

Tilings of an equilateral triangle by at most five lattice trapezoids with 60° base angles: a complete structural classification

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Abstract

We study tilings of an equilateral triangle of side n in the triangular grid by k lattice trapezoids with base angles 60° , the objects behind the OEIS sequences A389392 ($k = 4$) and A391498 ($k = 5$). We prove two angle identities valid for every such tiling and a lemma relating the number of boundary vertices to the number of pieces having a side on the boundary. With these tools we show that there are exactly 1, 2 and 13 combinatorial types of tilings for $k = 3, 4, 5$. For $k = 3$ every tiling is a pinwheel. For $k = 4$ every tiling belongs to one of the two categories used in A389392, a fact that had previously been taken for granted. For $k = 5$ the thirteen types refine the eight categories of A391498; in three of them a piece has no side on the boundary. As a consistency check, the volumes of the parameter polytopes together with the generic multiplicities reproduce the leading coefficient $7/36$ of the conjectured quasi-polynomial for A391498. All lemmas were checked against an exhaustive enumeration of the tilings with pairwise distinct pieces for $n \leq 15$. Finally, the classification turns the thirteen types into eight explicit families of sets of shapes, and an inclusion–exclusion over them reduces the conjectured generating function of A391498 to twelve elementary counting statements, and we settle all twelve: the resulting closed formula reproduces the sequence for every $n \leq 127$.

1 Setting and main result

Throughout, $n \geq 1$ is an integer and we use oblique coordinates: (x, y) denotes the point $x \mathbf{u} + y \mathbf{v}$, where $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let

$$T = T_n = \{(x, y) : x \geq 0, y \geq 0, x + y \leq n\},$$

with corners $A = (0, 0)$, $B = (n, 0)$ and $C = (0, n)$. *Lattice lines* are the lines $x = c$, $y = c$ and $x + y = c$, i.e. the lines parallel to a side of T ; two of them meet at angles of 60° and 120° . A *lattice segment* is a segment lying on a lattice line.

For integers $1 \leq s \leq b - 1$ we write $\{b \times s\}$ for the trapezoid with sides on lattice lines, long base b , short base $b - s$ and legs of length s . Its angles are 60° at the ends of the long base and 120° at the ends of the short base. A convex polygon with sides on lattice lines has all its angles in $\{60^\circ, 120^\circ\}$, and if it has m_{60} corners of the first kind and m_{120} of the second, then $2m_{60} + m_{120} = 6$. Hence it is an equilateral triangle, a quadrilateral with two angles of each kind (a trapezoid if its two 60° corners are adjacent, a parallelogram otherwise), a pentagon or a hexagon.

A k -tiling of T is a decomposition $T = Q_1 \cup \dots \cup Q_k$ into such trapezoids (the *pieces*) with pairwise disjoint interiors. The sequences A389392 and A391498 [1, 2] count, for $k = 4$ and $k = 5$ respectively, the sets of k pairwise different shapes $\{b \times s\}$ that occur as the shapes of the pieces of some k -tiling of T_n . Distinctness plays no role in the structural results below; it only matters for the counting.

The graph of a tiling. Let V consist of A , B , C and all corners of pieces. Every side of a piece is subdivided by the points of V that it contains; the resulting segments are the *edges* of a plane graph G with vertex set V , whose bounded faces are the pieces. A vertex is a *boundary vertex* if it lies on ∂T and is not a corner of T , and an *interior vertex* if it does not lie on ∂T . An edge is a *boundary edge* if it is contained in ∂T , and an *interior edge* otherwise.

Combinatorial types. The *combinatorial type* of a tiling is the plane graph G together with the lattice direction of each of its edges, up to the six symmetries of T . Thus two tilings have the same type if and only if, after a symmetry of T is applied to one of them, they differ only in the lengths of their edges. The angle of a piece at one of its corners is the angle between the two edges there; hence G , together with the angles of all pieces at all their corners, determines the directions of all edges as soon as the direction of one boundary edge is fixed, and therefore determines the combinatorial type.

Theorem 1.1. *For $k = 3, 4$ and 5 there are exactly 1, 2 and 13 combinatorial types of k -tilings of equilateral triangles. They are described in Theorems 6.1, 6.2, 7.1 and 7.2 and in Corollary 7.3.*

Sections 2–6 develop the tools and settle $k = 3$ and $k = 4$; they close the gap in the treatment of A389392, where the exhaustiveness of the two categories was assumed (Remark 6.3). Section 7 deals with $k = 5$, Section 8 compares the thirteen types with the eight categories of A391498, Section 9 describes the computer verification, Section 10 discusses the counting problem and Section 11 turns the classification into a closed formula for A391498.

2 Local structure and two angle identities

For a vertex v of G let $a(v)$ (resp. $c(v)$) be the number of pieces having a corner at v with angle 60° (resp. 120°), and let $h(v)$ be the number of pieces containing v in the relative interior of one of their sides. Since every piece has two corners of each kind,

$$\sum_v a(v) = \sum_v c(v) = 2k. \quad (1)$$

Lemma 2.1 (local structure). (a) *At a corner of T we have $(a, c, h) = (1, 0, 0)$.*

(b) *At a boundary vertex we have $h = 0$ and $(a, c) \in \{(1, 1), (3, 0)\}$; we say that the vertex is of type B_1 or B_3 . A B_1 vertex is incident with exactly one interior edge; a B_3 vertex is incident with exactly two, which form angles of 60° with the side of T and with each other.*

(c) *At an interior vertex we have $h \leq 1$ and $a + 2c + 3h = 6$. Hence (a, c, h) is one of $(1, 1, 1)$, $(3, 0, 1)$, $(0, 3, 0)$, $(2, 2, 0)$, $(4, 1, 0)$, $(6, 0, 0)$, and we say that the vertex is of type T , T' , Y , X , W , Z , respectively. The degrees of these vertices are 3, 4, 3, 4, 5, 6.*

At a vertex Q of type T two of the three edges are collinear; the straight segment they form is the line of Q , and the third edge is the stem of Q . The piece containing Q in the relative interior of a side lies on the side of the line opposite to the stem. At a vertex of type Y any two of the three edges form an angle of 120° .

Proof. The pieces containing v cover a neighbourhood of v in T . Each of them contributes an angle of 60° or 120° (if v is one of its corners) or of 180° (if v lies in the relative interior of one of its sides), and these contributions add up to 60° , 180° or 360° in cases (a), (b), (c), respectively. In (a) an angle of 60° cannot be split. In (b), a piece containing v in the relative interior of a side has this side on ∂T , because the piece lies in T ; it then covers a half-disc around v , so no piece has a corner at v , and v would not be a vertex. In (c), two such pieces would have their sides on a common line and lie on opposite sides of it, and again no piece would have a corner at

v . The remaining assertions are arithmetic: an interior vertex has degree $a + c + h$, the number of pieces around it, and a boundary vertex has degree $a + c + 1$; at a vertex of type T the piece with angle 180° determines the line, and the stem separates the two pieces with angles 60° and 120° . $\square$

We write $B_1, B_3, t, t', y, x, w, z$ for the numbers of vertices of the respective types, and $B = B_1 + B_3$ for the number of boundary vertices.

Lemma 2.2 (angle identities). *For every k -tiling of T ,*

$$y = 1 + B_3 + t' + w + 2z, \quad (2)$$

$$B_1 + 3B_3 + t + 3t' + 2x + 4w + 6z = 2k - 3. \quad (3)$$

In particular, every tiling has a vertex of type Y.

Proof. By Lemma 2.1, $a(v) - c(v)$ equals 1 at the corners of T ; 0 at vertices of types B_1, T, X ; 3 at vertices of types B_3, T', W ; 6 at vertices of type Z ; and -3 at vertices of type Y . By (1) the sum of all these numbers is 0, which is (2). Next we add up all angles at all vertices in two ways. Piece by piece, each piece contributes 360° at its corners and 180° for every vertex lying in the relative interior of one of its sides, so the total is $360k + 180(t + t')$ degrees. Vertex by vertex, the total is $3 \cdot 60 + 180B + 360(t + t' + y + x + w + z)$ degrees. Hence

$$2k = 1 + B_1 + B_3 + t + t' + 2(y + x + w + z),$$

and eliminating y by means of (2) gives (3). $\square$

3 Dissections of convex polygons, full strips and chords

Lemma 3.1. *If a convex polygon P is the union of two convex polygons P_1, P_2 with disjoint interiors, then there is a line ℓ such that P_1 and P_2 are the intersections of P with the two closed half-planes bounded by ℓ .*

Proof. By the separation theorem there is a line ℓ with $P_i \subseteq H_i$, where H_1, H_2 are the closed half-planes bounded by ℓ . A point of P in the interior of H_1 does not lie in P_2 , hence lies in P_1 ; taking closures, $P \cap H_1 \subseteq P_1 \subseteq P \cap H_1$. The same argument applies to P_2 . $\square$

Lemma 3.2 (lattice chords). *Let P be a convex polygon with sides on lattice lines, and let σ be a lattice segment joining two points of ∂P and meeting the interior of P .*

- (a) *If P is an equilateral triangle, then σ is parallel to a side and cuts off an equilateral triangle.*
- (b) *If $P = \{b \times s\}$, then exactly one of the following holds: (i) σ is parallel to the bases and joins the two legs, and both parts are trapezoids; (ii) σ is parallel to a leg λ and joins the two bases, the part adjacent to λ is a parallelogram and the other part is a trapezoid; (iii) σ cuts off an equilateral triangle at one of the two 60° corners.*
- (c) *If P is a parallelogram, then either σ is parallel to two sides of P and both parts are parallelograms, or σ lies in the third lattice direction and either joins two opposite sides (both parts are then trapezoids) or cuts off an equilateral triangle at a 60° corner.*

Proof. (a) The endpoints of σ lie on two different sides, so σ is parallel to the third side. (b) After a symmetry of the lattice, $P = \{0 \leq y \leq s, x \geq 0, x + y \leq b\}$. A lattice segment through an interior point lies on a line $y = c, x = c$ or $x + y = c$. A segment on $y = c$ joins the legs, as in (i). A segment on $x = c$ joins $y = 0$ to the short base $y = s$ if $c \leq b - s$, which is case (ii) with the parallelogram $\{0 \leq x \leq c\}$; otherwise it joins $y = 0$ to the leg $x + y = b$ and cuts

off a triangle at $(b, 0)$. Segments on $x + y = c$ are treated symmetrically. (c) After a symmetry, $P = \{0 \leq x \leq p, 0 \leq y \leq q\}$ with 60° corners at $(0, 0)$ and (p, q) ; segments on $x = c$ or $y = c$ give two parallelograms, and a segment on $x + y = c$ joins opposite sides if $\min(p, q) \leq c \leq \max(p, q)$ and cuts off a triangle at $(0, 0)$ or at (p, q) otherwise. $\square$

Corollary 3.3. (a) *No equilateral triangle is tiled by fewer than three trapezoids.*

(b) *If a trapezoid is tiled by two trapezoids, they are separated by a segment of type (i); we call such a segment a parallel cut.*

(c) *If a parallelogram is tiled by two trapezoids, they are separated by a segment in the third lattice direction joining two opposite sides.*

Proof. A trapezoid is not a triangle. By Lemma 3.1, a tiling of a convex polygon by two convex pieces is given by a single segment σ as in Lemma 3.2. In case (a) one of the parts is a triangle; in cases (b) and (c) the listed alternatives are the only ones in which both parts are trapezoids. $\square$

A *full strip* is a piece having 60° corners at two corners of T .

Lemma 3.4. *A full strip with corners at A and B is $\{n \times s\}$ with long base AB and legs on AC and BC ; its complement in T is the equilateral triangle of side $n - s$ at C , which is tiled by the other pieces. A tiling has at most one full strip.*

Proof. The long base of a trapezoid joins its two 60° corners, and its legs leave these corners along the sides of T . Two full strips would both contain a neighbourhood in T of a common corner of T . $\square$

A *chord* of a tiling is a segment contained in the union of the boundaries of the pieces, joining two points of ∂T , and not contained in ∂T .

