

# ON THE TERNARY PENTAGONAL NUMBERS CONJECTURE

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ABSTRACT. Communicated by Guy in 1994, the ternary pentagonal numbers conjecture of Blecksmith and Selfridge asserts that every integer larger than 33066 is the sum of three positive pentagonal numbers. Prior to this note, it was even unknown whether every sufficiently large integer is the sum of three positive pentagonal numbers. We resolve this in the affirmative, using the landmark work of Duke and Schulze-Pillot on ternary quadratic forms to handle all sufficiently large integers  $n$  with  $v_3(24n + 3) \leq 8$ , and present an explicit lift to handle the  $n$  with  $v_3(24n + 3) \geq 9$ .

## 1. INTRODUCTION AND RESULTS

For an integer  $k \geq 3$ , let  $p_k(x) = ((k - 2)x^2 - (k - 4)x)/2$  be the  $k$ -gonal polynomial. We say that  $n$  is a  $k$ -gonal number if  $n = p_k(m)$  for some nonnegative integer  $m$  and that  $n$  is a generalized  $k$ -gonal number if  $n = p_k(x)$  for some  $x \in \mathbb{Z}$ . Since Fermat conjectured in 1638 that every positive integer is the sum of  $k$   $k$ -gonal numbers (which is now known as the Fermat–Cauchy polygonal number theorem), additive bases of (generalized) polygonal numbers have endured as a topic of thorough study.

While the Fermat–Cauchy polygonal number theorem is sharp for every  $k$ , the question of how many  $k$ -gonal numbers one needs to represent every sufficiently large integer is subtler. In 1830, Legendre [Leg30] showed that every sufficiently large integer is the sum of four  $k$ -gonal numbers if  $k \not\equiv 0 \pmod{4}$ , and is the sum of five  $k$ -gonal numbers if  $k \equiv 0 \pmod{4}$  (and at least one of these five is 0 or 1). While Meng and Sun [MS17] showed that the  $k \equiv 0 \pmod{4}$  case of Legendre’s result is sharp for  $k \geq 8$  (the  $k = 4$  case is Lagrange’s four-square theorem), the question of whether only three  $k$ -gonal numbers suffice to represent every sufficiently large integer is still open in general. In fact, it is still unknown whether every sufficiently large integer is the sum of three *generalized*  $k$ -gonal numbers for general  $k \equiv 5, 11 \pmod{12}$ , though [HK18, Theorem 1.2] gives some control over the exceptional set.

In 1994, Guy [Guy94] posed numerous representation problems regarding (generalized) polygonal numbers. In particular, he communicated a conjecture of Richard Blecksmith and John Selfridge on the positive integers that are the sum of three pentagonal numbers, which was recently rediscovered by Sun [KS20, Remark 1.1].

**Conjecture 1.1** (Blecksmith–Selfridge, [Guy94, pg.171]). *Every integer larger than 33066 is the sum of three positive pentagonal numbers.*

See [OEI] for a list of all 210 currently known positive integers that are not the sum of three pentagonal numbers. While Guy quickly and elementarily shows that in fact every positive integer is the sum of three generalized pentagonal numbers, proving Conjecture 1.1 appears considerably more difficult, with no proof, elementary or otherwise, being found to date.

After stating Conjecture 1.1, Guy simply asks whether every sufficiently large integer is the sum of three pentagonal numbers, which has also remained open since the publication of [Guy94]. We resolve this question in the affirmative, and assert further that one can require each of the three pentagonal numbers to be positive.

**Theorem 1.2.** *Every sufficiently large integer is the sum of three positive pentagonal numbers.*

The proof is separated into two cases:  $v_3(24n + 3) \leq 8$  and  $v_3(24n + 3) \geq 9$ . For  $v_3(24n + 3) \leq 8$ , we recast the ternary pentagonal numbers conjecture to use the work of Duke and Schulze-Pillot [DSP90] to approximate the ternary representation function of the positive pentagonal numbers by the ternary representation function of the generalized pentagonal numbers, from which we ultimately show that the former is  $\Omega_\varepsilon(n^{1/2-\varepsilon})$ . We note that this asymptotic bound is ineffective (and thus so is Theorem 1.2) since it relies on Siegel’s estimate [Sie35b]. For all integers with  $v_3(24n + 3) \geq 9$ , we provide an explicit lift found by Chat-GPT 5.6 Sol. A statement of AI use may be found toward the end of the paper, immediately preceding the acknowledgments.

