

ON THE ESCAPE RATE OF FAVORITE SITES OF PLANAR RANDOM WALKS

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ABSTRACT

For planar simple random walk, the favorite sites at time n are the sites whose local time at time n is maximal. We prove that, almost surely, for every $\gamma > 1/2$ and every $c > 0$, all favorite sites lie outside the ball centered at the origin with radius $c\sqrt{n}/(\log n)^\gamma$ for all sufficiently large n . At the critical exponent $\gamma = 1/2$, almost surely, for every $c > 0$, the entire favorite sites lies within distance $c\sqrt{n/\log n}$ of the origin infinitely often.

AI USAGE

This manuscript was substantially generated and subsequently checked by AI systems. At the time of this version, the arguments and citations have not yet been independently verified in full by the human contributor. **A digested and more streamlined version will replace this preliminary version on both Hexagon and arXiv in the coming months.**

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1. Introduction

A favorite site of a random walk is a site at which the accumulated local time is maximal. Although the planar walk returns to the origin infinitely often, its favorite sites need not remain near the origin. The question considered here is how small a neighborhood of the origin can still contain a favorite at arbitrarily large times. Because several sites can tie for the maximum, there are two distinct observables: the distance of the nearest favorite and the distance of the farthest favorite.

Let $(S_n)_{n \geq 0}$ be simple symmetric random walk on $\mathbb{Z}^2$, started at the origin. Define

$$L_n(x) = \sum_{j=0}^n \mathbf{1}_{\{S_j=x\}}, \quad M_n = \max_{x \in \mathbb{Z}^2} L_n(x), \quad \mathcal{F}_n = \{x \in \mathbb{Z}^2 : L_n(x) = M_n\}. \quad (1.1)$$

The set $\mathcal{F}_n$ is finite and nonempty. Put

$$R_n^- = \min_{x \in \mathcal{F}_n} \|x\|_2, \quad R_n^+ = \max_{x \in \mathcal{F}_n} \|x\|_2, \quad a_\gamma(n) = \frac{\sqrt{n}}{(\log n)^\gamma}, \quad n \geq 3. \quad (1.2)$$

For example, along the path $(0, e_1, 0, e_1)$ the two sites tie, so $R_3^- = 0$ and $R_3^+ = 1$. Thus localizing an arbitrarily chosen favorite does not by itself localize the whole favorite set.

THEOREM 1.1 (Lower-class escape rate). *Almost surely, simultaneously for every real γ ,*

$$\liminf_{n \rightarrow \infty} \frac{R_n^-}{a_\gamma(n)} = \infty, \quad \gamma > \frac{1}{2}, \quad (1.3)$$

$$\liminf_{n \rightarrow \infty} \frac{R_n^+}{a_\gamma(n)} = 0, \quad \gamma \leq \frac{1}{2}. \quad (1.4)$$

In particular, for each $\gamma > 1/2$ and each fixed $A > 0$, eventually every favorite lies outside $B(0, Aa_\gamma(n))$. At the critical exponent, for every $\varepsilon > 0$ there are infinitely many times at which *all* favorites lie in $B(0, \varepsilon\sqrt{n/\log n})$. The conclusion identifies the boundary among logarithmic powers of the diffusive scale. It does not assert an integral test for arbitrary slowly varying perturbations of the critical gauge.

Dembo, Peres, Rosen and Zeitouni [7, Theorem 1.1] proved that every planar favorite lies at distance $n^{1/2+o(1)}$. This first-order exponent leaves the logarithmic scales in Theorem 1.1 undetermined. Erdős and Révész [8, Question 14] asked whether planar favorite sites escape to infinity. Révész later asked for their rate of escape [12, Chap. 19], as recorded by Dembo [5, p. 6]. Dembo's ICM survey [6, Open Problem 2.1] asks more broadly about the evolution of favorite sites in dimensions $d \geq 2$. Hao, Li, Okada and Zheng [9] recently proved that almost surely three favorite sites occur simultaneously infinitely often and eventually there is no simultaneous occurrence of four favorite sites. In [9, Remark 1.4(1)] they also highlight the planar escape-rate question. In one dimension the corresponding theorem has critical exponent 1; see Lifshits and Shi [11, Theorem 1.1] for the simple random walk and Bass [2, Theorem 2.2] for both Brownian motion and the walk. Our focus is the location of the full set of planar maximizers. The two assertions in Theorem 1.1 therefore retain the distinction between its nearest and farthest members throughout the proof.

1.1. NOTATION AND THE TWO LOCAL-TIME ESTIMATES. All logarithms are natural. We write P_x and E_x for the law and expectation of a walk started at x , and omit the subscript

when $x = 0$. Its transition operator is

$$Pf(x) = \frac{1}{4} \sum_{y: \|y-x\|_2=1} f(y).$$

By symmetry the same notation acts on lattice measures. For $r > 0$ let $D(x, r) = \{y \in \mathbb{Z}^2 : \|y - x\|_2 < r\}$ and $\tau_r = \inf\{n \geq 0 : S_n \notin D(0, r)\}$. Closed balls are denoted by $B(x, r)$. We use the same geometric notation in the plane when the ambient space is specified. A local time at a stopping time includes that time; inside a killed domain it agrees with occupation strictly before exit.

The Green-function normalization and the local-time levels are

$$g = \frac{2}{\pi}, \quad u_{J,c} = 2gJ^2 - cgJ \log J. \quad (1.5)$$

The logarithmic radius is J , so the corresponding time scale is e^{2J} . The coefficient of $J \log J$ will matter: knowing only the leading $2gJ^2$ term cannot distinguish the powers in Theorem 1.1.