Lemma 3.5 (no chords). *Every chord is parallel to a side of T and cuts off a corner triangle T_1 ; every piece lies in T_1 or in the trapezoid $T \setminus T_1$. If $k \leq 5$ and the tiling has no full strip, then it has no chord.*

Proof. The only lattice directions pointing from a corner of T into T run along the sides of T . Hence a chord ends at non-corner points of two different sides of T , so it is parallel to the third side and cuts off a corner triangle T_1 . The chord does not meet the interior of any piece, so every piece lies on one side of it. By Corollary 3.3(a), T_1 contains at least three pieces, so the trapezoid $T \setminus T_1 = \{n \times s'\}$ contains at most two. If it consists of one piece, this piece is a full strip. If it consists of two pieces, they are separated by a parallel cut (Corollary 3.3(b)), and the piece containing the long base of $T \setminus T_1$, which is a whole side of T , is a full strip. $\square$

4 Boundary pieces and frames

Lemma 4.1 (boundary pieces). *Assume that the tiling has neither a full strip nor a chord. Then:*

- (a) *the corners A, B, C lie in three different pieces, the corner pieces; each of them has exactly two sides on ∂T , namely its long base and one leg, lying on the two sides of T through its corner;*
- (b) *every other piece has at most one side on ∂T ;*
- (c) *no side of a piece lying on ∂T contains a boundary vertex in its relative interior;*
- (d) *every side of T contains a boundary vertex; in particular $B \geq 3$;*

(e) if m denotes the number of non-corner pieces having a side on ∂T , then

$$m = B - 3 \quad \text{and} \quad f := k - 3 - m \geq 0, \quad (4)$$

where f is the number of pieces without a side on ∂T .

Proof. (a) A piece containing two corners of T has 60° corners there (Lemma 2.1(a)), so it would be a full strip. The two sides of the corner piece Q_A through A are its long base and a leg, and they lie on AB and AC . The short base of Q_A is parallel to its long base and does not pass through A , so it does not lie on ∂T . If, say, the long base is $AD \subset AB$, the other leg starts at D in the direction of BC , and it lies on BC only if $D = B$, i.e. only if Q_A is a full strip. The case of a long base on AC is symmetric.

(b) A convex polygon has at most one side on a given line, so two sides of a piece lying on ∂T lie on different sides of T . If these two sides of the piece share an endpoint, this endpoint is a corner of T . Otherwise they are opposite sides of the piece; they are not its bases, which are parallel, so they are its legs. But then its short base joins two points on different sides of T and is a chord.

(c) This is Lemma 2.1(b).

(d) The side of Q_A on AB has length less than n (a long base of length n would make Q_A a full strip), so its endpoint other than A is a boundary vertex on AB .

(e) The boundary edges of G form a cycle with $3 + B$ edges, and each of them lies in exactly one piece. By (a)–(c), the corner pieces contain exactly six of them and every other piece contains at most one. $\square$

Lemma 4.2 (frames). *Assume that the tiling has neither a full strip nor a chord, and that $m = 0$. Then $B_3 = 0$ and $B_1 = 3$, and after a reflection of T the corner pieces are*

$$\begin{aligned} Q_A &= \{x \geq 0, 0 \leq y \leq s_A, x + y \leq n - s_B\}, \\ Q_B &= \{y \geq 0, x \geq s_C, n - s_B \leq x + y \leq n\}, \\ Q_C &= \{0 \leq x \leq s_C, y \geq s_A, x + y \leq n\}, \end{aligned}$$

where $s_A, s_B, s_C \geq 1$ are their legs; their long bases are $n - s_B$, $n - s_C$ and $n - s_A$. Put $S = s_A + s_B + s_C$. The remaining pieces tile the triangle $\{x \geq s_C, y \geq s_A, x + y \leq n - s_B\}$ of side $n - S$ if $S < n$, the inverted triangle $\{x \leq s_C, y \leq s_A, x + y \geq n - s_B\}$ of side $S - n$ if $S > n$, and there are no remaining pieces if $S = n$.

Proof. By (4) we have $B = 3$, so by Lemma 4.1(d) every side of T contains exactly one boundary vertex and is the union of the two sides of corner pieces lying on it. Suppose that the boundary vertex on AB is of type B_3 . Then Q_A and Q_B both have 60° corners there, so both have their long bases on AB . Since every corner piece has exactly one long base and one leg on ∂T , one of the sides AC, BC then consists of two legs, whose endpoints meet at the boundary vertex of that side with angles $120^\circ + 120^\circ > 180^\circ$, which is absurd. Hence the three boundary vertices are of type B_1 , and every side of T contains exactly one long base. There are only two ways of assigning bijectively to each corner piece a side of T through its corner, namely $(Q_A, Q_B, Q_C) \mapsto (AB, BC, CA)$ and $(Q_A, Q_B, Q_C) \mapsto (AC, BA, CB)$, and they are exchanged by a reflection. In the first case the corner pieces are as displayed, with their long bases b_A, b_B, b_C in place of $n - s_B, n - s_C, n - s_A$; the condition that each side of T is covered exactly once reads $b_A + s_B = b_B + s_C = b_C + s_A = n$.

Finally, a point $(x, y) \in T$ with $y > s_A$ lies in Q_C if and only if $x \leq s_C$, and otherwise it lies in Q_B if and only if $x + y \geq n - s_B$; a point with $y < s_A$ lies in Q_A if and only if $x + y \leq n - s_B$, and otherwise it lies in Q_B if and only if $x \geq s_C$. Hence the points not covered by the corner pieces are, up to boundaries, those with $x > s_C, y > s_A, x + y < n - s_B$ (a non-empty set if and only if $S < n$) and those with $x < s_C, y < s_A, x + y > n - s_B$ (non-empty if and only if $S > n$). $\square$

Corollary 4.3. *Assume that the tiling has neither a full strip nor a chord. If $k = 3$, then $m = 0$ and $S = n$. If $k \in \{4, 5\}$, then $m \geq 1$.*

Proof. By (4), $m \leq k - 3$. If $m = 0$, the $k - 3$ non-corner pieces tile the triangle of Lemma 4.2. For $k = 3$ this triangle is empty, so $S = n$. For $k \in \{4, 5\}$ it would be a triangle tiled by one or two trapezoids, contradicting Corollary 3.3(a). $\square$

5 Pinwheel skeletons and interior edges

Let P be a vertex of type Y. The three edge directions at P contain no pair of opposite directions, so each edge at P is contained in a maximal straight segment of G having P as an endpoint; we call these three segments the *arms* of P .

Lemma 5.1 (pinwheel skeleton). *Suppose that the three arms of a Y vertex P end at points D_1, D_2, D_3 of ∂T . Then the D_i are boundary vertices lying on the three different sides of T , and the arms divide T into three trapezoids R_1, R_2, R_3 , each containing one corner of T . Each arm is a leg of one of the two regions adjacent to it and the short base of the other.*

We call R_1, R_2, R_3 a *pinwheel skeleton*. For a region R_i , its two arms are its *leg-arm* and its *base-arm* (the latter contains its short base), and the leg of R_i lying on ∂T is its *boundary leg*.

Proof. An arm cannot end at a corner of T , since it would reach the corner along a side of T ; and two arms meet only at P . Let R be the region bounded by two arms PD_i, PD_j and by an arc of ∂T from D_i to D_j containing r corners of T . Then R is a convex polygon with angle 120° at P , angles $\alpha_i, \alpha_j \in \{60^\circ, 120^\circ\}$ at D_i and D_j , and angle 60° at each of the r corners, so that $120 + \alpha_i + \alpha_j + 60r = 180(r + 1)$, i.e. $\alpha_i + \alpha_j = 60 + 120r$. This forces $r = 1$ and $\{\alpha_i, \alpha_j\} = \{60^\circ, 120^\circ\}$; thus R is a quadrilateral whose two 60° corners are adjacent, i.e. a trapezoid. If $\alpha_i = 60^\circ$, the long base of R is the segment of ∂T between D_i and the corner, PD_i is a leg of R and PD_j is its short base. Finally, the two regions adjacent to an arm PD_i have angles adding up to 180° at D_i , so exactly one of them has an angle of 60° there. $\square$

For a vertex v let $\iota(v)$ be the number of interior edges at v ; thus $\iota(v) = 1$ or 2 if v is of type B_1 or B_3 , and $\iota(v) = \deg v$ if v is interior. Let H be the graph formed by the interior vertices and the edges joining two of them, and let e_H be its number of edges.

Lemma 5.2 (interior edges). *Assume that the tiling has no chord. Then*

$$\sum_{v \text{ interior}} \iota(v) - 2e_H = B_1 + 2B_3. \quad (5)$$

Moreover:

- (a) *two vertices are joined by at most one edge, and no vertex of type Y lies on a triangle of edges;*
- (b) *if two edges at a vertex Q of type T end at boundary vertices, then one of them is the stem of Q and the other one lies on the line of Q ; no vertex of type T has three such edges.*

Proof. An interior edge joining two boundary vertices would be a chord, so every interior edge has one or two interior endpoints, and counting the endpoints of interior edges gives (5). (a) Edges are straight segments. Three edges forming a triangle lie on lattice lines, so the triangle is equilateral and its angles are 60° , whereas two edges at a Y vertex form an angle of 120° . (b) Otherwise the line of Q would be a chord. $\square$

6 Three and four pieces

Theorem 6.1 (three pieces). *Every 3-tiling of T is a pinwheel: up to a reflection of T there are positive integers s_A, s_B, s_C with $s_A + s_B + s_C = n$ such that, with*

$$D = (n - s_B, 0), \quad E = (s_C, n - s_C), \quad F = (0, s_A), \quad P = (s_C, s_A),$$

the pieces are

$$ADPF = \{n - s_B \times s_A\}, \quad BEPD = \{n - s_C \times s_B\}, \quad CFPE = \{n - s_A \times s_C\},$$

with long bases AD, BE, CF . Each of the segments PD, PE, PF is a leg of one piece and the short base of another. Conversely, every such triple defines a 3-tiling.

Proof. A full strip would leave a triangle tiled by two trapezoids (Lemma 3.4), which is impossible by Corollary 3.3(a). Hence there is no full strip and, by Lemma 3.5, no chord. By Corollary 4.3, $m = 0$ and $S = n$, so the pieces are the three corner pieces of Lemma 4.2; using $n - s_B - s_A = s_C$ one checks that their corners are the listed points. The statement about PD, PE, PF is read off directly (or from Lemma 5.1), and the converse is immediate. $\square$

Theorem 6.2 (four pieces). *Every 4-tiling of T is of exactly one of the following two forms:*

- (A) *a full strip $\{n \times s\}$ together with a pinwheel of the triangle of side $n - s$;*
- (B) *a pinwheel in which one of the three pieces is divided into two by a parallel cut.*

Proof. If there is a full strip, the remaining triangle carries a 3-tiling, which is a pinwheel by Theorem 6.1; this is (A).

Assume that there is no full strip. Then there is no chord (Lemma 3.5), and Corollary 4.3 and (4) give $m = 1$, $f = 0$ and $B = 4$. Substituting $B_1 = 4 - B_3$ into (3) yields $2B_3 + t + 3t' + 2x + 4w + 6z = 1$. Hence $B_3 = 0$, $B_1 = 4$, $t = 1$, all the other counts vanish, and $y = 1$ by (2). Let P be the Y vertex and Q the T vertex. By (5), $3 + 3 - 2e_H = 4$, so $e_H = 1$: the segment PQ is an edge, and each of P and Q has two further edges, which end at boundary vertices. By Lemma 5.2(b), the edge QP lies on the line of Q , which continues beyond Q to a boundary vertex, and the stem of Q ends at a boundary vertex. Thus all three arms of P end on ∂T , and Lemma 5.1 provides a pinwheel skeleton. The stem of Q is the only interior edge not contained in an arm; it lies in one region, which it divides into two pieces, while the other two regions are single pieces. By Corollary 3.3(b), the stem is a parallel cut of its region; this is (B).