## 2. DISCUSSION

**2.1. Background and overview.** Seeking to determine when the integral points on the ellipsoid  $Q(x_1, x_2, x_3) = n$  are asymptotically uniformly distributed for  $Q$  a positive-definite integral ternary quadratic form, Duke and Schulze-Pillot [DSP90] prove several results of crucial use to us. Their setup is as follows: let  $N$  be the level<sup>1</sup> of  $Q$ , let  $\mathcal{F}$  be a convex subset of  $\mathcal{E} := \{\mathbf{t} \in \mathbb{R}^3 : Q(\mathbf{t}) = 1\}$  with a piecewise-smooth boundary, and let  $\mathbf{h} \in \mathbb{Q}^3$  be such that  $A\mathbf{h} \in \mathbb{Z}^3$  and  $Q(\mathbf{h}) \in \mathbb{Z}$ , where  $A$  is the matrix such that  $Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A \mathbf{x}$ . Now define

$$\begin{aligned} r(Q, \mathbf{h}, n) &:= \# \{ \mathbf{x} \in \mathbb{Q}^3 : \mathbf{x} \equiv \mathbf{h} \pmod{\mathbb{Z}^3}, Q(\mathbf{x}) = n \}, \\ r(Q, \mathcal{F}, \mathbf{h}, n) &:= \# \{ \mathbf{x} \in \mathbb{Q}^3 : \mathbf{x} \equiv \mathbf{h} \pmod{\mathbb{Z}^3}, Q(\mathbf{x}) = n, \mathbf{x}/\sqrt{n} \in \mathcal{F} \}. \end{aligned}$$

Letting  $\mu$  be the normalized measure on  $\mathcal{E}$ , the asymptotic uniform distribution statement sought is then  $r(Q, \mathcal{F}, \mathbf{h}, n) \sim \mu(\mathcal{F})r(Q, \mathbf{h}, n)$ .

The critical result of their paper, [DSP90, Theorem 1], when combined with the routine bound  $r(Q, \mathbf{h}, n) \ll_\varepsilon n^{1/2+\varepsilon}$ , asserts that for suitable  $n$ , indeed

$$r(Q, \mathcal{F}, \mathbf{h}, n) = \mu(\mathcal{F})r(Q, \mathbf{h}, n) + O_{\varepsilon, S, N}(n^{1/2-1/175+\varepsilon}), \quad (1)$$

where  $S$  is a parameter governing which  $n$  are “suitable”. The remainder of the paper is dedicated to showing when  $r(Q, \mathbf{h}, n)$  has the predicted growth of  $n^{1/2+o(1)}$ . The key breakthrough making (1) possible is the development of power-saving bounds for Fourier coefficients of cusp forms of half-integral weight, initiated by Iwaniec [Iwa87] and extended by Duke [Duk88].

The path toward proving [DSP90, Theorem 1] is separated into four steps. For  $\delta > 0$  to be specified later, one first tightly bounds the indicator function  $\mathbf{1}_{\mathcal{F}}(\mathbf{x})$  above and below by a series of homogeneous  $Q$ -harmonic polynomials  $P_v^\pm(\mathbf{x})$  of degree  $v$  with rapid decay in  $v$  and  $|P_0^\pm - \mu(\mathcal{F})| < \delta$ ; this is [DSP90, Lemma 1]. Next, since

$$\sum_{v \geq 0} \sum_{\substack{\mathbf{x} \in \mathbf{h} + \mathbb{Z}^3, \\ Q(\mathbf{x}) = n}} P_v^-(\mathbf{x}/\sqrt{n}) \leq r(Q, \mathcal{F}, \mathbf{h}, n) \leq \sum_{v \geq 0} \sum_{\substack{\mathbf{x} \in \mathbf{h} + \mathbb{Z}^3, \\ Q(\mathbf{x}) = n}} P_v^+(\mathbf{x}/\sqrt{n}),$$

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<sup>1</sup>The level  $N$  of a quadratic form  $\frac{1}{2}\mathbf{x}^T A \mathbf{x}$  (with  $A$  symmetric and integral with diagonal entries all even) is the smallest positive integer such that  $NA^{-1}$  is integral with diagonal entries all even.