The escape argument uses the following two estimates, proved in Propositions 6.3 and 7.2. For fixed $\beta > 0$, $c > 2$ and $A > 0$,

$$P\left(\max_{\|x\|_2 \leq e^J J^{-\beta}} L_{\tau_{eJ}}(x) \geq u_{J,c}\right) \leq J^{-2\beta+c-2+o(1)} + O(J^{-A}). \quad (1.6)$$

For every fixed $\eta > 0$, there is $p_\eta > 0$ such that

$$P(M_{\tau_{eJ}} \geq u_{J,2+\eta}) \geq p_\eta \quad \text{for all sufficiently large } J. \quad (1.7)$$

Since $a_\gamma(e^{2J}) = e^J / (2J)^\gamma$, the disk in (1.6) is at the scale relevant to the theorem. Taking $\beta = \gamma$ and $c = 2 + 2\delta$ makes its main term $J^{-2\gamma+2\delta+o(1)}$, summable over integer J when $\gamma > 1/2$ and $\delta > 0$ is small enough. The lower estimate is amplified over independent segments to give an eventual almost-sure bound at deterministic exponential times. Comparing it with the upper estimate excludes a favorite near the origin throughout each time block. More precisely, if $N_j = \lfloor e^{2j} \rfloor$ and

$$\max_{x \in B(0, Ce^{j+1} j^{-\gamma})} L_{N_{j+1}}(x) < M_{N_j},$$

then monotonicity excludes every site in this ball from $\mathcal{F}_n$ for all $N_j \leq n \leq N_{j+1}$. Section 8 makes this comparison with summable error probabilities. Recurrence at the critical exponent uses a separate geometric-time localization argument.

Constants denoted by c, C may change between occurrences. Dependence on a fixed parameter is indicated when relevant. Integer parts and fixed changes of radius are harmless in estimates with a $J \log J$ margin: replacing J by $J + O(1)$ changes $u_{J,c}$ by $O(J)$. Directed lattice crossings and conditional bridge laws are defined exactly when they are introduced.

2. Proof strategy

The two assertions of Theorem 1.1 have separate proofs. For critical recurrence we produce, at infinitely many times, a unique favorite in a small disk and then account for the local time inherited from earlier times. For escape we compare the local times in a small disk with the maximum over the whole walk. The recurrence argument is shorter and is completed in Section 4. The escape proof then has three parts: Section 5 controls traversal

counts, Section 6 combines those counts with terminal occupation to prove the localized upper bound, and Section 7 proves and amplifies the global lower bound. Section 8 makes the final comparison over time blocks.

2.1. CRITICAL RECURRENCE: FROM ONE FAVORITE TO THE WHOLE SET. Consider the walk at an independent geometric time K with $P(K = n) = (1 - z)z^n$ and time scale $N = (1 - z)^{-1}$. Choose one favorite using independent continuous priorities at the sites. Reversing the path through the last visit to this selected site separates it into completed excursions, a terminal arm, and a forward suffix. Once the excursions, suffix, and priorities are fixed, the selected-site condition imposes an upper capacity on the arm's occupation at every other site. The admissible arm is therefore a walk stopped at its first return to the selected site or its first capacity violation. This exact reversal, including the priority rule, is proved in Lemma 4.2.

The endpoint law of the arm is a positive mixture of stopped occupation measures. Each measure μ obeys a resolvent identity $(I - zP)\mu = \alpha\delta_0 - \nu$, where $\nu \geq 0$ and $\alpha - \nu(\mathbb{Z}^2) = (1 - z)\mu(\mathbb{Z}^2)$. Let w_R be the normalized autocorrelation of a square of radius R , put $G_z = \sum_{m \geq 0} z^m P^m$, and set $\phi_R = G_z * w_R$. Its nonnegative Fourier transform makes ϕ_R maximal at the origin, so the unknown exit measure has the favorable sign:

$$\mu(w_R) = \alpha\phi_R(0) - \nu(\phi_R) \geq (1 - z)\mu(\mathbb{Z}^2)\phi_R(0), \quad \phi_R(0) \gtrsim R^2 \left(1 + \log \frac{N}{R^2}\right).$$

Taking $R \asymp \sqrt{N}/f$ gives

$$\mu(B(0, C\sqrt{N}/f)) \geq c\mu(\mathbb{Z}^2) \frac{1 + \log f}{f^2}, \quad 2 \leq f \leq N^{1/8}. \quad (2.1)$$

If σ is the first visit to the selected favorite, restricting to $K - \sigma \geq N/8$ retains a fixed positive mass. This condition depends only on the exposed pieces and implies $K \geq N/8$, changing the radius in this estimate from $\sqrt{N}/f$ to $C\sqrt{K}/f$.

Locating the selected favorite does not yet locate the entire favorite set: other sites may tie it. Insert the four-step loop $(x, x + e_1, x, x + e_2, x)$ after its last visit to x . The new local time at x rises by two and those at the two visited neighbors rise by one, so x becomes the unique favorite. The loop can be recovered from the image path, and the injection costs only a fixed factor in probability. Nor does a fresh segment erase the preceding local times. If H is the old profile and the fresh profile ℓ has unique maximizer y , then

$$\arg \max(H + \ell) \subset \text{supp } H \cup \{y\}.$$

Thus we only need to keep the old range inside the target disk; no bound on the heights of H is needed.

For the final block construction, take $s_k = \lfloor e^{J_k} \rfloor$ with $J_{k+1} - J_k = 10 \log J_k$ and $f_k \asymp \sqrt{J_{k+1}}$. The fresh unique-favorite event has probability at least $d_k \asymp (\log J_k)/J_k$. Since $J_k \sim 10k \log k$, the sum $\sum_k d_k$ diverges. Retaining clocks $K_k \gtrsim d_k e^{J_{k+1}}/J_{k+1}^2$ and using the eventual old-range bound $\sqrt{s_k J_k}$ gives

$$\frac{\sqrt{s_k J_k}}{\sqrt{K_k/J_{k+1}}} = O\left(\frac{1}{J_k^2 \sqrt{\log J_k}}\right) \rightarrow 0.$$

Conditional Borel–Cantelli then gives the critical assertion for R_n^+ .