The two forms are disjoint, because only in (A) does a piece have long base n . $\square$

Remark 6.3. Theorem 6.2 is exactly the exhaustiveness statement that the constructive description of Categories A and B in OEIS A389392 takes for granted, and Theorem 6.1 is a complete proof of the pinwheel lemma. Only Sections 2–6 are needed for this purpose.

7 Five pieces

Theorem 7.1 (five pieces with a full strip). *If a 5-tiling of T contains a full strip $\{n \times s\}$, then this is its only full strip, and the other four pieces form a tiling of the triangle of side $n - s$ of form (A) or (B).*

Proof. This follows from Lemma 3.4 and Theorem 6.2. $\square$

Theorem 7.2 (five pieces without full strip). *Up to a symmetry of T , every 5-tiling of T without full strip is of exactly one of the following seven forms. In the first five forms the tiling has a pinwheel skeleton R_1, R_2, R_3 (Lemma 5.1), and the regions not mentioned are single pieces.*

- (D) Two regions are each divided into two pieces by a parallel cut.
- (S) One region is divided into three pieces by two parallel cuts.
- (V) One region R is divided into three pieces by two disjoint segments which start on its base-arm and end on its long base, each of them parallel to the leg of R farther from it; the middle piece is a trapezoid whose long base lies on the base-arm.
- (C_∂) One region R is divided by a segment γ , which starts on the base-arm of R and is parallel to its boundary leg λ , into a trapezoid, which is a piece, and a parallelogram Π adjacent to λ ; and Π is divided into two pieces by a segment in the third lattice direction joining γ to λ .
- (C_o) As in (C_∂), but with λ the leg-arm of R instead of its boundary leg.
- (F) Some piece F does not meet ∂T . Its corners are a Y vertex P and three T vertices V_1, V_2, V_3 (counterclockwise), with angles at P, V_1, V_2, V_3 equal to $120^\circ, 60^\circ, 60^\circ, 120^\circ$. The sides PV_1, V_1V_2 and V_2V_3 of F are prolonged beyond V_1, V_2 and V_3 , respectively, up to ∂T , and the third edge at P ends on ∂T .
- (Δ) Some piece M meets ∂T in a single point D^* , a vertex of type B_3 . The other corners of M are a T vertex Q and two Y vertices P_j, P_i ; the long base of M is D^*Q , which is the stem of Q , and its short base is P_jP_i . The side QP_j lies on the line of Q , whose other half ends on ∂T , and the third edges at P_i and P_j end on ∂T .

Each of these seven forms is a single combinatorial type.

Proof. Step 0: profiles. By Lemma 3.5 there is no chord, and by Corollary 4.3 and (4) we have $m \in \{1, 2\}$, $B = 3 + m$ and $f = 2 - m$. Substituting $B_1 = 3 + m - B_3$ into (3) gives

$$2B_3 + t + 3t' + 2x + 4w + 6z = 4 - m. \quad (6)$$

Together with (2) this leaves exactly the following seven possibilities (all counts not mentioned are 0):

$m = 2$			$m = 1$		
(i)	$B_1 = 5, t = 2$	$y = 1$	(iv)	$B_1 = 4, t = 3$	$y = 1$
(ii)	$B_1 = 5, x = 1$	$y = 1$	(v)	$B_1 = 4, t' = 1$	$y = 2$
(iii)	$B_1 = 4, B_3 = 1$	$y = 2$	(vi)	$B_1 = 4, t = x = 1$	$y = 1$
			(vii)	$B_1 = 3, B_3 = 1, t = 1$	$y = 2$

In the left-hand side of (5), interior vertices of types T, Y, T', X contribute 3, 3, 4, 4.

Step 1: cases (v) and (vi) are impossible. In both cases there are three interior vertices, one of them of type Y, and (5) reads $10 - 2e_H = 4$. Thus $e_H = 3$, and the three interior vertices form a triangle through a Y vertex, contrary to Lemma 5.2(a).

Step 2: case (ii) is impossible. Let P and X be the interior vertices. Now $7 - 2e_H = 5$, so PX is an edge, and the three other edges at X end at boundary vertices. At a vertex of type X the four angles are, in cyclic order, $60^\circ, 120^\circ, 60^\circ, 120^\circ$ or $60^\circ, 60^\circ, 120^\circ, 120^\circ$. In the first case the four edges lie on two lines through X ; in the second they lie on one line together with two rays on opposite sides of it. A line through X not containing PX would be a chord. Hence we are in the second case, and PX lies on the line, which is therefore an arm of P ending on ∂T . The other two arms of P are single edges ending on ∂T , so we have a pinwheel skeleton, and the two rays at X divide the two regions adjacent to this arm into two pieces each. By Corollary 3.3(b) both rays are parallel cuts. Since a parallel cut joins the two legs of its region, the arm is a leg of both adjacent regions, contrary to Lemma 5.1.

Step 3: case (iii) is impossible. The interior vertices are two Y vertices P_1, P_2 , and there is one vertex D^* of type B_3 . Now $6 - 2e_H = 4 + 2$, so $e_H = 0$, and the two interior edges at D^* end at interior vertices, one at P_1 and one at P_2 (two non-collinear segments issuing from D^* cannot end at the same point). Let M be the piece having a 60° corner at D^* between these two edges. The corners of M adjacent to D^* are P_1 and P_2 , where M has angles of 120° . But in a trapezoid each 60° corner is adjacent to the other 60° corner.

Step 4: case (i) leads to (D), (S), (V) or (C_∂) . Let P be the Y vertex and Q_1, Q_2 the T vertices. Now $9 - 2e_H = 5$, so $e_H = 2$ and H is a path.

(4a) P is the middle vertex of H . Each Q_i has two edges ending at boundary vertices, so by Lemma 5.2(b) the edge Q_iP lies on the line of Q_i , which continues up to ∂T , and the stem σ_i of Q_i ends on ∂T . Thus all arms of P end on ∂T and we have a pinwheel skeleton, with Q_1 and Q_2 on different arms. The only interior edges not contained in arms are σ_1 and σ_2 . If they lie in different regions, each of these two regions is divided into two pieces, the third region is a piece, and Corollary 3.3(b) gives (D). Suppose that they lie in the same region R ; then they start on the two different arms of R . Each σ_i is a lattice segment joining two points of ∂R , so it is of one of the types (i)–(iii) of Lemma 3.2(b). Type (iii) is excluded, because the triangle cut off would be a union of at most two pieces (Corollary 3.3(a)). A segment starting at an interior point of the leg-arm cannot be of type (ii), and a segment starting at an interior point of the base-arm cannot be of type (i). Hence one of σ_1, σ_2 joins the two legs of R and the other joins its two bases, so they cross; this is impossible, since there are no further interior vertices.

(4b) A T vertex, say Q_1 , is the middle vertex of H . Since Q_2 has two edges ending at boundary vertices, Q_1Q_2 lies on the line of Q_2 (Lemma 5.2(b)). Let γ be the segment from Q_1 through Q_2 to ∂T , and let σ_2 be the stem of Q_2 . The vertex Q_1 has one more edge, which ends at a boundary vertex D_1 . If the line of Q_1 consisted of Q_1D_1 and Q_1Q_2 , then D_1Q_1 together with γ would be a chord. Hence the line of Q_1 contains Q_1P , and there are two possibilities.

The line of Q_1 is $PQ_1 \cup Q_1D_1$. Then Q_1Q_2 is the stem of Q_1 . All arms of P end on ∂T ; let R be the region containing γ . All interior edges not contained in arms lie in R , so R consists of three pieces: the part of R on one side of γ , and the two pieces into which σ_2 divides the part R' on the other side. As in (4a), γ is of type (i) or (ii). If it were of type (i), R' would be a trapezoid with bases parallel to γ , and by Corollary 3.3(b) σ_2 would be parallel to γ , which is absurd because σ_2 starts on γ . So γ is of type (ii): Q_1 lies on the base-arm of R , and γ is parallel to a leg λ of R and divides R into a parallelogram Π adjacent to λ and a trapezoid Z . A parallelogram is not a piece, so $R' = \Pi$ and Z is a piece. By Corollary 3.3(c), σ_2 lies in the third lattice direction and joins γ to the opposite side λ of Π . Since σ_2 ends on ∂T , λ is the boundary leg of R : this is (C_∂) .

The line of Q_1 is $PQ_1 \cup Q_1Q_2$. Then the segment from P through Q_1 and Q_2 to ∂T is an arm of P , from which the stems $\sigma_1 = Q_1D_1$ and σ_2 branch off. The other two arms of P are single edges ending on ∂T , so we have a pinwheel skeleton. If σ_1 and σ_2 lay in the two different regions adjacent to this arm, each of them would divide its region into two pieces, and by Corollary 3.3(b) the arm would be a leg of both regions, contrary to Lemma 5.1. So they lie in one region R , which they divide into three pieces, and they are disjoint. If the arm is the leg-arm of R , both are parallel cuts (types (ii) and (iii) being excluded as in (4a)): this is (S). If the arm is the base-arm of R , both are of type (ii). After a symmetry of the lattice we may write $R = \{0 \leq y \leq s, x \geq 0, x + y \leq b\}$ with the base-arm on the line $y = s$; then each of the two segments lies on a line $x = p$ or on a line $x + y = q$. If the segment nearer to the leg $\{x = 0\}$ lay on a line $x = p$, the piece between them would be the parallelogram $R \cap \{x \leq p\}$; similarly for the leg $\{x + y = b\}$. Hence the two segments lie on lines $x + y = q$ and $x = p$ with $q < p$, and the middle piece $R \cap \{x \leq p, x + y \geq q\}$ is a trapezoid whose long base lies on the line $y = s$: this is (V).

Step 5: case (iv) leads to (F) or $(C_\circ)$. Let P be the Y vertex and Q_1, Q_2, Q_3 the T vertices. Now $12 - 2e_H = 4$, so $e_H = 4$, and the graph H , which has four vertices and four edges, contains

exactly one cycle. A triangle through P is excluded by Lemma 5.2(a). If the cycle were the triangle $Q_1Q_2Q_3$, then P , which would have two edges ending on ∂T , would lie outside it; the open triangle would contain no vertex and no edge, and it would be a triangular piece. Hence the cycle is a 4-cycle through all interior vertices. It bounds a region containing no vertex and no edge (such an edge would be a fifth edge of H), i.e. a piece F with corners P, Q_1, Q_2, Q_3 ; as F has four corners, none of these points lies in the relative interior of a side of F . The piece F does not meet ∂T , and each Q_i has exactly one edge ending on ∂T . Since Q_i is a corner of F , the two sides of F at Q_i are the stem of Q_i and one half of its line, and the edge ending on ∂T is the other half of the line. So at each Q_i exactly one of the two sides of F through Q_i is prolonged beyond Q_i up to ∂T .

Label the corners of F counterclockwise $V_0 = P, V_1, V_2, V_3$, put $e_j = V_jV_{j+1}$ (indices modulo 4), and let θ_j be the angle of F at V_j , so that $\theta_0 = 120^\circ$. No side of F is prolonged at both ends, since this would give a chord, and no side is prolonged at P . Hence the sides prolonged at V_1, V_2, V_3 are, respectively,

$$(a) e_0, e_1, e_2; \quad (b) e_0, e_1, e_3; \quad (c) e_0, e_2, e_3; \quad (d) e_1, e_2, e_3.$$

A reflection of T reverses the cyclic order and exchanges (a) with (d) and (b) with (c), so it suffices to treat (a) and (c). Let D be the endpoint on ∂T of the third edge at P .

Pattern (c). The sides e_0 and e_3 are prolonged at V_1 and V_3 , so all arms of P end on ∂T , and F lies in the region R bounded by the arms containing e_0 and e_3 . The prolongation of e_2 at V_2 yields a segment γ from V_3 through V_2 to ∂T , and the stem V_2V_1 of V_2 joins γ to the other arm of R . All interior edges not contained in arms lie in R , so R consists of three pieces: the part of R on the side of γ not containing P , and the two pieces into which V_1V_2 divides the part R' containing P . Exactly as in Step 4, γ is of type (ii), V_3 lies on the base-arm, γ is parallel to a leg λ of R , and R' is the parallelogram Π adjacent to λ . Since $P \in R'$ is the corner of R where the base-arm meets the leg-arm, λ is the leg-arm. By Corollary 3.3(c), V_1V_2 lies in the third lattice direction and joins γ to λ : this is (C_o) , and F is the piece of Π containing P .

