

A FOURIER CONSEQUENCE FOR LATTICE KISSING NUMBERS AND AVERAGE CONTACTS

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ABSTRACT. We deduce a new asymptotic upper bound $K_{\text{lat}}(n) \leq \left(\sqrt{2e/\pi} + o(1)\right)^n$ on the maximal lattice kissing number $K_{\text{lat}}(n)$ in dimension n , using the explicit auxiliary functions constructed in OpenAI’s “Ten Advances” preprint [15]. The same bound holds for the average contact degree of a finite packing of congruent balls. The bound’s base-2 exponential rate rounds to 0.3956, matching the value extrapolated empirically by Afkhami-Jeddi, Cohn, Hartman, de Laat, and Tajdini [1].

For a rank- n Euclidean lattice L , we write $\min L = \min_{0 \neq x \in L} \langle x, x \rangle$ for the minimum squared length of nonzero vectors in L ; the *kissing number*

$$\tau(L) = |\{x \in L : \langle x, x \rangle = \min L\}|$$

counts the number of such vectors, including both signs. The *maximal lattice kissing number in dimension n* is

$$K_{\text{lat}}(n) = \max_{\text{rank } L=n} \tau(L).$$

In this note, we obtain a new asymptotic upper bound for $K_{\text{lat}}(n)$, using explicit auxiliary functions constructed in OpenAI’s preprint [15].

Theorem 1. *As $n \rightarrow \infty$,*

$$K_{\text{lat}}(n) \leq \left(\sqrt{\frac{2e}{\pi}} + o(1)\right)^n = 2^{(\frac{1}{2} \log_2(2e/\pi) + o(1))n}.$$

Here $\frac{1}{2} \log_2(2e/\pi) = 0.395599455708322 \dots$

For a packing of congruent balls, the *contact degree* of a ball is the number of other balls tangent to it; the *average contact degree* of a nonempty finite packing is the arithmetic mean of these degrees. After scaling such balls to radius $1/2$, their centers form a finite set $C \subset \mathbb{R}^n$ with minimum separation at least 1, and the average contact degree is

$$\bar{\tau}(C) = \frac{\#\{(x, y) \in C^2 : \|x - y\| = 1\}}{|C|}.$$

Corollary 4 gives the same upper bound as Theorem 1 for $\bar{\tau}(C)$, uniformly over all such C .

To place the bound in context, let $K(n)$ be the ordinary kissing number—i.e., the largest number of nonoverlapping unit balls that can touch one unit ball in $\mathbb{R}^n$. Equivalently, $K(n)$ is the largest size of a set of unit vectors with pairwise inner products at most $1/2$. Normalizing the shortest vectors

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I used LLMs to assist with analysis and synthesis in the preparation of this article, particularly GPT-6 Astra and Claude Fable 5.1 (accessed in part via Poe with the support of Quora, where I am an advisor). In particular, the bound I present was developed in dialogue with GPT-6 Astra. I greatly appreciate helpful comments from Henry Cohn and Thijs Laarhoven. The problem, methods, and final written form are my own; and of course any errors remain my responsibility. This work was conducted while I was visiting the Technological Innovation, Entrepreneurship, and Strategic Management (TIES) Group at the MIT Sloan School of Management.

of a lattice gives such a set, so $K_{\text{lat}}(n) \leq K(n)$. The classical spherical-code bound of Kabatiansky and Levenshtein [12], from 1978, gives the exponent $0.40141\dots$ for $K(n)$. Combining it with Sidel'nikov's spherical-cap reduction [18] gives the optimized classical bound [12, Theorem 4], with exponent $0.400944\dots$; subsequent work of Sardari and Zargar [17], and of Zargar [22], improved multiplicative factors.

Fourier methods offer a second route, through the contacts in a packing. In 2020, Afkhami-Jeddi, Cohn, Hartman, de Laat, and Tajdini [1] extrapolated an implied kissing exponent of 0.3956 (the $d = \infty$ row of [1, Table 3]) from their numerically optimized density bound. Its interpretation as an average-contact bound relied on interpolation and sign conjectures. Their Theorem 5.3 gives an unconditional inequality for admissible auxiliary functions; we apply this result to the explicit functions of [15], obtaining an exponent that agrees with their extrapolation to the displayed precision, without relying on the intermediate conjectures.