2.2. **ESCAPE: THE MISSING POWER IN A LOCALIZED UPPER BOUND.** Write $u_{J,c} = 2gJ^2 - cgJ \log J$ and stop the walk on leaving a disk of radius e^J . The target disk has radius $e^J J^{-\beta}$. A geometric tail for one site's local time, combined with the chance of hitting a site in the logarithmic shell at distance about e^{J-k} , gives only

$$CJ^{c-1} \sum_{k \geq \beta \log J + O(1)} (k+1)e^{-2k} \leq J^{-2\beta+c-1+o(1)}. \quad (2.2)$$

As $c \downarrow 2$, summing this over exponential time blocks would require $\beta > 1$. The theorem needs $\beta > 1/2$. The localized estimate (1.6) gains the missing factor J^{-1} by keeping the traversal event and the terminal occupation in the same cell.

Cover the target disk by cells centered at y , with terminal radius $M = e^h$ and core $V_y = D(y, M/2)$; write $L = J - h$. Let T_j count the completed inward traversals around a cell at logarithmic level j . The upper argument uses $h = (C_h \log J)^4$, large enough that the first-crossing estimate applies through level L . Fix a positive barrier intercept z ; from this point z has this meaning in the traversal argument. The upper argument keeps the square-root traversal profile below $B_j = \rho_J j + z + (j \wedge (J - j))^{1/4}$, where $\rho_J = 2 - (\log J)/J$. The slope records the $J \log J$ shift in an extreme traversal count; the curved buffer permits a summable bound on early barrier crossings.

Fix a cell in source shell $\beta \log J + O(1) \leq k \leq C_0 \log J$. For a fixed unit endpoint band $[q, q+1]$ in the central range $\epsilon_0 L \leq q \leq B_L + 1$, let $p = B_L + 1 - q \geq 0$ be its deficit from the terminal barrier. Then the source-inclusive ballot estimate is

$$P(\text{good prefix, } q \leq \sqrt{2T_L} < q+1) \leq C(k+1)^C (1+p) L^{-2} e^{-q^2/(2L)}. \quad (2.3)$$

It bounds a joint event, rather than the endpoint conditional on a good prefix. The factor L^{-2} combines the source-height law and the ballot prefactor. The corresponding centered continuation and its lattice comparison are established in Sections 5 and 6.

The endpoint T_L counts completed pieces but does not determine their occupation of V_y . Expose the radial word w and the entrance and directed-exit data, then erase the interiors of the pieces. Conditional on these data, the interiors are independent killed bridges. Conditioning on the *complete* word also constrains later endpoints, so the proof compares its full likelihood, including connectors and the final suffix, with a sequentially normalized endpoint law. For an admissible centered word with b completed pieces, let $Y = \sqrt{2b}$ and $p_w = B_L + 1 - Y$. If $X_i(x)$ is the occupation of $x \in V_y$ by piece i , then, in the central range $\epsilon_0 L \leq Y \leq B_L + 1$, the terminal estimate has the form

$$P \left(\max_{x \in V_y} \sum_{i \leq b} X_i(x) \geq u_{J,c} \middle| w, b \right) \leq C e^{-2L} J^c e^{b/L} \times \exp \left\{ -\kappa \min \left(\frac{(p_w - p_*)^2}{h}, \frac{J^2}{h} \right) \right\}.$$

Here $p_* = O(\log J)$. For an endpoint in the unit band of (2.3), $b = T_L + O(1) = q^2/2 + O(L)$, so $Y = q + O(1)$ and $p_w = p + O(1)$. Hence $e^{b/L}$ cancels the factor $e^{-q^2/(2L)}$. The decay around the preferred deficit p_* controls the remaining endpoint sum:

$$\sum_{0 \leq p \leq B_L + 1} (1+p) \exp \left\{ -\kappa \min \left(\frac{(p - p_*)^2}{h}, \frac{J^2}{h} \right) \right\} = J^{o(1)}.$$

The cancellation alone would not control that sum. One original cell therefore contributes at most $(k+1)^C e^{-2L} J^{c-2+o(1)}$. There are $O(e^{2(L-k)})$ such cells in shell k , so

$$\sum_{k \geq \beta \log J + O(1)}^{C_0 \log J} e^{2(L-k)} (k+1)^C e^{-2L} J^{c-2+o(1)} \leq J^{-2\beta+c-2+o(1)}.$$

The small-endpoint range and distant source shells are handled separately in Proposition 6.3.

2.3. ESCAPE: RESTORING THE TERMINAL DEPTH IN THE GLOBAL MAXIMUM. For the global lower bound, the traversal construction reaches only $L = J - h$ and supplies a cell with

$$b = 2L^2 - 2L \log L + O(L) \quad (2.4)$$

completed pieces. Rosen's one- and two-center estimates give a positive probability that at least one Brownian root exists. The pair estimate is used before a one-sided strong coupling transfers the *existence* event to the lattice; adding coupling errors separately over exponentially many roots would lose this positive probability.

For the lower bound take $h = (\log J)^\xi$ with $1 < \xi < 2$. Conditional on regular endpoint moments, the centered bridge field has covariance close to a multiple of the killed Green function. A Gaussian approximation and the discrete Gaussian free field maximum yield, for the core V_y ,

$$\max_{x \in V_y} \sum_{i \leq b} X_i(x) = gb + 2\sqrt{2} g\sqrt{b} h + O_P \left(J\sqrt{h} + \frac{J \log J}{\log \log J} \right). \quad (2.5)$$

The second term is needed at the required precision. Indeed,

$$\begin{aligned} gb &= 2gJ^2 - 4gJh - 2gJ \log J + O(h^2 + h \log J + J), \\ 2\sqrt{2} g\sqrt{b} h &= 4gJh + O(h^2 + h \log J). \end{aligned}$$

The terminal maximum restores the $4gJh$ lost by truncating the traversal construction, leaving $2gJ^2 - 2gJ \log J + o_P(J \log J)$. Here $\xi > 1$ makes the terminal disk large enough for the lattice comparison, while $\xi < 2$ makes the moment screen and the error in (2.5) fit within the $J \log J$ margin. A conditional failure bound for each candidate cell, followed by a union bound, turns coarse-root existence into the positive probability estimate (1.7) without selecting a root using other bridge interiors.