Pattern (a). Now e_3 is the stem of V_3 , so the arm of P along e_3 ends at V_3 . Let D_1, D_2, D_3 be the endpoints on ∂T of the prolongations of e_0, e_1, e_2 . The interior edges are the four sides of F and the four segments $PD, V_1D_1, V_2D_2, V_3D_3$, which join F to ∂T and divide $T \setminus F$ into four pieces:

- O_b , bounded by PD , an arc of ∂T from D to D_1 , and the straight segment D_1V_1P (which contains V_1 in its interior);
- O_c , bounded by V_1D_1 , an arc from D_1 to D_2 , and the segment $D_2V_2V_1$;
- O_d , bounded by V_2D_2 , an arc from D_2 to D_3 , and the segment $D_3V_3V_2$;
- O_a , bounded by V_3D_3 , an arc from D_3 to D , and the segments DP and PV_3 .

Each of them is a quadrilateral whose corners, apart from corners of T , are P, D, D_1 for O_b ; V_1, D_1, D_2 for O_c ; V_2, D_2, D_3 for O_d ; and V_3, D_3, D, P for O_a . Hence the four arcs contain 1, 1, 1, 0 corners of T , respectively. The angle of O_c at V_1 is $180^\circ - \theta_1$. In O_c the corner K of T is a 60° corner whose neighbours are D_1 and D_2 , so the corner opposite to K , which is V_1 , has angle 120° ; hence $\theta_1 = 60^\circ$. In the same way $\theta_2 = 60^\circ$, and therefore $\theta_3 = 120^\circ$. In O_a the corner V_3 has angle 60° and its neighbours are D_3 and P (angle 120°), so O_a has angle 60° at D_3 and 120° at D . All boundary vertices are of type B_1 , so the two pieces meeting at each of D, D_1, D_2, D_3 have angles adding up to 180° there. Consequently O_b has angle 60° at D and 120° at D_1 ; O_c has 60° at D_1 and 120° at D_2 ; and O_d has 60° at D_2 and 120° at D_3 , in accordance with O_a . All angles are determined, and this is (F).

Step 6: case (vii) leads to (Δ) . Let P_1, P_2 be the Y vertices, Q the T vertex and D^* the vertex of type B_3 . Now $9 - 2e_H = 3 + 2$, so $e_H = 2$. The two interior edges at D^* end at two different

interior vertices. Let M be the piece with a 60° corner at D^* . By the argument of Step 3, these two vertices are not both Y vertices, so they are Q and some P_i , and M has angle 120° at its corner P_i . Consequently, if X denotes the corner of M following D^* along D^*Q , then M has angle 60° at X , and the fourth corner Y of M , adjacent to X and to P_i , has angle 120° .

Suppose first that D^*Q lies on the line of Q . If M contains Q in the relative interior of a side, then X is the vertex following Q on this line. It is not on ∂T , since the line would then be a chord, so it is a Y vertex, where M has angle 120° , a contradiction. Otherwise $X = Q$, and the side QY of M is the stem of Q . The vertex Y is not on ∂T : it is not a corner of T , only one interior edge arrives at a boundary vertex other than D^* , and the sides QY and YP_i of M are not contained in ∂T . So Y is an interior vertex, with angle 120° in M and different from Q and from P_i (otherwise M would be a triangle); thus Y is the other Y vertex P_j , and H consists of the edges QP_j and P_jP_i . Then the third edge at Q ends on ∂T , and together with D^*Q it forms the line of Q , which is a chord. This is impossible.

Hence D^*Q is the stem of Q . Then M lies on the side of the stem, $X = Q$, and the side QY of M lies on the line of Q . As before, $Y = P_j$ with $j \neq i$, and H consists of QP_j and P_jP_i . So $M = D^*QP_jP_i$ has long base D^*Q and short base P_jP_i . The other half of the line of Q and the third edges at P_j and at P_i end at three boundary vertices D' , D''' , D'' , all of type B_1 (the two interior edges at D^* are already known). These three segments and the sides of M are all the interior edges, and they divide $T \setminus M$ into four pieces: X_1 , bounded by D^*Q , QD' and an arc from D' to D^* ; X_H , bounded by the straight segment $D'QP_j$ (which contains Q in its interior), by P_jD''' and by an arc from D''' to D' ; X_2 , bounded by P_jD''' , an arc from D''' to D'' , $D''P_i$ and P_iP_j ; and X_3 , bounded by P_iD'' , an arc from D'' to D^* , and D^*P_i . Counting corners as in Step 5, these four arcs contain 1, 1, 0, 1 corners of T . By Lemma 2.1(b), X_1 and X_3 have angle 60° at D^* . The corners of X_1 are Q, D', K, D^* in cyclic order, with K a corner of T ; its two 60° corners are K and D^* , so X_1 has angle 120° at D' . Hence X_H has angle 60° at D' ; its corners are D', P_j, D''', K' , its 60° corners are D' and K' , and so it has angle 120° at D''' . Then X_2 , with corners D''', P_j, P_i, D'' , has angle 60° at D''' and hence also at D'' ; and X_3 has angle 120° at D'' , in accordance with its corners D'', P_i, D^*, K'' having angles $120^\circ, 120^\circ, 60^\circ, 60^\circ$. All angles are determined, and this is (Δ) .

Conclusion. The forms (D), (S), (V) and (C_∂) have profile (i), the forms (F) and $(C_\circ)$ have profile (iv), and (Δ) has profile (vii). Within profile (i), P is the middle vertex of H only in (D); the three interior vertices are collinear in (S) and (V) but not in (C_∂) ; and (S) differs from (V) in whether the arm containing them is the leg-arm or the base-arm of the divided region. Within profile (iv), all arms of P end on ∂T in $(C_\circ)$ but not in (F). So the seven forms are pairwise different. In each case the arguments above determine G and all angles up to a symmetry of T : for the first five forms the pinwheel skeleton is unique up to symmetry (Theorem 6.1), and the admissible choices of the divided regions are permuted transitively by the rotations of T . Hence each form is a single combinatorial type. Finally, each form occurs (Figure 1). $\square$

Corollary 7.3. *Up to symmetry there are exactly thirteen combinatorial types of 5-tilings. Six of them contain a full strip: the strip together with a 4-tiling of form (A) whose strip is parallel to it (one type) or lies on one of the other two sides of the remaining triangle (two types, distinguished by the chirality of the pinwheel), and the strip together with a 4-tiling of form (B) in which any one of the three pinwheel pieces is cut (three types). The other seven are the types of Theorem 7.2.*

Proof. By Theorem 7.1, a tiling with a full strip on AB is determined, up to edge lengths, by the form of the 4-tiling of the triangle T' at C and by its position; the reflection of T fixing AB acts on these data, and different orbits give different types because the full strip is unique. For (A), the second strip lies on the side of T' parallel to AB or on one of its other two sides, and the pinwheel has one of two chiralities; the reflection exchanges the two chiralities and the two non-parallel sides, which leaves $1 + 2 = 3$ orbits. For (B), the pinwheel has two chiralities and

the cut piece is one of three; the reflection acts freely on these six configurations, which leaves three orbits. $\square$

8 The thirteen types and the categories of A391498

Table 1 lists the thirteen types with the labels T1–T13 used in our programs (numbered in order of first appearance in the enumeration), their profiles, the smallest n for which each type occurs with pairwise distinct pieces, the generic multiplicity μ discussed in Section 10, and the category of A391498 containing the type. The last column was obtained for $n = 12$ by comparing, set by set, the output of Rodrigo’s program for the eight categories with our classification: each category consists exactly of the sets realized by the types listed (a set belongs to several categories when it is realized by tilings of different types).

Type	Form	(B_1, B_3, t, y)	first n	μ	Category
T1	full strip + (A), parallel strips	$(6, 0, 1, 1)$	6	9	AA1
T4, T5	full strip + (A), non-parallel strips	$(4, 0, 3, 1)$	6	9	AA1
T6, T9	full strip + (B)	$(5, 0, 2, 1)$	7	3	AA2
T11	full strip + (B)	$(4, 0, 3, 1)$	7	3	AA2
T2	(D)	$(5, 0, 2, 1)$	6	1	DD1
T8	(S)	$(5, 0, 2, 1)$	7	1	CC3
T13	(V)	$(5, 0, 2, 1)$	10	6	CC1
T7	(C_∂)	$(5, 0, 2, 1)$	7	4	CC2
T12	$(C_\circ)$	$(4, 0, 3, 1)$	7	4	CC2
T3	(F)	$(4, 0, 3, 1)$	6	1	BB2
T10	(Δ)	$(3, 1, 1, 2)$	7	1	BB1

Table 1: The thirteen types of 5-tilings. In all of them $t' = x = w = z = 0$. The column μ gives the number of tilings, up to symmetry, realizing a generic set of the type: for T1, T4, T5 the nine tilings are three of each type; for T6, T9, T11 one of each; for T7, T12 two of each; for T13 all six are of type T13.

Remark 8.1. (1) The forms (V), (C_∂) and $(C_\circ)$ arise in the same way: a region R of the skeleton is divided by a segment of type (ii), parallel to a leg λ , into a trapezoid and a parallelogram Π , and Π is divided in the third lattice direction. In (V) the second cut joins the two sides of Π lying on the bases of R (for either choice of λ); in (C_∂) and $(C_\circ)$ it joins the other two sides of Π , and λ is the boundary leg or the leg-arm, respectively. This agrees with CC1 consisting of (V) and CC2 of (C_∂) and $(C_\circ)$.

(2) A 5-tiling arises from a 4-tiling by one of the two operations underlying A389392 (adding a full strip, or dividing a piece by a parallel cut) if and only if its type is T1, T2, T4, T5, T6, T8, T9 or T11. Indeed, adding a full strip gives the six types with a full strip; and if two pieces form a trapezoid, merging them gives a tiling of form (A) or (B) by Theorem 6.2, and dividing one piece of such a tiling by a parallel cut gives T1 (cutting the strip of (A)), a full strip together with (B) (cutting a pinwheel piece of (A)), (D) (cutting an uncut piece of (B)) or (S) (cutting one half of the cut piece of (B)). The remaining five types T3, T7, T10, T12, T13 arise in neither way. In agreement with this, our program found no pair of pieces forming a trapezoid in any of the $2556 + 936 + 390 + 936 + 432$ tilings of these types for $n = 13$, whereas every tiling of types T1, T2, T6, T8, T9, T11 contains such a pair. This is why exhaustiveness for $k = 5$ cannot be proved by a simple induction on k and requires the global argument of Theorem 7.2.

(3) In (F) and $(C_\circ)$ one piece has its four corners in the interior of T , and in (Δ) one piece meets ∂T in a single point; (F) and (Δ) have no pinwheel skeleton at all.