we have

$$\begin{aligned} r(Q, \mathcal{F}, \mathbf{h}, n) - \mu(\mathcal{F})r(Q, \mathbf{h}, n) &\leq \sum_{v \geq 1} \sum_{\substack{\mathbf{x} \in \mathbf{h} + \mathbb{Z}^3 \\ Q(\mathbf{x}) = n}} P_v^+(\mathbf{x}/\sqrt{n}) + \delta r(Q, \mathbf{h}, n), \\ r(Q, \mathcal{F}, \mathbf{h}, n) - \mu(\mathcal{F})r(Q, \mathbf{h}, n) &\geq \sum_{v \geq 1} \sum_{\substack{\mathbf{x} \in \mathbf{h} + \mathbb{Z}^3 \\ Q(\mathbf{x}) = n}} P_v^-(\mathbf{x}/\sqrt{n}) - \delta r(Q, \mathbf{h}, n). \end{aligned}$$

Then for each  $v \geq 1$ , the methods of Iwaniec [Iwa87] and Duke [Duk88] are used, after refinement by [DSP90, Lemma 2], to bound the Fourier coefficients of the shifted (by  $\mathbf{h}$ ) theta series associated with  $Q$  weighted by  $P_v^\pm$ , since they are cusp forms of weight  $3/2 + v$ . These Fourier coefficients are precisely the  $v$ -summands above multiplied by  $n^{v/2}$  (recall that the  $P_v^\pm$  are homogeneous). Finally, [DSP90, Lemma 3] uses these Fourier coefficient bounds to prove that the  $\sum_{v \geq 1}$  sums above are at most on the order of  $\delta r(Q, \mathbf{h}, n) + \delta^{-21/4-\varepsilon} n^{1/2-1/28+\varepsilon}$ . Taking  $\delta = n^{-1/175}$  and noting  $r(Q, \mathbf{h}, n) \ll_\varepsilon n^{1/2+\varepsilon}$  then yields (1).

We now elucidate how this applies to our problem, first setting some notation. Let  $r_\diamond(n)$  be the number of representations of  $n$  as the sum of three generalized pentagonal numbers and  $r_\diamond^+(n)$  be the number of representations of  $n$  as the sum of three positive pentagonal numbers. Let  $q(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2$  be the sum of three squares quadratic form and  $r_\square(n)$  be the sum of three squares function, counting the number of solutions to  $q(\mathbf{x}) = n$  over  $\mathbf{x} \in \mathbb{Z}^3$ .

Now let  $\tilde{q}(\mathbf{x}) = 36q(\mathbf{x})$ , which has level  $N = 144$ , and set  $\mathbf{h} = (-1/6, -1/6, -1/6)$ . Define  $\mathcal{E} = \{\mathbf{t} \in \mathbb{R}^3 : \tilde{q}(\mathbf{t}) = 1\}$  and  $\mathcal{F} = \{\mathbf{t} \in \mathcal{E} : t_i > 0\}$ , wherein we have  $\mu(\mathcal{F}) = 1/8$ . Then, since  $\sum_{i=1}^3 p_5(m_i) = n$  is equivalent to  $\tilde{q}(\mathbf{m} + \mathbf{h}) = 24n + 3$  for  $\mathbf{m} \in \mathbb{Z}^3$ , we have

$$r_\diamond(n) = r(\tilde{q}, \mathbf{h}, 24n + 3), \quad r_\diamond^+(n) = r(\tilde{q}, \mathcal{F}, \mathbf{h}, 24n + 3).$$

Our proof now proceeds directly from (1). In our setting, the “suitable”  $n$  are exactly those for which  $v_3(24n + 3)$  is bounded; we must treat the unbounded case separately. Using the formula  $r_\diamond(n) = \frac{1}{8}(r_\square(24n + 3) - \mathbf{1}_{n \equiv 1(3)} r_\square((8n + 1)/3))$  [Rob12, Theorem 3] and the Siegel mass formula [Sie35a], we prove that  $r_\diamond(n) \geq \frac{1}{12} r_\square(24n + 3)$  for all  $n$ . Siegel’s (ineffective) estimate [Sie35b] implies  $r_\square(24n + 3) \gg_\varepsilon n^{1/2-\varepsilon}$ , whence it follows from (1) that  $r_\diamond^+(n) \gg_\varepsilon n^{1/2-\varepsilon}$  for bounded  $v_3(24n + 3)$ .