Finally, independent walk segments amplify disk success to an almost-sure lower bound at deterministic exponential times. Comparing that lower bound at the left end of a time block with the localized upper bound at its right end gives a block error with main term $j^{-2\gamma+2\delta+o(1)}$ and a summable exit error. For any $\gamma > 1/2$, choose $\delta > 0$ small enough to make the main term summable. This proves escape of R_n^- throughout the blocks and completes the other half of the theorem.

3. Elementary estimates

For a finite lattice domain D , write

$$G_D(x, y) = E_x \sum_{j < \tau_{D^c}} \mathbf{1}_{\{S_j = y\}}, \quad \tau_{D^c} = \inf\{j \geq 0 : S_j \notin D\}.$$

The following estimates fix the normalization used throughout the paper.

LEMMA 3.1 (Elementary walk estimates). *There are constants $c, C \in (0, \infty)$ with the following properties.*

(1) *If $\tau_R = \inf\{n : \|S_n\|_2 \geq R\}$, then*

$$\sup_{\|x\| < R} E_x \tau_R \leq CR^2, \quad \sup_{\|x\| < R} P_x(\tau_R > mCR^2) \leq 2^{-m}.$$

(2) *For all $N, R \geq 1$,*

$$P\left(\max_{k \leq N} \|S_k\|_2 \geq R\right) \leq C \exp\{-cR^2/N\}.$$

(3) *Uniformly for $1 \ll r < \|x\| < R$,*

$$P_x(\tau_{\partial D(0,R)} < \tau_{\partial D(0,r)}) = \frac{\log(\|x\|/r) + O(r^{-1})}{\log(R/r)}$$

and the complementary inner-hitting probability is obtained by replacing the numerator by $\log(R/\|x\|) + O(r^{-1})$.

(4) *For each fixed $\kappa \in (0, 1)$, uniformly for $\|x\| \leq (1 - \kappa)R$ in $D(0, R)$,*

$$G_{D(0,R)}(x, x) = g \log R + O_\kappa(1).$$

Conditional on first hitting x , the number of visits to x before leaving the disk is geometric with this mean.

Proof. For (1), optional stopping of $\|S_{n \wedge \tau_R}\|_2^2 - (n \wedge \tau_R)$, followed by the one-step overshoot bound, gives the expectation estimate. Markov's inequality at time $2CR^2$, iterated with the strong Markov property, gives the geometric tail. Part (2) follows by applying the usual exponential martingale separately to both coordinates and to both signs.

For the annulus estimate, we use [9, Lemma 2.1, (2.5)]: uniformly for $r < \|x\| < R$,

$$P_x(H_{\partial D(0,R)} < H_{\partial D(0,r)}) = \frac{\log(\|x\|/r) + O(r^{-1})}{\log(R/r)}.$$

Here H is the positive-time hitting time and the boundaries are the digital outer vertex boundaries defined in that paper. The later directed-edge convention can differ from this vertex-boundary convention only within a one-unit radial collar. Applying the estimate with r, R shifted by at most two units, equivalently repeating its potential-kernel proof at the directed crossing time, changes the logarithmic terms by $O(r^{-1})$. Hence the same estimate holds for the directed-edge rows below. The potential-kernel proof of that result also gives (4). The geometric assertion in (4) is the strong Markov property at successive positive returns to x . ■

4. Recurrence at the critical scale

We prove that the full favorite set lies in a disk of radius $o(\sqrt{n/\log n})$ at infinitely many times. The argument first localizes a selected favorite at a geometric time and then makes it unique by an injective four-step insertion. If a fresh segment has a unique favorite, every favorite after adding the past lies either at that site or in the old range. Thus a bound on the old range suffices even when its local-time heights are arbitrary.

4.1. A RESOLVENT ESTIMATE AND REVERSAL AT A FAVORITE.

LEMMA 4.1 (A stopped resolvent bound). *Let $0 < z < 1$, $N = (1 - z)^{-1}$, and let μ be a finite positive measure on $\mathbb{Z}^2$ of mass β . Suppose that*

$$(I - zP)\mu = \alpha\delta_0 - \nu, \quad \nu \geq 0, \quad \alpha - \nu(\mathbb{Z}^2) = (1 - z)\beta. \quad (4.1)$$

There are universal constants $c, C > 0$ such that, whenever $2 \leq f \leq N^{1/8}$,

$$\mu \left(B \left(0, C \frac{\sqrt{N}}{f} \right) \right) \geq c\beta \frac{1 + \log f}{f^2}. \quad (4.2)$$

Proof. We test the resolvent identity against a function whose maximum is at the origin. This makes the unknown positive measure ν contribute with the needed sign in (4.3).

Let $Q_R = [-R, R]^2 \cap \mathbb{Z}^2$ and set

$$w_R := |Q_R|^{-1} \mathbf{1}_{Q_R} * \mathbf{1}_{-Q_R}.$$

Then $0 \leq w_R \leq 1$, $w_R(0) = 1$, and w_R is supported in $B(0, 3R)$. Put

$$G_z := \sum_{n \geq 0} z^n P^n, \quad \phi_R := G_z * w_R.$$

Fourier inversion gives

$$\widehat{\phi_R}(\theta) = \frac{|\widehat{\mathbf{1}_{Q_R}}(\theta)|^2 / |Q_R|}{1 - z\widehat{P}(\theta)} \geq 0.$$

Consequently $\phi_R(y) \leq \phi_R(0)$ for every y . Pairing (4.1) with ϕ_R and using $(I - zP)\phi_R = w_R$ yields

$$\begin{aligned} \mu(w_R) &= \alpha\phi_R(0) - \nu(\phi_R) \\ &\geq (\alpha - \nu(\mathbb{Z}^2))\phi_R(0) = (1 - z)\beta\phi_R(0). \end{aligned} \quad (4.3)$$

For $\|\theta\| \leq c/R$, the numerator in the Fourier transform is at least cR^2 , while $1 - z\widehat{P}(\theta) \asymp N^{-1} + \|\theta\|^2$. Hence, for $R^2 \leq N/4$, polar integration gives

$$\begin{aligned} \phi_R(0) &\geq cR^2 \int_0^{c/R} \frac{t \, dt}{N^{-1} + t^2} \\ &\geq cR^2 \left(1 + \log \frac{N}{R^2} \right). \end{aligned} \quad (4.4)$$