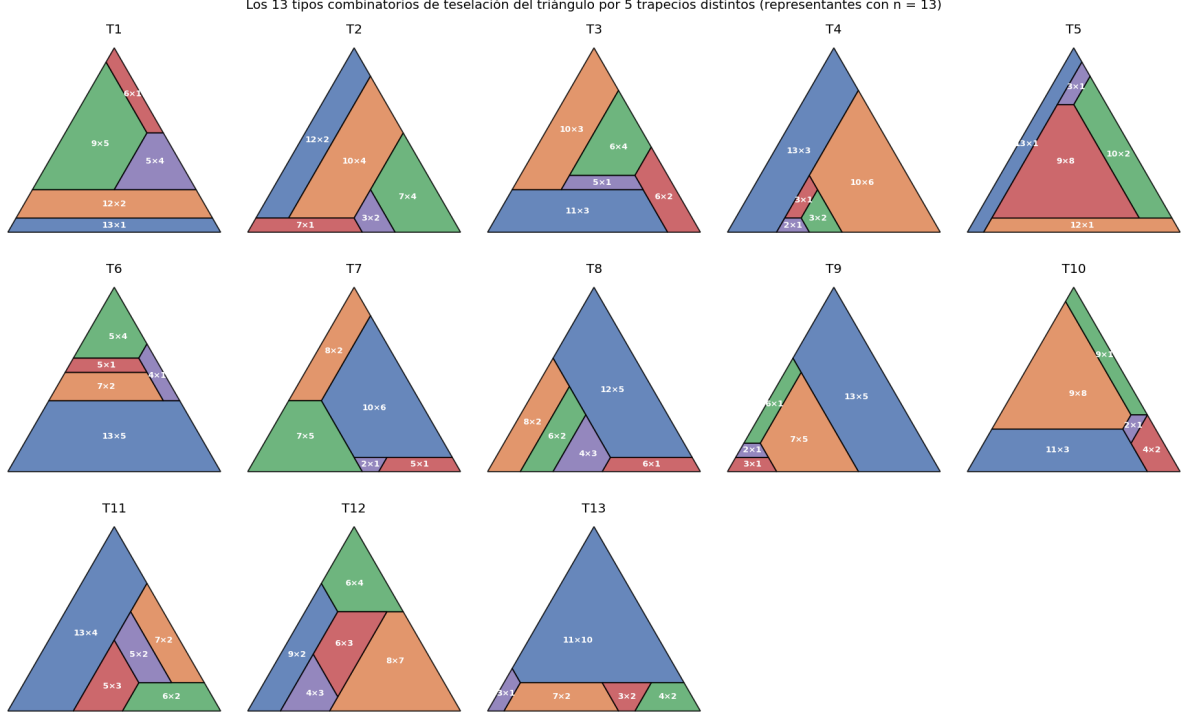


Figure 1: Representatives of the thirteen types for $n = 13$; the pieces are labelled $b \times s$.

9 Computational verification

All statements were checked against independent computations; the scripts (Python 3 and C) accompany these notes.

- **brute.py**, **brute.c**: exact-cover enumeration of all k -tilings of T_n by lattice trapezoids with pairwise distinct shapes, in any of the six orientations, using no structural information. For $k = 3$ it reproduces $\lfloor (n-1)(n-2)/6 \rfloor$, for $k = 4$ the terms of A389392 for $n \leq 12$, and for $k = 5$ the terms of A391498 for $n \leq 22$.
- **cats.c**: a port to C of Rodrigo's program for the eight categories. Its output agrees set by set with the exhaustive enumeration for $n \leq 14$, it reproduces the 47 terms of the b-file, and it extends A391498 to $n = 127$.
- **classify.py**: canonical codes of combinatorial types (the graph G with edge directions, minimized over the six symmetries of T) and multiplicities.
- **verify_structure.py**: for every tiling produced by the enumeration it builds G , computes (a, c, h) at every vertex and checks Lemma 2.1, the identities (2)–(3), and, when there is no full strip, Lemma 4.1(a), (b), (e), Corollary 4.3 and the absence of chords (Lemma 3.5). It was run on all 1110 tilings with $k = 3$ and $n \leq 16$, all 15630 tilings with $k = 4$ and $n \leq 15$, and all 162990 tilings with $k = 5$ and $n \leq 15$. No violation occurred; exactly 1, 2 and 13 types were found, and the profile of every type is the one predicted by Theorems 6.2 and 7.2.
- **signatures.py**: for $n = 13$, the list of types (with multiplicities) realizing each set, and the test of Remark 8.1(2).

The enumeration only covers tilings with pairwise distinct pieces; the theorems do not need this assumption.

10 Outlook: from types to the counting formula

For every type, the tilings of T_n of that type are parametrized by the positions of the maximal segments of G on their lattice lines, subject to linear equations and strict linear inequalities. Hence, for each type τ , the number $N_\tau(n)$ of tilings of T_n of type τ , counted up to symmetry, is a quasi-polynomial in n of degree at most 4. For instance, the tilings of form (D) with a fixed chirality and fixed divided regions correspond to 5-tuples of positive integers with sum n (the three legs of the pinwheel, two of which are split into two parts by the cuts), so there are $\binom{n-1}{4}$ of them before the distinctness condition is imposed.

Proposition 10.1 (structural coincidences). *Let $R = \{0 \leq y \leq s, x \geq 0, x + y \leq b\}$ be the divided region, with boundary leg on $\{x = 0\}$ and leg-arm on $\{x + y = b\}$.*

- (a) *In (C_∂) , with $\gamma = \{x = p\}$ and the cut of Π on $\{x + y = c\}$ ($1 \leq p \leq b - s - 1, p < c < s$), the three pieces in R are $\{b - p \times s\}$, $\{c \times p\}$ and $\{s - c + p \times p\}$. In $(C_\circ)$, with $\gamma = \{x + y = q\}$ and the cut on $\{x = c'\}$ ($s + 1 \leq q \leq b - 1, b - s < c' < q$), they are $\{q \times s\}$, $\{b - c' \times b - q\}$ and $\{c' - q + s \times b - q\}$. The maps $(p, c) \mapsto (p, s + p - c)$ and $(p, c) \mapsto (q, c') = (b - p, b - c)$ are bijections preserving the set of shapes. Hence every set of type (C_∂) is realized by at least two tilings of type (C_∂) and two of type $(C_\circ)$.*
- (b) *In (V) the three pieces in R are $\{\beta_1 \times s\}$, $\{\beta_2 \times s\}$, $\{\beta_3 \times s\}$ with $\beta_1 + \beta_2 + \beta_3 = b + s$ and $\beta_i \geq s + 1$, and every ordering of $(\beta_1, \beta_2, \beta_3)$ occurs. Hence a set of type (V) with distinct β_i is realized by six tilings.*
- (c) *A set of type “full strip + (A)” is realized by nine tilings (three positions of the second strip and six placements of the pinwheel, modulo the reflection fixing the full strip), and a set of type “full strip + (B)” by three (six placements of the cut pinwheel, modulo the same reflection).*

Proof. (a) and (b) are direct computations of the pieces; the constraints are preserved by the maps (for instance $p < c < s$ if and only if $p < s + p - c < s$, and $b - s < b - c < b - p$ if and only if $p < c < s$). In (b), $q = \beta_1$ and $p = \beta_1 + \beta_2 - s$ recover the segments. (c) The pinwheel pieces depend only on its legs, so any of the six images of the pinwheel under the symmetries of its triangle gives the same shapes. $\square$

The data for $n = 13$ confirm that these are the only coincidences on the generic part: the 1940 sets split as 428 + 6 sets realized by one tiling of each of T6, T9, T11 (respectively two), 159 sets realized by three tilings of each of T1, T4, T5, 78 sets realized by two tilings of each of T7 and T12, 12 sets realized by six tilings of T13, 378, 363, 354 and 65 sets realized by a single tiling of type T3, T8, T2 and T10, and 97 sets realized by several tilings of types T2, T3, T8 through non-generic coincidences.

The leading coefficient. The parameter polytopes have the following normalized volumes, i.e. $N_\tau(n) = v_\tau n^4 + O(n^3)$: $v_\tau = \frac{1}{24}$ for T1, T2, T3, T4, T5, T6, T8, T9, T11 (for T3, in a fixed position, the tilings correspond to integers with $1 \leq v_1 < v_2, w \geq 1$ and $w + v_2 < u < n$, where $y = v_1, x = w, y = v_2$ and $x + y = u$ are the lines of four of its five interior maximal segments, the fifth being determined by them); $v_\tau = \frac{1}{48}$ for T7, T12 and T13 (for instance, for T13 the number of choices in a region with legs s_A and short base s_C is about $\frac{1}{2}(s_C - s_A)_+^2$, and $\iint_{a+c \leq 1} \frac{1}{2}(c - a)_+^2 da dc = \frac{1}{48}$); and $v_\tau = 0$ for T10, whose tilings depend on only three parameters besides n . Coincidences other than those of Proposition 10.1, as well as repeated shapes, are governed by additional linear equations and only affect lower-order terms. Dividing by the multiplicities of Table 1 gives

$$\frac{3 \cdot \frac{1}{24}}{9} + \frac{3 \cdot \frac{1}{24}}{3} + 3 \cdot \frac{1}{24} + \frac{2 \cdot \frac{1}{48}}{4} + \frac{\frac{1}{48}}{6} = \frac{4 + 12 + 36 + 3 + 1}{288} = \frac{7}{36},$$

which is exactly the leading coefficient $70/360$ of the conjectured quasi-polynomial $a(n) = (70n^4 - 695n^3 + \dots)/360$ for A391498. Turning this heuristic into a proof, and obtaining the lower-order terms, requires counting the tilings of each type with pairwise distinct pieces (by inclusion–exclusion over the equations “two pieces have the same shape”) and controlling the non-generic coincidences; this is the next step towards a proof of the conjectured generating function.

11 The counting programme

By Theorems 7.1 and 7.2 the shapes of the five pieces of a 5-tiling are given by explicit formulas in the parameters of its type. Writing them out, the thirteen types produce only eight different families of 5-element sets of shapes: types T1, T4, T5 share their formula, so do T6, T9, T11, and so do T7 and T12 by Proposition 10.1(a). For a pinwheel of a triangle of side m with legs (u_1, u_2, u_3) , $u_1 + u_2 + u_3 = m$, write

$$\mathcal{P}(m; u) = \{\{m - u_2 \times u_1\}, \{m - u_3 \times u_2\}, \{m - u_1 \times u_3\}\},$$

and for a region $\{b_i \times u_i\}$ of $\mathcal{P}$ note that $b_i - u_i = u_{i+2}$ (indices modulo 3).

Family	Types	Parameters and shapes
A	T1, T4, T5	$a, c \geq 1$, $m = n - a - c$, u a composition of m : $\{n \times a\}, \{n - a \times c\}, \mathcal{P}(m; u)$
B	T6, T9, T11	$a \geq 1$, u a composition of $m = n - a$, i , $1 \leq h < u_i$: $\{n \times a\}, \mathcal{P} \setminus \{P_i\}, \{b_i \times h\}, \{b_i - h \times u_i - h\}$
D	T2	u a composition of n , $i < j$, $1 \leq h_i < u_i$, $1 \leq h_j < u_j$: P_k and the two splits of P_i, P_j
S	T8	u, i , $h_1 + h_2 + h_3 = u_i$: $P_j, P_k, \{b_i \times h_1\}, \{b_i - h_1 \times h_2\}, \{b_i - h_1 - h_2 \times h_3\}$
C	T7, T12	u, i , $1 \leq p \leq u_{i+2} - 1$, $p < c < u_i$: $P_j, P_k, \{b_i - p \times u_i\}, \{c \times p\}, \{u_i - c + p \times p\}$
V	T13	u, i , $\beta_1 \leq \beta_2 \leq \beta_3$ with $\beta_r \geq u_i + 1$ and $\sum \beta_r = b_i + u_i$: $P_j, P_k, \{\beta_r \times u_i\}$
F	T3	$1 \leq v_1 < v_2$, $w \geq 1$, $w + v_2 < u \leq n - 1$, $p = u - v_2$: $\{u \times v_1\}, \{n - v_1 \times w\}, \{n - p \times n - u\}, \{n - w - v_2 \times p - w\}, \{u - v_1 - w \times v_2 - v_1\}$
Δ	T10	$1 \leq \mu < m < \beta \leq n - \mu - 1$: $\{m \times \mu\}, \{\beta \times m\}, \{n - \beta \times \mu\}, \{n - m \times \beta - m + \mu\}, \{n - \beta + m - 2\mu \times n - \beta - \mu\}$

Thus $a(n)$, the n -th term of A391498, is the number of elements of $A \cup B \cup D \cup S \cup C \cup V \cup F \cup \Delta$ whose five shapes are pairwise different. Generating all these sets for $n \leq 24$ reproduces the terms of A391498 exactly; this is a verification of the classification that uses no geometry at all.