We remark that one could alternatively demonstrate the existence of a primitive  $\mathbf{x} \in \mathbb{Z}^3$  with  $\sum_{i=1}^3 p_5(x_i) = n$  to use [DSP90, Theorem 3(ii)]; we have chosen to present the above argument instead since we suspect it to be more adaptable to a complete resolution of Conjecture 1.1 (see Subsection 2.2).

For unbounded  $v_3(24n + 3)$ , we require a different approach. For this regime, we present in Theorem 3.2 (a slight reformulation of) an explicit lift found by ChatGPT 5.6 Sol. In particular, noting that  $v_3(24n + 3) \geq 9$  is equivalent to  $n \equiv (3^8 - 1)/8 \pmod{3^8}$ , we find that a ternary *generalized* pentagonal number representation  $\mathbf{x} \in \mathbb{Z}^3$  of  $3^{-8}(n - (3^8 - 1)/8)$ , which we recall always exists [Guy94, pg.171], may be lifted by one of five affine mappings to give a ternary positive pentagonal number representation of  $n$ . The specific affine map employed depends on the positivity of the  $x_i$ .

**2.2. Tackling the full ternary pentagonal numbers conjecture.** It is plausible that one could prove Conjecture 1.1 using the methodology in [DSP90], provided that one assumes an explicit lower bound  $|L(1, \chi)| \gg n^{-\alpha}$  for some sufficiently small  $\alpha > 0$ , where  $\chi$  is the quadratic

character associated with  $\mathbb{Q}(\sqrt{-24n-3})$ . The mechanism we suggest is essentially to prove an explicit version of [DSP90, Theorem 1] for  $r_{\diamond}^+(n)$ , wherein we require the explicit lower bound on  $|L(1, \chi)|$  to get a sufficiently strong explicit lower bound on  $r_{\diamond}(n)$ .

For every  $v \geq 0$ , let  $P_v^-$  be a (symmetric) homogeneous harmonic polynomial of degree  $v$  as in [DSP90, Lemma 1], with  $\mathcal{F} = \{\mathbf{t} \in \mathcal{E} : t_i > 0\}$  as before. The primary hurdle of getting an explicit version of [DSP90, Theorem 1] is proving a bound of the form  $|c_v(n)| \leq A_{\beta,v} n^{v/2+1/4+\beta}$  for every  $v \geq 1$  and some  $0 < \beta < 1/4$ , where  $\sum_{v \geq 1} 24^{-v/2} A_{\beta,v} < \infty$  and

$$c_v(n) := [\omega^{24n+3}] \sum_{\mathbf{x} \in \mathbf{h} + \mathbb{Z}^3} P_v^-(\mathbf{x}) \omega^{\tilde{q}(\mathbf{x})} = [\omega^n] \sum_{\mathbf{x} \in \mathbb{Z}^3} P_v^-(\mathbf{x} - \mathbf{1}/6) \omega^{\sum_{i=1}^3 p_5(x_i)}, \quad (2)$$

where  $\omega = e^{2\pi iz}$  and  $[\omega^m]F(\omega)$  denotes the  $m^{\text{th}}$  Fourier coefficient of  $F(z)$ . Indeed, by the construction of the  $P_v^-$ , we have

$$r_{\diamond}^+(n) = r(\tilde{q}, \mathcal{F}, \mathbf{h}, 24n+3) = \sum_{\substack{\mathbf{x} \in \mathbf{h} + \mathbb{Z}^3 \\ \tilde{q}(\mathbf{x}) = 24n+3}} \mathbf{1}_{\mathcal{F}}(\mathbf{x}/\sqrt{24n+3}) \geq \sum_{v \geq 0} (24n+3)^{-v/2} c_v(n), \quad (3)$$

wherein the  $v = 0$  term will be the main term (which is  $r_{\diamond}(n)$  times some constant close to  $\mu(\mathcal{F}) = 1/8$ ) and the sum  $\sum_{v \geq 1}$  will be the error term, as described in the previous subsection. Of course, since the theta series on the left-hand side of (2) is a cusp form for  $\Gamma_1(144)$  of weight  $3/2 + v$  if  $v \geq 1$  [DSP90, pg.52], the Ramanujan–Petersson conjecture for cusp forms of half-integral weight predicts that  $c_v(n) \ll_{\varepsilon,v} n^{v/2+1/4+\varepsilon}$ , though a bound of this strength is not necessary, and in particular the implied constants twisted by  $24^{-v/2}$  are not necessarily summable in  $v$ .