Since $\mu(w_R) \leq \mu(B(0, 3R))$, take $R = \lfloor \sqrt{N}/(4f) \rfloor$ in (4.3)–(4.4). This proves (4.2), after changing the absolute radius constant. ■

LEMMA 4.2 (Localization of a selected favorite). *Let $0 < z < 1$, $N = (1 - z)^{-1}$, and let K be independent of the walk with*

$$P(K = n) = (1 - z)z^n, \quad n \geq 0.$$

Attach independent uniform $[0, 1]$ priorities $(U_x)_{x \in \mathbb{Z}^2}$ to the sites and let X_K^ be the least-priority member of $\mathcal{F}_K$. Then*

$$P \left(\|X_K^*\|_2 \leq C \frac{\sqrt{K}}{f} \right) \geq c \frac{1 + \log f}{f^2}, \quad 2 \leq f \leq N^{1/8}. \quad (4.5)$$

More precisely, the proof gives a stopped-resolvent representation of the endpoint law of $-X_K^$. The representation remains a positive mixture satisfying (4.1) after restriction by any event measurable with respect to the base tuple defined there.*

Proof. The purpose of reversal is to leave one path piece free: once the other pieces and priorities are fixed, its endpoints form a stopped occupation measure. We include the priority rule in the bijection so that ties impose fixed constraints on this free piece.

Step 1. Decompose and reconstruct the path.

Fix a clock atom $K = n$, a path $s = (s_0, \dots, s_n)$, and first insert

$$1 = \sum_x \mathbf{1}_{\{x=X_n^*\}}.$$

Thus we consider each selected site x separately, with x fixed when translating the priorities. Put $r = M_n - 1$, and let σ, ℓ be the first and last visits to x . Reverse and center the prefix through ℓ :

$$A_i = s_{\ell-i} - x, \quad 0 \leq i \leq \ell.$$

It has exactly r positive returns to zero by time $\ell - \sigma$. Hence it decomposes uniquely as

$$A = E_1 \oplus \dots \oplus E_r \oplus C, \quad (4.6)$$

where each E_i is a nearest-neighbor excursion from zero to zero with no intermediate visit to zero. Here $\oplus$ joins paths without counting their shared endpoints twice. Their total length is $\rho_r = \ell - \sigma$. The remaining path C starts at zero, has no positive return to zero, and ends at $a = -x$. The centered forward suffix

$$B_i = s_{\ell+i} - x, \quad 0 \leq i \leq n - \ell,$$

also has no positive return to zero.

Write $H_r(y)$ for the number of visits to $y \neq 0$ in the completed excursions and $N_C(y), N_B(y)$ for the corresponding positive-time counts in C, B . Translate priorities by the fixed sector shift, $V_y = U_{x+y}$. The statement that the root is the priority-selected favorite is exactly

$$H_r(y) + N_C(y) + N_B(y) \leq r + \mathbf{1}_{\{V_y > V_0\}}, \quad y \neq 0. \quad (4.7)$$

Indeed, a better-priority site must be strictly below the root's value $r + 1$, while a worse-priority site may tie it.

We now fix all pieces except the terminal arm C , together with the translated priorities:

$$\mathcal{B} = (r, E_1, \dots, E_r, B, V).$$

The local time still available to C at $y \neq 0$ is the capacity

$$c_{\mathcal{B}}(y) := r + \mathbf{1}_{\{V_y > V_0\}} - H_r(y) - N_B(y), \quad y \neq 0. \quad (4.8)$$

Only tuples with $c_{\mathcal{B}}(y) \geq 0$ for every $y \neq 0$ can occur. For such a tuple, run a new walk C from zero and let τ_c be its first positive return to zero or its first time at which some occupation count $N_C(y)$ exceeds $c_{\mathcal{B}}(y)$. Condition (4.7) says exactly that the admissible terminal arms are the prefixes $(C_0, \dots, C_q)$ with $q < \tau_c$.

Conversely, a feasible base tuple and one such prefix reconstruct the original path: concatenate (4.6), reverse it, translate by $-a = x$, and append the translated suffix B . Inequality (4.7) makes x a favorite and excludes every better-priority tie; avoidance of zero by B makes ℓ its last visit. The endpoint a recovers the sector $x = -a$, so both the path and the priority translation are unique. The forward and inverse maps are therefore genuine bijections, including $r = 0$.

Step 2. Identify the stopped occupation measure.

Write $q = |C|$ and $k = |B|$ for the terminal-arm and suffix lengths, so that $n = \rho_r + q + k$. Reversal and symmetry preserve the path weight, and

$$z^{\rho_r} 4^{-\rho_r} z^q 4^{-q} (1 - z) z^k 4^{-k} = (1 - z) z^n 4^{-n}. \quad (4.9)$$

For each fixed tuple, the factors outside $z^q 4^{-q}$ are fixed. Summing over its admissible terminal arms therefore gives their stopped occupation measure, with no multiplicity or extra normalization.

For a fixed capacity let

$$g_c(u) = E_0 \sum_{q < \tau_c} z^q \mathbf{1}_{\{C_q = u\}}.$$

By the Markov property,

$$z P g_c(u) = E_0 \sum_{q < \tau_c} z^{q+1} \mathbf{1}_{\{C_{q+1} = u\}} = g_c(u) - \mathbf{1}_{\{u=0\}} + E_0 [z^{\tau_c} \mathbf{1}_{\{C_{\tau_c} = u\}}].$$

Thus the occupation terms cancel except at the initial and stopped endpoints:

$$(I - zP)g_c = \delta_0 - E_0 [z^{\tau_c} \delta_{C_{\tau_c}}]. \quad (4.10)$$

The arm endpoint is $a = -x$. Thus the bijection and (4.9) identify the law of $-X_K^*$ as a positive mixture of the g_c 's, integrating also over the translated i.i.d. priorities. Mixing (4.10) gives (4.1); its mass relation follows by summing over $\mathbb{Z}^2$. Restricting the tuples by any event depending only on $\mathcal{B}$ merely removes positive mixture weights, so the same identity holds for the restricted endpoint measure.