Two experimental facts organize the count (both checked for all $n \leq 63$, resp. $n \leq 40$):

- (i) *Disjointness.* The only families that meet are D, S and F. Writing E_{DS} , E_{DF} , E_{SF} , E_{DSF} for the numbers of sets lying in exactly those families,

$$a(n) = \sum_{\mathcal{F}} |\mathcal{F}| - E_{DS} - E_{DF} - E_{SF} - 2E_{DSF}. \quad (7)$$

- (ii) *Fibres.* The number of parameter tuples with five different shapes that produce a given set is 3 for A, V, 6 for C and 1 for F and Δ ; for B, D, S it is 3 except for finitely many accidental coincidences per n (the quotient tuples/sets is 3.003 for $n = 40$).

The fibres of A, V, B, D, S come from the cyclic rotation of (u, i) , and the extra factor 2 for C from the involution of Proposition 10.1(a).

Lemma 11.1. *For every n ,*

$$|A| = \frac{1}{3} \left[\binom{n-1}{4} - \sum_{j \geq 1} (n-3j-1)_+ \right], \quad \text{i.e.} \quad \sum_n |A| x^n = \frac{x^6}{(1-x)^5 (1+x+x^2)}.$$

Proof. In a set of the family A the shape with long base n is unique, since $n-a < n$ and the bases of $\mathcal{P}(m; u)$ are at most $m-1 < n-a$; so a is determined by the set, and then so are c and $m = n-a-c$, and the three remaining shapes are those of $\mathcal{P}(m; u)$. Two shapes of the set coincide only if $u_1 = u_2 = u_3$: indeed $\{m-u_2 \times u_1\} = \{m-u_3 \times u_2\}$ forces $u_2 = u_3$ and $u_1 = u_2$, and similarly for the other two pairs. Conversely, the multiset $\mathcal{P}(m; u)$ determines (u_1, u_2, u_3) up to a cyclic rotation: the multiset of legs is $\{u_1, u_2, u_3\}$, and if the u_r are pairwise different, the two cyclic classes of a multiset give different multisets of pairs, because the shape with leg u_1 is $\{m-u_2 \times u_1\}$ in one and $\{m-u_3 \times u_1\}$ in the other; if two of the u_r are equal there is only one cyclic class. Hence the tuples (a, c, u) with five different shapes fall into orbits of exactly three elements under the rotation of u , and $|A|$ is a third of their number. Finally, the number of tuples is $\binom{n-1}{4}$ (compositions of n into five positive parts), of which those with $u_1 = u_2 = u_3$, i.e. with $m \equiv 0 \pmod{3}$, number $\sum_{j \geq 1} (n-3j-1)_+$. The generating function follows from $\sum_n \binom{n-1}{4} x^n = x^5/(1-x)^5$ and $\sum_n \sum_j (n-3j-1)_+ x^n = x^5/((1-x^3)(1-x)^2)$. $\square$

Lemma 11.2 (family Δ). *Write $(x)_+ = \max(x, 0)$. For every n ,*

$$|\Delta| = \sum_{a \geq 1} \binom{n-1-2a}{2} - \binom{\lfloor (n-1)/2 \rfloor}{2} - \sum_{\mu \geq 1} (n-3\mu-1)_+ - [3 \mid n] \left(\frac{n}{3} - 1 \right) + \left\lfloor \frac{n-1}{4} \right\rfloor.$$

Proof. Let $P_1, \dots, P_5$ be the five shapes in the order of the table. Putting $\mu = a$, $m = \mu + b$, $\beta = m + c$ and $d = n - \mu - 1 - \beta$ identifies the parameter set with $\{(a, b, c, d) : a, b, c \geq 1, d \geq 0, 2a + b + c + d = n-1\}$, which has $\sum_{a \geq 1} \binom{n-1-2a}{2}$ elements. Comparing the shapes pairwise, the only possible coincidences are

$$\begin{aligned} P_1 = P_3 &\iff \beta = n - m, & P_1 = P_5 &\iff \beta = n - 2\mu, \\ P_2 = P_4 &\iff \beta = n - m, \ n = 3m - \mu, & P_2 = P_5 &\iff \beta = n - m - \mu, \ n = 3m, \\ P_3 = P_5 &\iff m = 2\mu, \ \beta = n - 2\mu, & P_4 = P_5 &\iff \beta = 2m - 2\mu, \ n = 3m - 2\mu, \end{aligned}$$

whereas $P_1 = P_2$, $P_1 = P_4$, $P_2 = P_3$ and $P_3 = P_4$ contradict $\mu < m < \beta$. The loci of $P_2 = P_4$ and of $P_4 = P_5$ are contained in the locus $\beta = n - m$, and that of $P_3 = P_5$ in the locus $\beta = n - 2\mu$; so the union of the six is the union of $\{\beta = n - m\}$, $\{\beta = n - 2\mu\}$ and $\{\beta = n - m - \mu, \ n = 3m\}$. On the first one the constraints reduce to $1 \leq \mu < m < n/2$, which gives $\binom{\lfloor (n-1)/2 \rfloor}{2}$ points; on the second to $\mu \geq 1, \mu < m < n - 2\mu$, which gives $\sum_{\mu} (n-3\mu-1)_+$; the third is non-empty only if $3 \mid n$, and then $m = n/3$ and $1 \leq \mu < n/3$. The first two meet exactly when $m = 2\mu < n/2$, in $\lfloor (n-1)/4 \rfloor$ points, and the other intersections are empty. As the fibre is 1, $|\Delta|$ is the number of triples whose five shapes are different. $\square$

Lemma 11.3 (family C). Write $s = u_i$, $j = u_{i+1}$, $k = u_{i+2}$, so that the five shapes are $\{s+j \times j\}$, $\{j+k \times k\}$, $\{s+k-p \times s\}$, $\{c \times p\}$ and $\{s+p-c \times p\}$, with $1 \leq p \leq k-1$ and $p < c < s$. Two of them coincide exactly in the four cases

$$E_1 : 2c = s + p, \quad E_2 : s = j = k, \quad E_3 : j = s, p = k - s, \quad E_4 : k = s, p = s - j,$$

of which the last three are pairwise disjoint, and

$$6|C| = 3(T - E_1 - E_2 - E_3 - E_4 + I_{12} + I_{13} + I_{14})$$

with, all sums being over the compositions $n = s + j + k$,

$$\begin{aligned} T &= \#\{(\alpha, \gamma, \delta, \varepsilon, \eta) \geq 0 : 2\alpha + \gamma + \delta + \varepsilon + \eta = n - 6\}, \\ E_1 &= \sum \#\{p : 1 \leq p \leq \min(k-1, s-2), p \equiv s \pmod{2}\}, \quad E_2 = [3 \mid n] \binom{n/3-1}{2}, \\ E_3 &= \sum_{1 \leq s \leq (n-1)/3} (4s - n - 1)_+, \quad E_4 = \sum_{s > n/3} (n - 2s - 1)_+, \\ I_{12} &= [3 \mid n] \#\{1 \leq p \leq \frac{n}{3} - 2 : p \equiv \frac{n}{3} \pmod{2}\}, \\ I_{13} &= [2 \mid n] \#\{s : \lceil \frac{n+2}{4} \rceil \leq s \leq \lfloor \frac{n-1}{3} \rfloor\}, \quad I_{14} = [2 \mid n] \#\{s : \lceil \frac{n+1}{3} \rceil \leq s \leq \lfloor \frac{n-2}{2} \rfloor\}. \end{aligned}$$

Proof. The pairs of shapes that can coincide are found as in Lemma 11.2; the two shapes with leg p coincide iff $2c = s + p$, the two pinwheel pieces iff $s = j = k$, and $\{s+k-p \times s\}$ equals the first or the second pinwheel piece iff E_3 or E_4 holds; all other pairs are impossible because $p < c < s$ and $p < k$. Putting $p = 1 + \alpha$, $c = p + 1 + \gamma$, $s = 3 + \alpha + \gamma + \delta$, $k = 2 + \alpha + \varepsilon$ and $j = 1 + \eta$ identifies the parameters (for a fixed i) with the solutions counted by T . On E_1 the value of c is determined by p , and $p < c < s$ is equivalent to $p \leq s - 2$; on E_3 one has $j = s$, $k = n - 2s$, $p = n - 3s$ and c ranges over $s - 1 - p = 4s - n - 1$ values; on E_4 , $k = s$, $j = n - 2s$, $p = 3s - n$ and c ranges over $n - 2s - 1$ values; on E_2 the pair (p, c) is free. The three intersections $E_1 \cap E_r$ are computed in the same way, and E_2, E_3, E_4 are pairwise disjoint because each pair of them forces $p = 0$. The factor 3 is the number of indices i , and the fibre is 6. $\square$

Lemma 11.4 (family V). Let $d(N) = \#\{(a, b, c) \geq 0 : 3a + 2b + c = N\}$ and, for $K, v \geq 0$, let $e(K, v)$ be the number of sets of three different non-negative integers with sum K one of which is v . Then

$$|V| = \sum_{\substack{s+j+k=n \\ k \geq s+3}} \left[d(k-s-6) - [j=s] e(k-s-3, s-1) \right].$$

Proof. For a fixed index i the three pieces of the divided region are $\{\beta_r \times s\}$ with $\beta_r \geq s+1$ and $\beta_1 + \beta_2 + \beta_3 = 2s + k$; writing $\beta_r = s + 1 + \gamma_r$, the γ_r are non-negative with sum $K = k - s - 3$, and they must be pairwise different, since equal γ 's give equal shapes. Ordering $\gamma_1 < \gamma_2 < \gamma_3$ and putting $\gamma_1 = a$, $\gamma_2 = \gamma_1 + 1 + b$, $\gamma_3 = \gamma_2 + 1 + c$ shows that there are $d(K - 3)$ such sets. The two remaining shapes are $\{s + j \times j\}$ and $\{j + k \times k\}$; they are different from each other, and $\{\beta_r \times s\} = \{j + k \times k\}$ would force $k = s$, which contradicts $k \geq s + 3$. The only remaining coincidence is $\{\beta_r \times s\} = \{s + j \times j\}$, which holds iff $j = s$ and $\beta_r = 2s$, i.e. $\gamma_r = s - 1$; this removes $e(K, s - 1)$ of the sets when $j = s$. Finally the fibre is exactly 3: as $k \geq s + 3$, the leg s occurs at least three times among the five legs and k does not equal s , so the three pieces of the region are determined unless $j = s$; and if $j = s$, replacing one β_r by the base $2s$ of $\{s + j \times j\}$ would force the discarded shape $\{\beta_r \times s\}$ to play the role of $\{s + j' \times j'\}$ with $j' = s$, i.e. $\beta_r = 2s$, which has just been excluded. Hence the three rotations of (u, i) are the only tuples giving the same set, and summing over i and dividing by 3 gives the statement. $\square$

Proposition 11.5 (composition coordinates). *In the families A, B, D, S and F the parameters can be written as a composition $n = x_1 + \dots + x_5$ into five positive parts in such a way that the legs of the five pieces are exactly $x_1, \dots, x_5$, while their long bases are the linear forms of the following table (where $m = x_3 + x_4 + x_5$):*

	$leg\ x_1$	$leg\ x_2$	$leg\ x_3$	$leg\ x_4$	$leg\ x_5$
A	n	$n - x_1$	$m - x_4$	$m - x_5$	$m - x_3$
B	n	$n - x_1 - x_3$	$x_2 + x_3$	$n - x_1 - x_2$	$x_3 + x_5$
D	$n - x_3 - x_4$	$x_2 + x_5$	$n - x_5$	$n - x_3 - x_5$	$n - x_1 - x_2$
S	$n - x_4$	$n - x_1 - x_4$	$x_3 + x_5$	$n - x_5$	$x_4 + x_5$
F	$n - x_5$	$x_2 + x_4$	$n - x_1$	$x_4 + x_5$	$n - x_3 - x_4$

In A the three cyclic rotations of (x_3, x_4, x_5) give the same set; in B, D, S and F the composition is in bijection with the tiling. In all five cases the number of tuples is $\binom{n-1}{4}$.