Provided that  $\alpha + \beta < 1/4$ , one will see from (3) that  $r_{\diamond}^+(n) > 0$  for  $n \geq C$  for some explicit constant  $C$ , leaving the integers less than  $C$  to be checked by computer. Because we thus require  $C$  not to be unreasonably large, we suggest proving an explicit bound for each  $c_v(n)$  for a particular  $\{P_v^-\}_v$ , rather than proving a fully general explicit version of [DSP90, Theorem 1]. We note that we have written the theta series on the right-hand side of (2) in this manner due to its close connection to a family of quasimodular forms that Ramanujan studied in his last notebook [Ram88, pg.369], which have recently garnered significant attention; we refer the reader to [ABS25, AOS25, BP25, Mat25].

### 3. PROOF

We first set some standard notation. For a prime  $p$  and an integer  $m$ , let  $v_p(m)$  denote the  $p$ -adic valuation of  $m$ . Let  $\mathbb{Q}_p$  and  $\mathbb{Z}_p$  denote the  $p$ -adic numbers and  $p$ -adic integers, respectively. We use the Vinogradov notations  $\ll$  and  $\gg$  interchangeably with the big  $O$  and big  $\Omega$  notations, respectively. When  $\varepsilon$  appears in such an asymptotic notation, we mean that the asymptotic statement holds for all  $\varepsilon > 0$ , and write  $\varepsilon$  in the subscript of the asymptotic notation to indicate that the implied constant depends on  $\varepsilon$ . If the implied constant depends on other variables, we also include them in the subscript. We write  $M^T$  to denote the transpose of a matrix  $M$ .

**3.1. Handling Small  $v_3(24n+3)$ .** To approximate  $r_{\diamond}^+(n)$  by  $\frac{1}{8}r_{\diamond}(n)$  using [DSP90, Theorem 1], we require that  $24n+3$  has bounded  $p$ -adic valuation for each  $p \mid N$  such that  $\tilde{q}$  is isotropic over  $\mathbb{Q}_p$  and  $\mathbf{h} \notin \mathbb{Z}_p^3$ ; the only such prime is  $p = 3$ . With this noted, it remains to verify that  $r_{\diamond}(n) \gg_{\varepsilon} n^{1/2-\varepsilon}$ . It is the requisite use of this bound that makes Theorem 3.1 ineffective.

**Theorem 3.1.** *Every sufficiently large  $n$  with  $v_3(24n + 3) \leq 8$  is the sum of three positive pentagonal numbers.*

*Proof.* By [DSP90, Theorem 1], if  $v_3(24n + 3) \leq C$  for some fixed  $C > 0$ , then, after noting the routine upper bound  $r(Q, \mathbf{h}', n) \ll_\varepsilon n^{1/2+\varepsilon}$  for  $Q$  an arbitrary positive-definite ternary quadratic form, we have the estimate

$$r_\diamond^+(n) = \frac{1}{8}r_\diamond(n) + O_{\varepsilon,C}(n^{1/2-1/175+\varepsilon}). \quad (4)$$

We take  $C = 8$  in particular since the explicit lift given in Subsection 3.2 covers  $v_3(24n + 3) \geq 9$ .