Step 3. Compare the deterministic and random clock scales.

Lemma 4.1 gives a radius proportional to $\sqrt{N}/f$. To obtain $\sqrt{K}/f$, we restrict the mixture to an event of fixed positive probability on which $K \geq N/8$. This restriction must depend only on $\mathcal{B}$, so that the resolvent identity still applies. Whole-path reversal, translation to the new starting point, and transport of the i.i.d. priorities preserve the joint law and send

$$(\sigma, \ell) \mapsto (n - \ell, n - \sigma).$$

Conditionally on $K = n$, the pair has this symmetry. In particular, $E[\sigma + \ell \mid K = n] = n$. Since $\sigma \leq \ell$ and $0 \leq n - \sigma \leq n$, it follows that

$$E[n - \sigma \mid K = n] = E[\ell \mid K = n] \geq n/2, \quad P(n - \sigma \geq n/4 \mid K = n) \geq 1/3. \quad (4.11)$$

Also $P(K \geq N/2) \geq 1/4$ for all large N . Hence the restriction

$$\Gamma = \{n - \sigma \geq N/8\}$$

has mixture mass $\beta \geq 1/12$. In the tuple,

$$n - \sigma = (\ell - \sigma) + (n - \ell) = \rho_r + |B|,$$

so Γ depends only on the fixed tuple. Apply Lemma 4.1 to this restricted endpoint measure, whose mass is $\beta \geq 1/12$. It gives probability at least $c(1 + \log f)/f^2$ in a disk of radius $C_1 \sqrt{N}/f$. On Γ we have $K \geq N/8$, so this radius is at most $\sqrt{8} C_1 \sqrt{K}/f$. Since $\| -X_K^* \|_2 = \| X_K^* \|_2$, absorbing the fixed factor proves (4.5). ■

4.2. REMOVING TIES AND ABSORBING THE PAST.

LEMMA 4.3 (Localization with a unique favorite). *For the same geometric clock, uniformly for $2 \leq f \leq N^{1/8}$,*

$$P \left(|\mathcal{F}_K| = 1, R_K^+ \leq C \frac{\sqrt{K}}{f} \right) \geq c \frac{1 + \log f}{f^2}. \quad (4.12)$$

The conclusion remains valid, up to a universal change of c , after restricting the output clock to

$$aN \leq K \leq N \log N, \quad a = \frac{c}{8} \frac{1 + \log f}{f^2}. \quad (4.13)$$

Proof. For every atom counted in (4.5), insert immediately after the last visit to its priority-selected favorite x the fixed loop

$$x, x + e_1, x, x + e_2, x. \quad (4.14)$$

The local time at x rises by two, those at the two neighbors rise by one, and all other local times are unchanged. Thus x is the unique favorite of the new path. The old suffix avoids x , so in the image the last visit to the unique favorite is the final vertex of (4.14). Removing the four preceding prescribed steps is therefore the unique inverse. On geometrically weighted path atoms the image-to-source weight ratio is $z^4/4^4$, bounded below for z close to one. This proves (4.12).

Now $P(K < aN) \leq 2a$, while $P(K > N \log N) \leq \exp\{-(\log N)/2\}$. With the constant in (4.13) chosen after the constant in (4.12), these discarded masses total at most half the right-hand side of (4.12). Four added steps are absorbed by changing the universal constants. ■

LEMMA 4.4 (Adding an inherited local-time profile). *Let $H \geq 0$ be any finitely supported integer profile and let $\ell \geq 0$ be a second integer profile with a unique maximizer y . Every maximizer of $H + \ell$ lies in $\text{supp } H \cup \{y\}$.*

Proof. For every $x \notin \text{supp } H \cup \{y\}$, uniqueness of the maximum of ℓ gives

$$(H + \ell)(x) = \ell(x) < \ell(y) \leq (H + \ell)(y).$$

Such a site cannot maximize $H + \ell$. ■

4.3. INFINITELY MANY SUCCESSFUL BLOCKS.

PROPOSITION 4.5 (Recurrence at the critical scale). *Almost surely,*

$$\liminf_{n \rightarrow \infty} \frac{R_n^+}{\sqrt{n/\log n}} = 0. \quad (4.15)$$

Consequently, for every $\gamma \leq \frac{1}{2}$,

$$\liminf_{n \rightarrow \infty} \frac{R_n^+}{a_\gamma(n)} = 0. \quad (4.16)$$

Proof. Step 1. Choose independent fresh segments.

Fix $\varepsilon > 0$. We need fresh segments long enough that the old range is small compared with $\sqrt{K_k/\log K_k}$, while each segment still ends before the next deterministic block. Choose J_1 sufficiently large and define

$$J_{k+1} = J_k + 10 \log J_k, \quad s_k = \lfloor e^{J_k} \rfloor,$$

Here J_k is the logarithmic block time. For the fresh block beginning at s_k , choose the geometric time scale and localization divisor as

$$N_k = \frac{e^{J_{k+1}}}{J_{k+1}^2}, \quad f_k = C\varepsilon^{-1} \sqrt{J_{k+1}}.$$

Choose C large enough that localization gives the radius below; for all large k , $2 \leq f_k \leq N_k^{1/8}$. Apply Lemma 4.3 to the fresh increment walk, with an independent geometric clock with parameter $1/N_k$. It gives a unique fresh favorite Y_k satisfying

$$\|Y_k\| \leq \frac{\varepsilon}{8} \sqrt{\frac{K_k}{J_{k+1}}} \quad (4.17)$$

with conditional probability at least

$$d_k = c_\varepsilon \frac{\log J_{k+1}}{J_{k+1}}, \quad (4.18)$$

even after

$$\frac{d_k}{8} N_k \leq K_k \leq J_{k+1} N_k. \quad (4.19)$$

Since $e^{J_{k+1}} = e^{J_k} J_k^{10}$ and $J_{k+1} \sim J_k$, (4.18)–(4.19) give, uniformly over the clock restriction,

$$c'_\varepsilon e^{J_k} J_k^7 \log J_k \leq K_k \leq \frac{e^{J_{k+1}}}{J_{k+1}}$$

for all sufficiently large k . The lower bound makes the fresh segment dominate the past, while the upper bound keeps it within its deterministic block: $K_k/s_k \rightarrow \infty$ and $s_k + K_k < s_{k+1}$. The singleton estimate already includes the four inserted steps.