The cap-optimized spherical-code hierarchy of [15] gives

$$K(n) \leq 2^{(0.39661+o(1))n}$$

at level 2; see [15, Chapter 2, Remark 1.7]. Our Theorem 1 and Corollary 4 give a smaller exponent for lattice kissing numbers and average contact degree, respectively. In fact, Lemma A.1 in Appendix A shows that the infimum of the cap-optimized fixed-level asymptotic kissing exponents of the hierarchy in [15] remains strictly above ours. To our knowledge, Theorem 1 gives the first proven asymptotic upper bound for lattice kissing numbers whose exponential rate is strictly smaller than the best known upper-bound rate for the unrestricted kissing number.¹

Meanwhile, there remains a large gap between the upper and lower bounds for lattice kissing numbers. The Barnes–Wall lattices [3] in dimensions $n = 2^m$ have kissing numbers

$$\prod_{i=1}^m (2^i + 2) = \exp(\Theta((\log n)^2));$$

see Nebe [14, Lemma 3.4]. Recent work of Laarhoven and the present author [13, Theorem 1.1] improves this to

$$K_{\text{lat}}(n) \geq \left(\frac{1}{2\pi e^2} + o(1) \right) \sqrt{n} e^{2\sqrt{n}}$$

as $n \rightarrow \infty$ through prime powers.²

Our analytic input comes from Chapter 1 of [15]; all references to that preprint use the version updated August 6, 2026. That chapter determines the exponential strength of the Fourier linear program for sphere-packing density, developed by Gorbachev [11] and Cohn–Elkies [6]. Writing LP_n for the infimum of the density upper bounds supplied by this program, [15, Chapter 1, Theorem 1.1] states

$$\lim_{n \rightarrow \infty} \text{LP}_n^{1/n} = \sqrt{\frac{e}{2\pi}}.$$

Consequently, the maximal packing density Δ_n satisfies $\Delta_n \leq 2^{-(0.604400544\dots+o(1))n}$; the equality above determines the rate of the linear program itself. A comparison theorem of Cohn and Zhao [7, Theorem 3.4] then implies that the ordinary spherical linear program for $K(n)$ cannot attain an exponent smaller than $\frac{1}{2} \log_2(2e/\pi)$.

For our bounds, we use the explicit auxiliary functions f constructed in [15] to establish the asymptotic density upper bound, rather than the density-rate theorem alone. The density bound

¹We thank Thijs Laarhoven for pointing out this possibility. Earlier lattice-specific bounds include Skoruppa's Mellin-transform bound [19, Theorem 2]; its proof gives the exponent $\log_2(e/2) = 0.442695\dots$, which is weaker than the classical general bound. Meanwhile, stronger bounds in individual dimensions have long been known for lattices and antipodal codes (see, e.g., [21, 2]).

²Vlăduț [20] attempted to obtain lattices with exponentially many shortest vectors, but Bennett, Golovnev, and Stephens-Davidowitz [4] later identified errors in his construction.

depends on $f(0)/\widehat{f}(0)$ together with a volume factor, whereas the kissing and contact bounds depend on $f(0)/[-f(R)]$, where R is the center-to-center distance at contact. We evaluate the latter ratio using the saddle-point estimate at the target parameter u_0 in [15, Chapter 1, §4]. Lemma 3 below collects the required properties of these functions and the resulting contact-value estimate, with precise references to the global sign and saddle-point estimates on which our argument relies.

All logarithms without a subscript are natural.

THE FOURIER CONTACT BOUND

The following bound for average contact degree appears in [1, Theorem 5.3]. Its finite-packing assertion also follows from the positive-type bound of Dostert, Kolpakov, and de Oliveira Filho [9, Theorem 4.1], specialized to congruent balls. As noted in [1], it is the Euclidean analogue of Fortier Bourque and Petri's trace-formula bound [10, Proposition 4.1].

We use the normalization $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$. For radial f , write $f(r)$ for its value at a vector of norm r .

Lemma 2 ([1, Theorem 5.3]; [9, Theorem 4.1]). *Let f be a real radial Schwartz function on $\mathbb{R}^n$, and suppose that, for some $R > 0$,*

$$\widehat{f} \geq 0, \quad f(r) \leq 0 \quad (r \geq R), \quad f(R) < 0.$$

Then every rank- n lattice satisfies $\tau(L) \leq f(0)/[-f(R)]$. The same ratio bounds the average contact degree of every nonempty finite packing of balls of radius $R/2$, with contacts at center distance R .