By (4), it remains to show that  $r_\diamond(n) \gg_\varepsilon n^{1/2-\varepsilon}$ . We proceed in doing so by first invoking the formula  $r_\diamond(n) = \frac{1}{8}(r_\square(24n + 3) - r_\square((8n + 1)/3))$  recorded in [Rob12, Theorem 3], where  $r_\square((8n + 1)/3)$  is understood to be zero if  $n \not\equiv 1 \pmod{3}$ . Then, recalling that  $r_\square(n) = r(\text{gen } q, n)$  since  $q$  is the only form in its genus up to integral equivalence, we use the Siegel mass formula [Sie35a] to write  $r_\diamond(n)$  in terms of  $r_\square(24n + 3)$  times a factor that we verify to be bounded away from zero. Let

$$\alpha_p(q, n) := \lim_{k \rightarrow \infty} p^{-2k} \# \left\{ \mathbf{x} \in (\mathbb{Z}/p^k\mathbb{Z})^3 : q(\mathbf{x}) \equiv n \pmod{p^k} \right\}$$

be the usual  $p$ -adic density of the equation  $q(\mathbf{x}) = n$ . By appealing to Legendre's three-square theorem and the Siegel mass formula [Sie35a], since  $24n + 3$  is odd and  $24n + 3 \not\equiv 7 \pmod{8}$ , we have that  $r_\square(24n + 3) \neq 0$  and each  $\alpha_p(q, 24n + 3) \neq 0$ . Observe that  $\alpha_p(q, m^2n) = \alpha_p(q, n)$  for  $p \nmid m$ , in view of the bijection  $\mathbf{x} \mapsto m\mathbf{x}$  on  $(\mathbb{Z}/p^k\mathbb{Z})^3$ . So

$$\frac{r_\square((8n + 1)/3)}{r_\square(24n + 3)} = \frac{\mathbf{1}_{n \equiv 1(3)} \alpha_3(q, (8n + 1)/3) \sqrt{(8n + 1)/3}}{\alpha_3(q, 24n + 3) \sqrt{24n + 3}} = \frac{\mathbf{1}_{n \equiv 1(3)} \alpha_3(q, (8n + 1)/3)}{3\alpha_3(q, 24n + 3)}.$$

Thus, invoking  $r_\diamond(n) = \frac{1}{8}(r_\square(24n + 3) - r_\square((8n + 1)/3))$  from [Rob12, Theorem 3] yields

$$r_\diamond(n) = \frac{1}{8} \left( 1 - \frac{\mathbf{1}_{n \equiv 1(3)} \alpha_3(q, (8n + 1)/3)}{3\alpha_3(q, 24n + 3)} \right) r_\square(24n + 3). \quad (5)$$

In view of the formula  $\alpha_3(q, m) = 4/3 - 3^{-1-\lceil (v_3(m)-1)/2 \rceil} - (\frac{m_3}{3})^{v_3(m)+1} 3^{-\lfloor (v_3(m)+3)/2 \rfloor}$  (see [Jon21, Theorem 1.3], for example) with  $m_3 = 3^{-v_3(m)}m$ , we see that  $\alpha_3(q, m) \leq \alpha_3(q, 9m)$ . So

$$1 - \frac{\mathbf{1}_{n \equiv 1(3)} \alpha_3(q, (8n + 1)/3)}{3\alpha_3(q, 24n + 3)} \geq \frac{2}{3}. \quad (6)$$

We now conclude by remarking the ineffective (due to its reliance on Siegel's estimate [Sie35b]) bound  $r_\square(24n + 3) \gg_\varepsilon n^{1/2-\varepsilon}$ , which may be seen in a multitude of ways. Since 2 is the only prime  $p$  such that  $q$  is anisotropic over  $\mathbb{Q}_p$ , we have that  $24n + 3$  is in

$$R_1(\text{gen } q) := \{m \in \mathbb{N} : \alpha_p(q, m) \neq 0 \text{ for all primes } p, p \nmid m \text{ if } q \text{ is anisotropic over } \mathbb{Q}_p\}.$$

Thus, upon noting that  $r(\text{gen } q, n)$  and  $r(\text{gen}_4 q, n)$  coincide since we have no congruence conditions, by [DSP90, Lemma 5(i)], we have  $r_\square(24n + 3) = r(\text{gen } q, 24n + 3) \gg_\varepsilon n^{1/2-\varepsilon}$ .

Therefore, by (5) and (6),  $r_\diamond(n) \gg_\varepsilon n^{1/2-\varepsilon}$ , whence it follows from (4) that  $r_\diamond^+(n) \gg_\varepsilon n^{1/2-\varepsilon}$  for  $v_3(24n + 3) \leq 8$ .  $\square$

**3.2. Handling Large  $v_3(24n + 3)$ .** With the integers  $n$  with  $v_3(24n + 3) \leq 8$  handled in Subsection 3.1 using [DSP90], we now turn to the integers with  $v_3(24n + 3) \geq 9$ , presenting the following lift.