Step 2. Include the inherited profile.

Let

$$H_k(x) = \sum_{j=0}^{s_k-1} \mathbf{1}_{\{S_j=x\}}, \quad \ell_k(z) = \sum_{i=0}^{K_k} \mathbf{1}_{\{S_{s_k+i}-S_{s_k}=z\}}.$$

The old profile stops before s_k , and the fresh profile includes its time-zero visit. Hence, for every $x \in \mathbb{Z}^2$,

$$H_k(x) + \ell_k(x - S_{s_k}) = \sum_{j=0}^{s_k-1} \mathbf{1}_{\{S_j=x\}} + \sum_{i=0}^{K_k} \mathbf{1}_{\{S_{s_k+i}=x\}} = L_{s_k+K_k}(x).$$

On a successful block, ℓ_k has the unique maximizer Y_k . Lemma 4.4, applied after translation by S_{s_k} , therefore gives, at $t_k = s_k + K_k$,

$$\mathcal{F}_{t_k} \subset \{S_j : 0 \leq j < s_k\} \cup \{S_{s_k} + Y_k\}.$$

Part (2) of Lemma 3.1 and Borel–Cantelli give

$$\max_{j \leq s_k} \|S_j\| \leq \sqrt{s_k} J_k$$

eventually. The clock bounds above give

$$\frac{s_k J_k^2 J_{k+1}}{K_k} = O_\varepsilon \left(\frac{J_k^{-4}}{\log J_k} \right), \quad \log(s_k + K_k) = J_{k+1} + O_\varepsilon(\log J_{k+1})$$

uniformly over (4.19). Consequently,

$$\sqrt{s_k} J_k = o\left(\sqrt{\frac{K_k}{J_{k+1}}}\right), \quad \log(s_k + K_k) \sim J_{k+1}, \quad s_k = o(K_k). \quad (4.20)$$

On every sufficiently late successful block, the favorite-set inclusion above now gives

$$\begin{aligned} R_{t_k}^+ &\leq \max_{j \leq s_k} \|S_j\|_2 + \|Y_k\|_2 \\ &\leq \sqrt{s_k} J_k + \frac{\varepsilon}{8} \sqrt{\frac{K_k}{J_{k+1}}} \leq \frac{\varepsilon}{2} \sqrt{\frac{t_k}{\log t_k}}. \end{aligned}$$

Here (4.20) makes the first term negligible and identifies the fresh and full time scales. Thus every full favorite at t_k lies in

$$B\left(0, \frac{\varepsilon}{2} \sqrt{\frac{t_k}{\log t_k}}\right) \quad (4.21)$$

for all sufficiently large successful k .

Step 3. Obtain infinitely many witness times.

The recursion gives $J_k \sim 10k \log k$, so $d_k \asymp_\varepsilon k^{-1}$. In particular,

$$\sum_k d_k = \infty. \quad (4.22)$$

The fresh increments occupy disjoint deterministic blocks. On the product space enlarged by the independent clocks and priorities, let E_k be the fresh singleton-success event (4.17), with clock restriction (4.19). The clock restriction ensures that E_k uses only increments within the k th deterministic block. Its conditional probability, given the walk and auxiliary variables in earlier blocks, is at least d_k . Lévy's conditional Borel–Cantelli lemma and (4.22) imply that E_k occurs infinitely often almost surely. The old-range bound and the deterministic comparisons (4.20) hold eventually almost surely; on their intersection, every sufficiently late E_k implies (4.21). Thus (4.21) occurs infinitely often. Fubini removes the auxiliary clocks and priorities: for almost every walk path, their probability of producing infinitely many witness times is one, so that path itself has infinitely many such integer witness times.

Intersecting over rational $\varepsilon > 0$ proves (4.15). Finally,

$$\frac{a_{1/2}(n)}{a_\gamma(n)} = (\log n)^{\gamma-1/2} \leq 1$$

eventually for $\gamma \leq 1/2$, which proves (4.16). $\blacksquare$

5. Continuum barriers and lattice traversal counts

Traversal counts separate the large-scale accumulation of excursions from occupation inside the last annulus. We first record the continuum barrier estimates in the form needed below. We then compare rare upper events by multiplying one-step lattice probabilities. For the lower event, a single strong coupling transfers the existence of a suitable Brownian center. The two comparisons have different purposes: the first preserves probabilities on their exponential scale, while the second preserves a positive probability of existence. Together they give the three lattice inputs used later: an upper bound on barrier crossings, a joint bound for a barrier-respecting traversal endpoint, and a positive probability of

finding a lower root. The first two enter the localized upper bound; the third enters the lower bound for the disk maximum.

5.1. SPHERICAL TRAVERSAL ESTIMATES. We use Rosen's spherical traversal construction [13]. Write d for geodesic distance on the sphere. We denote the geodesic radii by ϑ_j (Rosen's h_j), reserving h for the terminal logarithmic radius $\log M$. The upper and lower arguments use separate fixed choices of the initial radius. We keep the customary notation r_0 in each argument and state the change explicitly; the two choices are never used in the same pathwise comparison.