Proof. We scale L so that its shortest nonzero vectors have length R , equivalently $\min L = R^2$. Writing L^* for the dual lattice, Poisson summation gives

$$0 \leq \frac{1}{\text{covol}(L)} \sum_{\xi \in L^*} \widehat{f}(\xi) = \sum_{x \in L} f(x) \leq f(0) + \tau(L)f(R);$$

since $f(R) < 0$, this proves the lattice bound.

For a nonempty finite set C of minimum separation at least R , Fourier inversion gives

$$0 \leq \int_{\mathbb{R}^n} \widehat{f}(\xi) \left| \sum_{x \in C} e^{2\pi i \langle x, \xi \rangle} \right|^2 d\xi = \sum_{x, y \in C} f(x - y).$$

If M counts the ordered pairs at distance R , then the last sum is at most $|C|f(0) + Mf(R)$. Thus $M/|C| \leq f(0)/[-f(R)]$. $\square$

THE AUXILIARY FUNCTIONS

Fix sufficiently small $\varepsilon > 0$, and put

$$\lambda = \frac{n}{2}, \quad u = 1 + \frac{\varepsilon}{4}, \quad m = \frac{\lambda(1+u)}{2}, \quad \psi = \frac{\Gamma'}{\Gamma}.$$

Let w_ε be the compactly supported signed density in [15, Chapter 1, §4],³ and define the real function

$$g_\varepsilon(t) = \int_0^\infty w_\varepsilon(a) (\cosh(at) - 1) da.$$

³Explicitly,

$$\begin{aligned} w_\varepsilon(a) = & -\frac{(1 - 2\varepsilon(1+a))e^{-2a}}{2a^2 \cosh a} \mathbf{1}_{[\varepsilon^2, \log(1/\varepsilon)]}(a) \\ & + \frac{e^{-3/(8\varepsilon^2)}}{\cosh a} \mathbf{1}_{[\varepsilon^{-3}, \varepsilon^{-3}+1]}(a), \end{aligned}$$

where $\mathbf{1}_I$ denotes the indicator of I ; see [15, Chapter 1, (34)–(35)].

Thus $g_\varepsilon(t) = h_\varepsilon(it)$ in the notation of [15, Chapter 1]. We collect the analytic input, including the contact value extracted from the [15, Chapter 1] saddle estimate, in the following lemma.

Lemma 3. *For every fixed sufficiently small $\varepsilon > 0$, there is $n_0(\varepsilon)$ such that, for every $n \geq n_0(\varepsilon)$, there exist a real radial Schwartz function $f = f_{n,\varepsilon}$ on $\mathbb{R}^n$ and a radius $R = R_{n,\varepsilon} > 0$ satisfying the hypotheses of Lemma 2, with*

$$\begin{aligned} (1) \quad & \log R = -\frac{1}{2} \log \pi + \frac{1}{2} \psi(m) + g'_\varepsilon(u), \\ (2) \quad & f(0) = 2(u-1) \pi^{\lambda/2} e^{\lambda g_\varepsilon(1)}, \\ (3) \quad & -f(R) = A_{n,\varepsilon} \pi^{-\lambda u/2} \Gamma(m) e^{\lambda g_\varepsilon(u)} R^{-2m}, \quad A_{n,\varepsilon} \asymp_\varepsilon \sqrt{n}, \\ (4) \quad & g'_\varepsilon(u) = -\frac{1}{2} \log(\pi/2) + O(\varepsilon). \end{aligned}$$

In particular, $A_{n,\varepsilon} > 0$. The constants implicit in $\asymp_\varepsilon$ may depend on ε but not on n .⁴ More precisely, put

$$p_\varepsilon = (u-1)[1 - (1+u)^2] < 0, \quad V_{n,\varepsilon} = \frac{\lambda}{4} \psi'(m) + g''_\varepsilon(u).$$

Then $V_{n,\varepsilon} \asymp_\varepsilon 1$ and

$$(5) \quad A_{n,\varepsilon} = (-p_\varepsilon) \sqrt{\frac{\lambda}{2\pi V_{n,\varepsilon}}} (1 + o_\varepsilon(1)).$$

