventually giving up.

On September 29, 2026, motivated by the launch of the hexagonmath.org website, I made a new attempt. Supplied with my old notes, GPT-6 Pro produced in a single attempt the proofs of Proposition 1.5 and Proposition 1.6, notably by identifying Phillips' lemma as the key ingredient. The following days, I rewrote this text by hand and with the help of Opus 5.5 to give appropriate credit to prior works and to make the proof easier to understand. On October 2, I became aware of the independent preprint [2] posted on September 30 to arXiv, and revised the manuscript accordingly. GPT-6 Pro and Opus 5.5 formalized in Lean the statements and proofs of Proposition 1.5 and Proposition 1.6. The formalizations are uploaded to the Palomar registry.

What a time!

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