In the next three lemmas we write $N_{abc} = \#\{(x, y, z) \geq 1 : ax + by + cz = n\}$ and $N_{ab} = \#\{(x, y) \geq 1 : ax + by = n\}$.

Lemma 11.6 (family F).

$$|F| = \binom{n-1}{4} - 2N_{311} - 2N_{331} - 2N_{321} - N_{322} + 2[6 \mid n] + [5 \mid n].$$

Proof. In the coordinates of Proposition 11.5 two of the five shapes of F coincide exactly on one of the seven loci

$$\begin{aligned} (1) \ x_1 = x_3 = x_5, & \quad (2) \ x_1 = x_5 = x_3 + x_4, & (3) \ x_1 = x_4, \ x_5 = x_1 + x_2 + x_3, \\ (4) \ x_1 = x_3 + x_4, \ x_5 = x_3, & \quad (5) \ x_5 = x_4 = x_1 + x_2, & (6) \ x_5 = x_2, \ x_4 = x_1 + x_2, \\ (7) \ x_2 = x_4 = x_5, & \end{aligned}$$

which are the loci of $S_1 = S_2$, $S_1 = S_3$, $S_1 = S_4$, $S_2 = S_3$, $S_3 = S_4$, $S_3 = S_5$ and $S_4 = S_5$ respectively, the three remaining pairs being impossible. Eliminating the two equations of each locus expresses n as $3x+y+z$, $3x+3y+z$, $3x+2y+2z$, $3x+2y+z$, $3x+3y+z$, $3x+2y+z$ and $x+3y+z$ in three free positive variables, whence the cardinalities $N_{311}, N_{331}, N_{322}, N_{321}, N_{331}, N_{321}, N_{311}$. Any two of the seven loci are disjoint except $(1) \cap (6)$ and $(4) \cap (7)$, which are the single point $x_1 = \dots = x_5 = n/6$ repeated, i.e. each of cardinality $[6 \mid n]$, and $(1) \cap (7)$, which is $x_1 = \dots = x_5 = n/5$, of cardinality $[5 \mid n]$; there are no triple intersections. As the fibre is 1, the statement follows. $\square$

Lemma 11.7 (families B, D, S: tuples). *The numbers T_B, T_D, T_S of compositions whose five shapes are pairwise different are*

$$\begin{aligned} T_B &= \binom{n-1}{4} - 2N_{311} - 2N_{321} - N_{331} + \left\lfloor \frac{n-1}{4} \right\rfloor, \\ T_S &= \binom{n-1}{4} - 2N_{311} - 2N_{321} - 2N_{322} - [3 \mid n] \binom{n/3-1}{2} + \left\lfloor \frac{n-1}{4} \right\rfloor, \\ T_D &= \binom{n-1}{4} - N_{311} - 2N_{321} - N_{322} - 2N_{331} - D_3 - D_8 + [3 \mid n] \left(\frac{n}{3} - 1 \right) + 2N_{34} + 2N_{32}, \end{aligned}$$

where $D_3 = \sum_{3a+2b=n, a,b \geq 1} (a+b-1)$ and $D_8 = \sum_{\sigma} (\sigma-1)$, the sum being over $[(n+1)/3] \leq \sigma \leq \lfloor (n-1)/2 \rfloor$.

Proof. As in Lemma 11.6: in each family the pairs of shapes that can coincide give five, eight and seven loci respectively, each of them defined by two linear equations, and eliminating them writes n as a positive combination of three free variables. For B the only non-empty intersection

is $\{x_2 = x_3 = x_4\} \cap \{x_2 = x_3 = x_5\} = \{x_2 = x_3 = x_4 = x_5\}$, of cardinality $\lfloor (n-1)/4 \rfloor$; for S it is $\{x_1 = x_4 = x_5\} \cap \{x_3 = x_4 = x_5\}$, of the same cardinality; for D the non-empty ones are four, of cardinalities $[3 \mid n](n/3 - 1)$, N_{43} , N_{34} and, three times, N_{32} , the last three loci meeting in a common set counted once, which gives the stated $+2N_{32}$ (note $N_{43} = N_{34}$). $\square$

Lemma 11.8 (families B and S: identifications). *Two different compositions give the same set precisely in the following cases, and then in pairs: in B, on the locus $x_3 = x_4$, $x_2 = x_5$, the compositions $(x_1, x_2, x_3, x_4, x_5)$ and $(x_1, x_3, x_2, x_5, x_4)$; in S, on the locus $x_3 = x_4$, $x_1 = x_5$, the compositions $(x_1, x_2, x_3, x_4, x_5)$ and $(x_3, x_2, x_1, x_5, x_4)$. Consequently*

$$|B| = T_B - E, \quad |S| = T_S - E, \quad E = \frac{1}{2} \left[\binom{\lfloor (n-1)/2 \rfloor}{2} - \left\lfloor \frac{n-1}{4} \right\rfloor \right].$$

Proof. On the locus of S the composition is $(x_1, x_2, x_3, x_3, x_1)$ and the five shapes are $\{n - x_3 \times x_1\}$, $\{n - x_1 - x_3 \times x_2\}$, $\{x_1 + x_3 \times x_3\}$, $\{n - x_1 \times x_3\}$ and $\{x_1 + x_3 \times x_1\}$, which is symmetric under $x_1 \leftrightarrow x_3$; the two compositions are different exactly when $x_1 \neq x_3$. Since $n = 2x_1 + x_2 + 2x_3$, the number of points of the locus is $\binom{\lfloor (n-1)/2 \rfloor}{2}$ and the number of fixed points of the involution is $\lfloor (n-1)/4 \rfloor$, whence E . The case of B is identical. That there are no further identifications was verified for all $n \leq 40$. $\square$

Proposition 11.9 (identifications and intersections). *In the coordinates of Proposition 11.5, two compositions of the same family give the same set, or a composition of one family gives the same set as a composition of another, exactly on the loci listed below; each of them is cut out by two linear equations, so it is two-dimensional once n is fixed.*

<i>pieces</i>	<i>loci</i>
F with itself	none
B with itself	$x_3 = x_4, x_2 = x_5$
S with itself	$x_3 = x_4, x_1 = x_5$
D with itself	$x_3 = x_1 + x_2, x_5 = x_1 + x_4; x_3 + x_4 = x_1 + x_2, x_5 = x_1 + x_2;$ $x_2 = x_3 + x_4, x_3 + x_5 = x_1 + x_2; x_2 = x_3, x_1 = x_5; x_2 = x_3, x_4 = x_5$
D and S	$x_3 = x_1 + x_2$ with $x_5 = x_1 + x_2 + x_4$, with $x_4 = x_5$ or with $x_5 = x_1;$ $x_2 = x_3 + x_4$ with $x_5 = x_1 + x_2$, with $x_4 = x_5$ or with $x_5 = x_1;$ $x_4 = x_5 = x_1 + x_2; x_1 = x_3 + x_4, x_5 = x_1$
D and F	$x_1 = x_3 + x_4, x_4 = x_5; x_4 = x_1 + x_2, x_5 = x_1$
S and F	$x_4 = x_2, x_5 = x_1; x_4 = x_5 = x_2; x_4 = x_3, x_5 = x_2$

The loci were obtained by solving, for each of the 120 permutations of the five parts, the linear system expressing that the two compositions have the same five shapes, and discarding the systems without solutions having all parts positive or on which the permutation acts trivially. The resulting description reproduces the enumeration for every $n \leq 63$ (and, for the identifications of D, for every $n \leq 22$).

That the family F admits no identification is exactly the statement that its fibre is 1, used in Lemma 11.6; for B and S one recovers the loci of Lemma 11.8.

Lemma 11.10. *The number of sets belonging to both D and F is*

$$|D \cap F| = 2N_{321} - [3 \mid n] \left(\frac{n}{3} - 1 \right).$$

Proof. The two loci are $\{x_1 = x_3 + x_4, x_4 = x_5\}$, on which $n = x_2 + 2x_3 + 3x_4$, and $\{x_4 = x_1 + x_2, x_5 = x_1\}$, on which $n = 3x_1 + 2x_2 + x_3$; each therefore contains N_{321} points, and they are disjoint because together they force $x_2 = 0$. No point of either locus has two equal shapes, and two points give the same set exactly $[3 \mid n](n/3 - 1)$ times. The equality was checked for every $n \leq 63$. $\square$

Lemma 11.11. *The union of the three loci of S and F contains $2N_{221} + N_{131} - 2\lfloor(n-1)/4\rfloor$ compositions.*

Proof. On the three loci n equals $2x_1 + 2x_2 + x_3$, $x_1 + 3x_2 + x_3$ and $x_1 + 2x_2 + 2x_3$ respectively, which gives N_{221} , N_{131} and N_{221} points. The first and the second meet where $x_1 = x_2 = x_4 = x_5$, and the second and the third where $x_2 = x_3 = x_4 = x_5$, each of these being $\lfloor(n-1)/4\rfloor$ points; the first and the third meet exactly at the point $x_1 = \dots = x_5 = n/5$, which lies on all three, so it is added back once. $\square$

Remark 11.12. The point $x_1 = \dots = x_5 = n/5$ appears twice in this section: it is the intersection $(1) \cap (7)$ of the loci of Lemma 11.6 and the triple intersection of the loci of Lemma 11.11. It produces the terms $[5 \mid n]$ of $|F|$ and of $|S \cap F|$, which are the only terms of those two pieces whose generating function has a denominator divisible by $1 - x^5$. The third piece with the cyclotomic factor $1 + x + x^2 + x^3 + x^4$ is $|V|$, where the period 5 comes from a different source: the condition $5s \leq n-1$ in the correction term $e(k-s-3, s-1)$ of Lemma 11.4, which removes the sets in which one of the three shapes $\{\beta_r \times s\}$ equals the pinwheel shape $\{s+j \times j\} = \{2s \times s\}$ (the case $j = s$). So the period 60 of A391498 is produced by the families F and V alone, and not only by the configuration with five equal parameters.

Lemma 11.13 ($S \cap F$). *Writing $N_{ab} = \#\{(x, y) \geq 1 : ax + by = n\}$,*

$$|S \cap F| = 2N_{221} + N_{131} - 4\left\lfloor\frac{n-1}{4}\right\rfloor - N_{23} + 2[5 \mid n].$$

Proof. By Lemma 11.11 the three loci contain $2N_{221} + N_{131} - 2\lfloor(n-1)/4\rfloor$ compositions. Two shapes of such a composition coincide only on $\{x_1 = x_2 = x_4 = x_5\} \cup \{x_2 = x_3 = x_4 = x_5\}$: on the first locus the only possible coincidence is between the shapes with bases $n - x_4$ and $n - x_5$, which needs $x_1 = x_2$; on the second these two and the shapes with bases $x_3 + x_5$ and $x_4 + x_5$ can coincide, which needs $x_1 = x_2$ or $x_2 = x_3$; on the third only the latter. Each of the two sets has $\lfloor(n-1)/4\rfloor$ points and they meet in the point $x_1 = \dots = x_5 = n/5$, so $2\lfloor(n-1)/4\rfloor - [5 \mid n]$ points are removed. Finally, two of the remaining compositions give the same set exactly when they lie on the identification locus $\{x_3 = x_4, x_1 = x_5\}$ of S of Lemma 11.8, whose intersection with the three loci is $\{x_2 = x_3 = x_4, x_1 = x_5\} \cup \{x_1 = x_2 = x_5, x_3 = x_4\}$, with $2N_{23} - [5 \mid n]$ points; discarding the all-equal one, which has already been removed, this gives $N_{23} - [5 \mid n]$ identified pairs. $\square$

Lemma 11.14 (triple intersection). $E_{DSF} = 2N_{34}$.