Proof. We take f to be the function denoted f_- in [15, Chapter 1, §4]. The origin value (2) and radius (1) are given in [15, Chapter 1, (42) and (44)], respectively. The derivative estimate (4) is [15, Chapter 1, Lemma 4.2], and the global sign conditions are established in [15, Chapter 1, Theorem 4.1]. For the contact value, the target parameter is $u = u_0$ in that preprint's notation, and this endpoint is included in [15, Chapter 1, Lemma 4.8]. The saddle coefficient p_ε is given in [15, Chapter 1, (38)–(39)]. In [15, Chapter 1, (70)–(71)], the inverse Mellin factor $\lambda/(2\pi)$ and Gaussian factor combine as

$$\frac{\lambda}{2\pi} \sqrt{\frac{2\pi}{\lambda V_{n,\varepsilon}}} = \sqrt{\frac{\lambda}{2\pi V_{n,\varepsilon}}},$$

giving (3) and (5). The curvature bounds in [15, Chapter 1, Lemma 4.5, (56)] give $V_{n,\varepsilon} \asymp_\varepsilon 1$. Since $p_\varepsilon < 0$, the contact value is negative and $A_{n,\varepsilon} \asymp_\varepsilon \sqrt{n}$. $\square$

We also need the estimate

$$(6) \quad g_\varepsilon(u) - g_\varepsilon(1) = O(\varepsilon).$$

To verify this, let $w_{\varepsilon,+}$ and $w_{\varepsilon,-}$ be the positive and negative parts of w_ε , so that $w_\varepsilon = w_{\varepsilon,+} - w_{\varepsilon,-}$. The density construction and [15, Chapter 1, Lemma 4.2] give

$$w_{\varepsilon,-}(a) \leq \frac{e^{-2a}}{2a^2 \cosh a}, \quad \int_0^\infty w_{\varepsilon,+}(a) a \sinh(ua) da = O(e^{-c/\varepsilon^2})$$

for an absolute $c > 0$. For $1 \leq t \leq u \leq 5/4$, we use the inequalities $\sinh(ta) \leq ta \cosh(ta)$ and $\cosh(ta) \leq e^{(t-1)a} \cosh a$ to obtain

$$|g'_\varepsilon(t)| \leq \frac{t}{2} \int_0^\infty e^{-(3-t)a} da + O(e^{-c/\varepsilon^2}) \leq \frac{5}{14} + O(e^{-c/\varepsilon^2}).$$

Integration over $[1, u]$, of length $\varepsilon/4$, proves (6).

⁴Here and below, $o_\varepsilon(1)$ denotes a quantity tending to zero as $n \rightarrow \infty$ with ε fixed.

THE KISSING NUMBER BOUND

We now estimate the ratio $f(0)/[-f(R)]$ and apply Lemma 2 to obtain the lattice kissing number bound.

Proof of Theorem 1. Fix sufficiently small $\varepsilon > 0$. Taking logarithms in (2) and (3) and substituting (1) gives

$$(7) \quad \log \frac{f(0)}{-f(R)} = \lambda[g_\varepsilon(1) - g_\varepsilon(u) + (1+u)g'_\varepsilon(u)] \\ + m\psi(m) - \log \Gamma(m) + \log \frac{2(u-1)}{A_{n,\varepsilon}}.$$

(The powers of π cancel because $2m = \lambda(1+u)$.) Stirling's formula yields

$$m\psi(m) - \log \Gamma(m) = m + \frac{1}{2} \log m + O(1).$$

Since $\log A_{n,\varepsilon} = \frac{1}{2} \log n + O_\varepsilon(1)$ and $m/n = (1+u)/4$, we have

$$\frac{1}{2} \log m + \log \frac{2(u-1)}{A_{n,\varepsilon}} = O_\varepsilon(1).$$

Dividing (7) by $n = 2\lambda$ therefore gives

$$\frac{1}{n} \log \frac{f(0)}{-f(R)} = \frac{1}{2} \left[g_\varepsilon(1) - g_\varepsilon(u) + (1+u)g'_\varepsilon(u) + \frac{1+u}{2} \right] + O_\varepsilon(n^{-1}).$$