**Theorem 3.2.** *Let  $n$  be such that  $v_3(24n + 3) \geq 9$ , or equivalently, let  $n \equiv (3^8 - 1)/8 \pmod{3^8}$ . Let  $\mathbf{x} \in \mathbb{Z}^3$  be a solution to*

$$p_5(x_1) + p_5(x_2) + p_5(x_3) = 3^{-8} (n - (3^8 - 1)/8)$$

*and suppose without loss of generality that  $p_5(x_1) \geq p_5(x_2) \geq p_5(x_3)$ . Defining  $\delta = (\mathbf{1}_{x_1 > 0}, \mathbf{1}_{x_2 > 0}, \mathbf{1}_{x_3 > 0})$ , now*

- (i) *set  $\mathbf{y} = (76x_1 - 16x_2 - 23x_3 - 6, 28x_1 + 41x_2 + 64x_3 - 22, x_1 + 68x_2 - 44x_3 - 4)$  if  $\delta = (1, 1, 1)$  or  $\delta = (1, 1, 0)$ ,*
- (ii) *set  $\mathbf{y} = (76x_1 + 16x_2 + 23x_3 - 19, 28x_1 - 41x_2 - 64x_3 + 13, x_1 - 68x_2 + 44x_3 + 4)$  if  $\delta = (1, 0, 1)$  or  $\delta = (1, 0, 0)$ ,*
- (iii) *set  $\mathbf{y} = (-44x_1 + 40x_2 - 55x_3 + 10, -20x_1 + 55x_2 + 56x_3 - 15, -65x_1 - 44x_2 + 20x_3 + 15)$  if  $\delta = (0, 1, 1)$  or  $\delta = (0, 1, 0)$ ,*
- (iv) *set  $\mathbf{y} = (-17x_1 - 56x_2 + 56x_3 + 3, -56x_1 + 49x_2 + 32x_3 - 4, -56x_1 - 32x_2 - 49x_3 + 23)$  if  $\delta = (0, 0, 1)$ , and*
- (v) *set  $\mathbf{y} = (-44x_1 - 40x_2 + 55x_3 + 5, -20x_1 - 55x_2 - 56x_3 + 22, -65x_1 + 44x_2 - 20x_3 + 7)$  if  $\delta = (0, 0, 0)$ .*

*Then  $\mathbf{y} \in \mathbb{Z}_+^3$  and is a solution to  $p_5(y_1) + p_5(y_2) + p_5(y_3) = n$ .*

*Proof.* To see that each  $y_i$  is positive for all  $\delta$ , see Table 1. Toward showing that  $\sum_{i=1}^3 p_5(y_i) = n$ , define the matrices

$$M_1 = \begin{pmatrix} 76 & -16 & -23 \\ 28 & 41 & 64 \\ 1 & 68 & -44 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 76 & 16 & 23 \\ 28 & -41 & -64 \\ 1 & -68 & 44 \end{pmatrix}, \quad M_3 = \begin{pmatrix} -44 & 40 & -55 \\ -20 & 55 & 56 \\ -65 & -44 & 20 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} -17 & -56 & 56 \\ -56 & 49 & 32 \\ -56 & -32 & -49 \end{pmatrix}, \quad M_5 = \begin{pmatrix} -44 & -40 & 55 \\ -20 & -55 & -56 \\ -65 & 44 & -20 \end{pmatrix}. \quad (7)$$

We then note that each  $M_i$  satisfies  $M_i^T M_i = 3^8 I$ , and  $6\mathbf{y} - \mathbf{1} = M(6\mathbf{x} - \mathbf{1})$  for all  $\delta$ , where

$$M = \begin{cases} M_1 & \delta \in \{(1, 1, 1), (1, 1, 0)\}, \\ M_2 & \delta \in \{(1, 0, 1), (1, 0, 0)\}, \\ M_3 & \delta \in \{(0, 1, 1), (0, 1, 0)\}, \\ M_4 & \delta = (0, 0, 1), \\ M_5 & \delta = (0, 0, 0). \end{cases} \quad (8)$$