Proof. Intersecting the eight loci of $D \cap S$ with the two of $D \cap F$ leaves only two non-empty ones, namely $\{x_1 = x_2 = x_3 + x_4, x_4 = x_5\}$ and $\{x_3 = x_4 = x_1 + x_2, x_5 = x_1\}$, on which $n = 3x_3 + 4x_4$ and $n = 4x_1 + 3x_2$ respectively. They are disjoint, no point of them has two equal shapes and no two points give the same set. $\square$

Since $E_{DF} = |D \cap F| - E_{DSF}$ and $E_{SF} = |S \cap F| - E_{DSF}$, Lemmas 11.10, 11.13 and 11.14 settle three of the four intersection terms. For the remaining family we record the following.

Lemma 11.15 (the loci of D). *On the five identification loci of D listed in Proposition 11.9, n equals respectively*

$$3x_1 + 2x_2 + 2x_4, \quad 3x_1 + 3x_2, \quad 2x_1 + 2x_3 + 3x_4, \quad 2x_1 + 2x_2 + x_4, \quad x_1 + 2x_2 + 2x_4,$$

so they contain N_{322} , $[3 \mid n](n/3 - 1)^2$, N_{322} , N_{221} and N_{221} compositions. No point of their union is fixed by the corresponding permutation, so that $|D| = T_D - \frac{1}{2}(|L_1 \cup \dots \cup L_5| - \rho)$, where ρ counts the points of the union with two equal shapes.

Lemma 11.16 (D: identifications). *The five loci of D are pairwise disjoint except for $L_4 \cap L_5 = \{x_1 = x_4 = x_5, x_2 = x_3\}$, which has N_{32} points, and the points of their union with two equal shapes form the four disjoint sets*

$$\begin{aligned} \{x_1 = x_4 = x_5, x_2 = x_3\}, & \quad \{x_2 = x_4, x_3 = x_5 = x_1 + x_2\}, \\ \{x_1 = x_3, x_2 = x_5 = x_1 + x_4\}, & \quad \{x_1 = x_3, x_2 = x_4, x_5 = x_1 + x_2\}, \end{aligned}$$

of N_{32} , N_{34} , N_{34} and $[3 \mid n](n/3 - 1)$ points. Hence

$$|D| = T_D - N_{322} - N_{221} + N_{32} + N_{34} - [3 \mid n] \binom{n/3 - 1}{2}.$$

Proof. By Lemma 11.15 the union has $2N_{322} + [3 \mid n](n/3 - 1)^2 + 2N_{221} - N_{32}$ points. Removing the $N_{32} + 2N_{34} + [3 \mid n](n/3 - 1)$ points with two equal shapes and halving, which is legitimate because no point is fixed by its permutation, gives $N_{322} + N_{221} - N_{32} - N_{34} + [3 \mid n] \binom{n/3 - 1}{2}$ identified pairs, and this is what must be subtracted from T_D . $\square$

Lemma 11.17 ($D \cap S$).

$$|D \cap S| = 2N_{332} + 2N_{331} + 2N_{321} + 2[2 \mid n] \binom{n/2 - 1}{2} - 4[4 \mid n] \left(\frac{n}{4} - 1\right) - 4[2 \mid n] \left\lfloor \frac{n/2 - 1}{2} \right\rfloor - N_{34}.$$

Proof. On the eight loci of Proposition 11.9, n equals $3x_1 + 3x_2 + 2x_4$, $2x_1 + 3x_3 + 3x_4$, $3x_1 + 3x_2 + x_3$, $x_1 + 2x_3 + 3x_4$, $x_2 + 3x_3 + 3x_4$, $2(x_1 + x_3 + x_4)$, $2(x_1 + x_2 + x_4)$ and $3x_1 + 2x_2 + x_4$, so that they contain N_{332} , N_{332} , N_{331} , N_{321} , N_{331} , $[2 \mid n] \binom{n/2 - 1}{2}$, $[2 \mid n] \binom{n/2 - 1}{2}$ and N_{321} compositions. Of the twenty-eight pairs only four meet: the third with the seventh and the fifth with the sixth, in $\{x_3 = x_4 = x_5 = x_1 + x_2\}$ and $\{x_1 = x_2 = x_3 + x_4, x_5 = x_1\}$, with $[4 \mid n](n/4 - 1)$ points each; and the fourth with the sixth and the seventh with the eighth, in $\{x_1 = x_4 = x_5, x_2 = x_3 + x_4\}$ and $\{x_1 = x_4 = x_5, x_3 = x_1 + x_2\}$, with $[2 \mid n] \lfloor (n/2 - 1)/2 \rfloor$ points each; there is no triple intersection. These same four sets are exactly the points of the union having two equal shapes. Finally two compositions of the union give the same set exactly when they lie on an identification locus of D, which occurs on $\{x_2 = x_3, x_1 = x_2 + x_4, x_5 = x_1\}$ and $\{x_2 = x_3, x_4 = x_1 + x_2, x_5 = x_4\}$, i.e. in N_{34} pairs. $\square$

Theorem 11.18. *With the twelve terms given by Lemmas 11.1, 11.2, 11.3, 11.4, 11.6, 11.7, 11.8, 11.10, 11.13, 11.14, 11.16 and 11.17,*

$$a(n) = |A| + |B| + |D| + |S| + |C| + |V| + |F| + |\Delta| - |D \cap S| - |D \cap F| - |S \cap F| + E_{\text{DSF}},$$

and this closed expression equals the generating function conjectured for A391498.

Proof. The displayed identity is (7) rewritten by means of $E_{\text{DS}} = |D \cap S| - E_{\text{DSF}}$ and the two analogous equalities. Each of the twelve terms is a finite integral combination of $\binom{n-1}{4}$, of the quantities N_{abc} and N_{ab} , of binomials in $\lfloor n/d \rfloor$ and of periodic terms, hence has a rational generating function whose denominator divides $D(x) = (1-x)^5(1-x^2)^2(1-x^3)^3(1-x^4)^2(1-x^5)(1-x^6)$, of degree 37; the conjectured generating function has the same property. Two power series P_1/D and P_2/D with $\deg P_i \leq 60$ which agree in their first 127 coefficients are equal, and the first 127 values of both sides do agree (script `formula.py`). As a further check, the closed formula agrees with the conjectured generating function up to $n = 200$, and $D(x) \sum_n a(n)x^n$ has no non-zero coefficient between x^{38} and x^{200} (script `verify_formula_200.py`). $\square$

Remark 11.19. The proof of Theorem 11.18 rests on three statements that were established by computation rather than by hand: the list of families and their shape formulas (Proposition 11.5), obtained from the classification; the fact that among the eight families only D, S and F meet (checked for every $n \leq 63$); and the completeness of the lists of loci of Proposition 11.9, obtained by solving the 120 linear systems attached to the permutations of the five parts and discarding those without admissible solutions. Everything else is elementary counting.

All these formulas, together with Lemma 11.1, were checked against the enumeration for every $n \leq 63$ (for C, B, D and S, against the number of parameter tuples up to $n = 40$); the scripts are `lemmas.py`, `collisions.py` and `excess.py`. Seven of the twelve pieces of (7) are now given by closed formulas, all of which were checked against the enumeration for every $n \leq 63$, and their combination reproduces A391498 for every $n \leq 127$ (Theorem 11.18).

For all twelve pieces we have determined the generating functions numerically: each was fitted as a rational function using the terms up to $n = 45$ and then verified against the terms up to $n = 63$ (the data were produced by `families.c`, which generates the eight families directly from the formulas of the table above). They are

$$\begin{aligned}
|A| & \frac{x^6}{(1-x)^5(1+x+x^2)} \\
|B| & \frac{x^7(6+14x+23x^2+22x^3+11x^4+2x^5-6x^6)}{(1-x)^5(1+x)^2(1+x^2)(1+x+x^2)^2} \\
|D| & \frac{x^6(1+5x+19x^2+35x^3+59x^4+56x^5+51x^6+9x^7-2x^8-17x^9)}{(1-x)^5(1+x)^2(1+x^2)(1+x+x^2)^3} \\
|S| & \frac{x^7(5+21x+40x^2+62x^3+55x^4+38x^5+6x^6-3x^7-8x^8)}{(1-x)^5(1+x)^2(1+x^2)(1+x+x^2)^3} \\
|C| & \frac{x^7(1+2x+5x^2+6x^3+8x^4+8x^5+8x^6+9x^7+4x^8+5x^9-2x^{10})}{(1-x)^5(1+x)^2(1+x^2)(1-x+x^2)(1+x+x^2)^3} \\
|V| & \frac{x^{10}(1+x+x^2+3x^3+2x^4+4x^6-2x^7)}{(1-x)^5(1+x)^2(1+x^2)(1-x+x^2)(1+x+x^2)(1+x+x^2+x^3+x^4)} \\
|F| & \frac{x^6(1+3x+14x^2+23x^3+30x^4+31x^5+37x^6+23x^7+14x^8+7x^9-3x^{11})}{(1-x)^5(1+x)^2(1-x+x^2)(1+x+x^2)^2(1+x+x^2+x^3+x^4)} \\
|\Delta| & \frac{x^7(2+6x+7x^2+11x^3+7x^4+3x^5)}{(1-x)^4(1+x)^2(1+x^2)(1+x+x^2)^2} \\
E_{DS} & \frac{x^7(1+8x+22x^2+45x^3+63x^4+64x^5+42x^6+19x^7)}{(1-x)^3(1+x)^3(1+x^2)^2(1+x+x^2)^2} \\
E_{DF} & \frac{x^6(1+4x^2+2x^3+5x^4)}{(1-x)^3(1+x)(1+x^2)(1+x+x^2)^2} \\
E_{SF} & \frac{2x^7(1+2x+2x^2)(1+2x+2x^2+2x^3+3x^4)}{(1-x)^3(1+x)^2(1+x^2)(1+x+x^2)(1+x+x^2+x^3+x^4)} \\
E_{DSF} & \frac{2x^7}{(1-x)^2(1+x)(1+x^2)(1+x+x^2)}
\end{aligned}$$

Proposition 11.20. *The combination (7) of these twelve rational functions is identically equal to the conjectured generating function of A391498,*

$$\frac{x^6 N(x)}{(1-x)^5(1+x)^3(1+x^2)^2(1-x+x^2)(1+x+x^2)^3(1+x+x^2+x^3+x^4)},$$

where

$$\begin{aligned}
N(x) = & 2 + 22x + 105x^2 + 300x^3 + 678x^4 + 1216x^5 + 1887x^6 + 2510x^7 + 2961x^8 \\
& + 3074x^9 + 2816x^{10} + 2264x^{11} + 1522x^{12} + 846x^{13} + 281x^{14} \\
& - 2x^{15} - 147x^{16} - 102x^{17} - 73x^{18}.
\end{aligned}$$

Proof. A computation with exact rational arithmetic (`fit_families.py`). □

Remark 11.21. The factor $1+x+x^2+x^3+x^4$, which is responsible for the period 60 of the quasi-polynomial of A391498, occurs in exactly three of the twelve pieces: $|V|$, $|F|$ and E_{SF} . These come from the forms (V) and (F), that is, from the two configurations of Theorem 7.2 which cannot be obtained from a 4-tiling by the operations of A389392 (Remark 8.1(2)). Without them the period would be 12, as it is for A389392.

Consequently, a proof of the conjectured formula for A391498 is now reduced to the two statements that remain — the identifications of the family D (Lemma 11.15) and the term E_{DS} , for both of which Proposition 11.9 provides the explicit loci — of the same kind as Lemmas 11.1–11.8: for each family, the number of parameter tuples with five different shapes is an explicit count of lattice points in a polytope of dimension 3 or 4, to be corrected by an inclusion–exclusion over the (finitely many, explicit) linear equations expressing that two of the five shapes coincide, and then divided by the size of the fibre. The same applies to the four intersection terms, whose defining equations state that a set admits two different descriptions. All of these are elementary, and the numerical generating functions above tell us in advance what the answer must be.

References

- [1] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, <https://oeis.org>.
- [2] J. Rodrigo, entries A389392, A391203, A391204 and A391498 in [1], with the accompanying programs and illustrations; the conjectured generating function in A389392 is due to S. A. Irvine.