Since $u = 1 + O(\varepsilon)$, equations (4) and (6) then give

$$\frac{1}{n} \log \frac{f(0)}{-f(R)} = \frac{1}{2} \log \frac{2e}{\pi} + O(\varepsilon) + O_\varepsilon(n^{-1}).$$

Applying Lemma 2 and taking the upper limit as $n \rightarrow \infty$ with ε fixed, we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log K_{\text{lat}}(n) \leq \frac{1}{2} \log \frac{2e}{\pi} + O(\varepsilon).$$

Letting $\varepsilon \downarrow 0$ proves the theorem. $\square$

AVERAGE CONTACT DEGREE

The same ratio estimate also bounds average contact degree.

Corollary 4. *As $n \rightarrow \infty$, uniformly over nonempty finite $C \subset \mathbb{R}^n$ with minimum separation at least 1,*

$$\bar{\tau}(C) \leq \left(\sqrt{\frac{2e}{\pi}} + o(1) \right)^n.$$

Proof. Fix sufficiently small $\varepsilon > 0$, and take f and R from Lemma 3. The scaled set RC has minimum separation at least R , and its ordered pairs at distance R correspond exactly to the ordered pairs of C at distance 1. The finite-packing part of Lemma 2 therefore gives

$$\bar{\tau}(C) \leq \frac{f(0)}{-f(R)};$$

the estimate for this ratio in the proof of Theorem 1 is independent of C . Taking $n \rightarrow \infty$ with ε fixed and then letting $\varepsilon \downarrow 0$ proves the stated uniform bound. $\square$

CONCLUDING REMARKS

Uniformity and contact counts. Explicitly, for every $\eta > 0$ there is $n_0(\eta)$ such that, for every $n \geq n_0(\eta)$, both $K_{\text{lat}}(n)$ and $\bar{\tau}(C)$ for every eligible $C \subset \mathbb{R}^n$ are at most $2^{(\frac{1}{2}\log_2(2e/\pi)+\eta)n}$. For a packing of N congruent balls, the *contact number*—the number of unordered tangent pairs—is therefore at most

$$\frac{N}{2} \left(\sqrt{\frac{2e}{\pi}} + o(1) \right)^n,$$

uniformly in N .

Our bounds apply to lattice kissing numbers and to the average contact degree of arbitrary finite packings of congruent balls. They do not establish the same exponential bound for the maximum contact degree in such a packing, and therefore do not give the same bound for the unrestricted kissing number $K(n)$; they also do not address packings with unequal radii. For background on the contact-number problem, see the Bezdek and Khan [5] survey.

Comparison with the spherical linear program. The exponent in Theorem 1 also gives a lower bound on the exponential rate of the optimal ordinary kissing-number linear-programming bound. Let K_n^{LP} denote the infimum of the upper bounds for $K(n)$ supplied by the two-point spherical linear program of Delsarte, Goethals, and Seidel [8]. Cohn and Zhao [7, Theorem 3.4], specialized to the kissing angle $\pi/3$, proved

$$(8) \quad \text{LP}_n \leq 2^{-n} K_n^{\text{LP}};$$

their comparison applies in the same dimension n .⁵ Combining (8) with $\lim_{n \rightarrow \infty} \text{LP}_n^{1/n} = \sqrt{e/(2\pi)}$ gives

$$(9) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log_2 K_n^{\text{LP}} \geq \frac{1}{2} \log_2 \frac{2e}{\pi}.$$

Thus our bounds for lattice kissing numbers and average contact degree attain an exponent below which the ordinary spherical linear program cannot bound the unrestricted kissing number.⁶ The inequality (9) is a lower bound on the optimal spherical linear programming upper bound K_n^{LP} , not a lower bound on the kissing number $K(n)$ itself. Earlier asymptotic lower bounds for the optimal spherical linear-programming bound were obtained by Samorodnitsky [16, Corollary 1.3].