Therefore

$$\begin{aligned} 24 \sum_{i=1}^3 p_5(y_i) + 3 &= \sum_{i=1}^3 (6y_i - 1)^2 = \|M(6\mathbf{x} - \mathbf{1})\|^2 = (6\mathbf{x} - \mathbf{1})^T M^T M (6\mathbf{x} - \mathbf{1}) \\ &= (6\mathbf{x} - \mathbf{1})^T 3^8 I (6\mathbf{x} - \mathbf{1}) = 3^8 \sum_{i=1}^3 (6x_i - 1)^2 = 3^8 \left( 24 \sum_{i=1}^3 p_5(x_i) + 3 \right). \end{aligned}$$

So  $\sum_{i=1}^3 p_5(y_i) = 3^8 \sum_{i=1}^3 p_5(x_i) + (3^8 - 1)/8 = n$ .  $\square$

TABLE 1. Lower bounds on each  $y_i$  in terms of  $x_1$  for every  $\delta$ , and the inequality chains of the  $x_i$  used to obtain them.

| $\delta$    | Inequalities on $x_i$             | Inequalities on $y_i$                                                      |
|-------------|-----------------------------------|----------------------------------------------------------------------------|
| $(1, 1, 1)$ | $x_1 \geq x_2 \geq x_3$           | $y_1 \geq 37x_1 - 6$<br>$y_2 \geq 28x_1 + 83$<br>$y_3 \geq x_1 + 20$       |
| $(1, 1, 0)$ | $x_1 \geq x_2 \geq -x_3 + 1$      | $y_1 \geq 60x_1 - 6$<br>$y_2 \geq 5x_1 + 42$<br>$y_3 \geq x_1 + 64$        |
| $(1, 0, 1)$ | $x_1 \geq -x_2 + 1 \geq x_3 + 1$  | $y_1 \geq 60x_1 + 20$<br>$y_2 \geq 5x_1 + 36$<br>$y_3 \geq x_1 + 116$      |
| $(1, 0, 0)$ | $x_1 \geq -x_2 + 1 \geq -x_3 + 1$ | $y_1 \geq 37x_1 + 20$<br>$y_2 \geq 28x_1 + 13$<br>$y_3 \geq x_1 + 4$       |
| $(0, 1, 1)$ | $-x_1 \geq x_2 \geq x_3$          | $y_1 \geq -29x_1 + 10$<br>$y_2 \geq -20x_1 + 96$<br>$y_3 \geq -21x_1 + 35$ |
| $(0, 1, 0)$ | $-x_1 \geq x_2 \geq -x_3 + 1$     | $y_1 \geq -44x_1 + 50$<br>$y_2 \geq -19x_1 + 41$<br>$y_3 \geq -x_1 + 35$   |
| $(0, 0, 1)$ | $-x_1 \geq -x_2 \geq x_3$         | $y_1 \geq -17x_1 + 115$<br>$y_2 \geq -7x_1 + 28$<br>$y_3 \geq -39x_1 + 23$ |
| $(0, 0, 0)$ | $-x_1 \geq -x_2 \geq -x_3$        | $y_1 \geq -29x_1 + 5$<br>$y_2 \geq -20x_1 + 22$<br>$y_3 \geq -21x_1 + 7$   |

## AI USE

Having proven that every positive integer  $n$  with bounded  $v_3(24n + 3)$  is the sum of three positive pentagonal numbers, we asked ChatGPT 5.6 Sol to prove that every integer congruent to  $(9^{\lceil k/2 \rceil} - 1)/8 \pmod{3^k}$  (which is equivalent to  $v_3(8n + 1) \geq k$ ) is the sum of three positive pentagonal numbers for  $k$  sufficiently large. ChatGPT initiated a search for, and succeeded in finding, the five matrices  $M_1, \dots, M_5$  given in (7) satisfying the properties  $M_i^T M_i = 3^8 I$  and  $6\mathbf{y} - \mathbf{1} = M(6\mathbf{x} - \mathbf{1})$ , where  $M$  is as in (8).

The proof of Theorem 3.2 is based on the argument provided by ChatGPT, though we have reformulated the statement and proof. All of the writing in the document is the author's own, and we take full responsibility for its mathematical correctness.

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