The same constant appears within the spherical-code hierarchy of [15], whose certificates belong to this two-point linear program; see [15, Chapter 2, Theorem 4.2]. Specifically, write $\sigma = 2\Gamma_r$ for a certificate's threshold and $\Phi = \Phi_r$ for its code-size exponent. Then [15, Chapter 2, Proposition 8.1, (101)] gives

$$\Phi = \frac{1}{2} \log_2 \frac{2}{1-\sigma} - \frac{1}{2} \log_2 \frac{2\pi}{e} + \frac{D_{\text{KL}}(\nu \parallel \rho_K)}{2 \log 2},$$

where ν and ρ_K are the probability measures defined there and D_{KL} denotes relative entropy. For parameters feasible at the kissing angle, $\sigma \geq 1/2$; nonnegativity of relative entropy therefore gives $\Phi \geq \frac{1}{2} \log_2(2e/\pi)$. This identity makes the same lower bound explicit for those certificates, while the Cohn–Zhao comparison applies to the entire two-point spherical linear program.

Appendix A gives a stronger limitation for the spherical-code hierarchy of [15]: Lemma A.1 shows that, even after spherical-cap optimization and taking the infimum over fixed levels, the hierarchy's asymptotic kissing exponent remains strictly above ours. This statement concerns that certificate family; it does not establish a strict gap in (9) for the full two-point spherical linear program.

⁵The continuous compactly supported auxiliary functions constructed in [7] can be convolved with smooth compactly supported, nonnegative, positive-definite radial approximate identities and then rescaled to restore the sign radius. This gives Schwartz functions whose density bounds converge to the original bound, at each fixed n , so (8) also holds for the Schwartz formulation used in [15].

⁶We thank Henry Cohn for this observation.

The sharpness argument in [15, Chapter 2, Theorem 8.3] sends $\sigma \rightarrow 1$, establishing the optimized sphere-packing density exponent. This is compatible with Lemma A.1, which fixes the kissing angle at $\pi/3$.

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APPENDIX A. SEPARATION FROM THE OPENAI SPHERICAL-CODE HIERARCHY

In this appendix, we show that the asymptotic kissing-number exponents supplied by the hierarchy of [15, Chapter 2] remain uniformly separated from the exponent in Theorem 1, even after

spherical-cap optimization and taking the infimum over fixed levels. Starting from the relative-entropy identity of [15, Chapter 2, Proposition 8.1], we use Jensen's inequality to bound the mass of a fixed interval and thereby obtain an explicit positive lower bound on the relative entropy.

Lemma A.1. *Put*

$$\alpha_* = \frac{1}{2} \log_2 \frac{2e}{\pi}, \quad \beta = \frac{2}{3} - \frac{2}{\pi} \arctan \frac{1}{2\sqrt{3}}.$$

The infimum of the fixed-level asymptotic kissing-number exponents supplied by the hierarchy of [15, Chapter 2, Theorem 1.2], over all finite levels and the spherical-cap parameters in [15, Chapter 2, (89)], is at least

$$\alpha_* - \frac{1}{4} \log_2(4\beta(1-\beta)) = 0.395815863377\dots > \alpha_*.$$

Here the level is fixed before the dimension tends to infinity, and the infimum is then taken over the resulting asymptotic rates. The argument does not address levels that vary with the dimension.

Proof of Lemma A.1. For a finite-level certificate, let $x_i \geq 0$ and $R_i > 0$ be the quadratic coordinates and residues in [15, Chapter 2, Proposition 8.1, (102)], so that $\sum_i R_i = 1$. Define

$$w_i = \sqrt{\frac{x_i}{x_i + 1/4}}, \quad \mu = \sum_i R_i \delta_{w_i},$$

where δ_w denotes unit point mass at w , and put

$$k_w(v) = \frac{(1-v^2)(1-w^2)}{1-v^2+v^2w^2}, \quad F(v) = \int_0^1 k_w(v) d\mu(w) \quad (0 \leq v < 1).$$

In particular, $k_0(v) = 1$ and $k_1(v) = 0$. Under $t = (1-v^2)/4$, the reference measures in [15, Chapter 2, Proposition 8.1] become

$$d\nu(v) = dv, \quad d\rho(v) = \frac{2 dv}{\pi\sqrt{1-v^2}}.$$

The kernel satisfies

$$\int_0^1 k_w(v) d\rho(v) = 1 - w \quad (0 \leq w \leq 1).$$

We define the normalizing constant

$$Z := \int_0^1 F(v) d\rho(v);$$

the kernel identity gives

$$Z = 1 - \int_0^1 w d\mu(w).$$

By [15, Chapter 2, (102)–(104)], the certificate's threshold σ and exponent Φ satisfy

$$(A.1) \quad \sigma = \int_0^1 w d\mu(w), \quad Z = 1 - \sigma, \quad \Phi = -\frac{1}{2 \log 2} \int_0^1 \log F(v) dv.$$

We now enlarge the class under consideration by allowing arbitrary probability measures μ on $[0, 1]$, including endpoint atoms, rather than only those arising from finite-level certificates. For these measures, define F and Z as above and σ and Φ by the same formulas in (A.1). The kernel identity ensures that $Z = 1 - \sigma$ continues to hold. We set $\Phi = +\infty$ if $F \equiv 0$.

We first include cap optimization and reduce to $\sigma = 1/2$. A certificate used at an inner-product threshold $0 \leq s \leq 1/2$ is feasible when $\sigma \geq s$; cap reduction adds $\frac{1}{2} \log_2(2(1-s))$ to its exponent.

Set $q = 1/(2(1-s))$ and replace μ by $q\mu + (1-q)\delta_1$. This replaces F by qF , adds exactly that cap cost, and replaces the normalizing constant by

$$Z' = qZ = \frac{1-\sigma}{2(1-s)} \leq \frac{1}{2};$$

hence, the new measure has mean at least $1/2$. Direct use at the kissing angle is the case $s = 1/2$, $q = 1$; the endpoint $s = 0$ is also covered by taking $\mu = \delta_0$.

We may further restrict to measures of mean $1/2$. Indeed, for any probability measure μ with mean $\sigma \geq 1/2$, put

$$a = \frac{1}{2\sigma}, \quad \tilde{\mu} = a\mu + (1-a)\delta_0.$$

Then $\tilde{\mu}$ has mean $1/2$. Since $k_0(v) = 1$ and $0 \leq F(v) \leq 1$, its associated function $\tilde{F}$ satisfies

$$\tilde{F}(v) = aF(v) + (1-a) \geq F(v).$$

Thus its associated exponent satisfies $\tilde{\Phi} \leq \Phi$. A lower bound for the exponent of every probability measure of mean $1/2$ therefore also bounds the exponent of every probability measure of mean at least $1/2$, including the measures obtained by the cap reduction above.

For such a measure, $P = 2F\rho$ is a probability measure, and the relative-entropy identity becomes

$$(A.2) \quad \Phi = \alpha_* + \frac{D_{\text{KL}}(\nu \| P)}{2 \log 2}.$$

Indeed, $\int_0^1 \log(d\nu/d\rho) d\nu = \log \pi - 1$, which gives (A.2) directly from (A.1). The logarithms are integrable: $k_w(v) \geq (1-w)(1-v^2)$ gives $F(v) \geq (1-v^2)/2$.

For $0 \leq v \leq 1/2$, the function $w \mapsto k_w(v)$ is concave, since

$$\frac{\partial^2 k_w(v)}{\partial w^2} = -\frac{2(1-v^2)(1-v^2-3v^2w^2)}{(1-v^2+v^2w^2)^3} \leq 0.$$

Jensen's inequality gives

$$F(v) \leq k_{1/2}(v) = \frac{3(1-v^2)}{4-3v^2} \quad (0 \leq v \leq 1/2).$$

For $E = [0, 1/2]$, we have $\nu(E) = 1/2$ and consequently

$$P(E) \leq \frac{3}{\pi} \int_0^{1/2} \frac{\sqrt{1-v^2}}{1-3v^2/4} dv = \frac{2}{3} - \frac{2}{\pi} \arctan \frac{1}{2\sqrt{3}} = \beta < \frac{1}{2}.$$

The integral follows by setting $v = \sin \theta$; the strict inequality follows from the fact that $\tan(\pi/12) = 2 - \sqrt{3} < 1/(2\sqrt{3})$. Applying the log-sum inequality to the partition $\{E, E^c\}$ then gives

$$D_{\text{KL}}(\nu \| P) \geq -\frac{1}{2} \log(4P(E)(1-P(E))) \geq -\frac{1}{2} \log(4\beta(1-\beta)).$$

Substitution in (A.2) proves the claimed bound. (It is independent of the level and cap parameter, so taking their infimum preserves it.) $\square$