Let X be spherical Brownian motion started at v , stopped at its first hit τ of $\partial B_d(v, r_*)$. When defining source stopping times we extend it by an independent Brownian continuation; the safety event below will imply that every portion actually used lies before τ . Put

$$r_j = r_0 e^{-j}, \quad \vartheta_j = 2 \arctan(r_j/2). \quad (5.1)$$

For the upper estimates, choose the fixed radii as in [13, Section 2, (2.2)], in particular $2r_* \leq \vartheta_0$. Let F_j be a maximal $d_0 \vartheta_j$ -separated set, augmented if necessary to a $d_0 \vartheta_j$ -cover; thus $|F_j| \asymp e^{2j}$. For $y \in F_L$, $T_{y,j}$ is the number, before τ , of completed excursions from $\partial B_d(y, \vartheta_{j-1})$ to $\partial B_d(y, \vartheta_j)$, with the initial and final incomplete pieces omitted. Its source shell is

$$k_y = \inf\{k : y \notin B_d(v, \vartheta_k)\}. \quad (5.2)$$

Pairs have separation shell k when $2\vartheta_k < d(y, y') \leq 2\vartheta_{k-1}$.

The barriers have an affine part and a buffer determined by the distance to the nearer endpoint. Set $\rho_L = 2 - (\log L)/L$ and define

$$\alpha_{z,\pm}(l) = \rho_L l + z \pm (l \wedge (L - l))^{1/4}. \quad (5.3)$$

Fix an admissible $z \geq z_0$ in Rosen's upper argument. The upper first-crossing event $U(y, l, z)$ means

$$\sqrt{2T_{y,j}} \leq \alpha_{z,+}(j) \quad (k_y < j < l), \quad \sqrt{2T_{y,l}} > \alpha_{z,+}(l).$$

Rosen's representative construction is deterministic. With fixed tie-breaking, every fine center is sent to its closest point of the predetermined coarser net at scale $\vartheta_{l+\frac{1}{4}(l \wedge (L-l))^{1/4}}$; equations (2.26)–(2.33) give the buffered ball inclusions and domination of its traversal count. Proposition 2.5, specifically equations (2.34)–(2.35), then proves the complete estimates

$$\#\{\text{representatives with source } k\} \leq C \exp\{2(l - k) + \frac{1}{2}(l \wedge (L - l))^{1/4}\}, \quad (5.4)$$

$$P(U(y, l, z)) \leq C \exp\{-2l - (l \wedge (L - l))^{1/4}\} e^{-2z}. \quad (5.5)$$

They hold pointwise for $1 \leq k \leq l \leq L - (4 \log L)^4$. Only the choice $F_L \cap B_d(v, \vartheta_{\log L})^c$, and not the proof of (5.4)–(5.5), imposed $k \leq \log L$. Hence, for any $s \leq C_0 \log L$, summing just the representatives whose source shell is at least s gives the bound below. Indeed, the product of (5.4) and (5.5) is $C e^{-2k} e^{-(l \wedge (L-l))^{1/4}/2} e^{-2z}$, and

$$\sum_{k \geq s} e^{-2k} \sum_{l=k}^{L-(4 \log L)^4} e^{-(l \wedge (L-l))^{1/4}/2} \leq C e^{-2s}.$$

Here the inner sum is uniformly bounded by the two endpoint tails. Consequently,

$$P(\exists y : k_y \geq s, \exists l \leq L - (4 \log L)^4 : U(y, l, z)) \leq C e^{-2s+o(\log L)} e^{-2z}. \quad (5.6)$$

5.1.1. *The endpoint estimate for complete centered excursions.* The endpoint calculation must be applied to a complete centered excursion forest. To specify that law, start spherical Brownian motion at v with $\vartheta_{k_y} \leq d(v, y) < \vartheta_{k_y-1}$ and continue it until its first exit from $B_d(y, \vartheta_0)$. Let $T_{y,j}^c$ count its completed traversals at level j , and let $\mathcal{G}_{y,L,z}^c$ require the upper barrier in (5.3) from $m = k_y + 1$ through $L - 1$. Put $Y_y^c = \sqrt{2T_{y,L}^c}$ and $B_L = \rho_L L + z$. Uniformly for $k_y \leq C_0 \log L$, every fixed $\epsilon_0 > 0$, $\epsilon_0 L \leq q \leq B_L + 1$, and fixed z , the source law in [13, (2.8)] and its Appendix Theorem 9.1(a) give

$$P(\mathcal{G}_{y,L,z}^c, Y_y^c \in [q, q+1]) \leq C(k_y + 1)^C (2 + B_L - q) L^{-2} e^{-q^2/(2L)}. \quad (5.7)$$

This assertion concerns the centered stopping rule just defined. We do not identify the descendants of a population stopped at the off-center boundary $\partial B_d(v, r_*)$ with an independent Galton–Watson forest. The later lattice endpoint estimate will be proved directly for its centered continuation.

The following version of [13, Appendix, Theorem 9.1(a)] is sufficient: if T is critical Galton–Watson with $P(\xi = m) = 2^{-m-1}$, $T_0 = x^2/2$, and

$$\begin{aligned} \delta > 0, \quad C \geq 0, \quad \eta > 1, \quad \epsilon \in (0, 1/2), \\ 2 \leq x, y \leq \eta L, \quad x^2/2 \in \mathbb{N}, \\ 0 \leq x \leq a, \quad 0 \leq y \leq b, \end{aligned}$$

$$\begin{aligned} P(\sqrt{2T_j} \leq a + (b-a)j/L + C(j \wedge (L-j))^{1/2-\epsilon}, \quad 1 \leq j < L; \\ \sqrt{2T_L} \in [y, y+\delta]) \leq C' \frac{(1+a-x)(1+b-y)}{L} \sqrt{\frac{x}{yL}} e^{-(x-y)^2/(2L)}. \end{aligned} \quad (5.8)$$

Write $m = k_y + 1$ and $N = L - m$. Conditional on $T_{y,m}^c = t$, the t complete source excursions give the critical geometric Galton–Watson chain in (5.8), started from t . Indeed each excursion from level m to level $m - 1$ is explored completely before the centered path can exit at level 0; its neighboring logarithmic side choices have probability $1/2$ independently of the entrance angle. The logarithmic gambler–ruin calculation (2.8) gives

$$P(T_{y,m}^c \geq t) \leq C m e^{-t/m},$$

and hence, after grouping the integer values of t for which $x = \sqrt{2t}$ lies in a unit band,

$$P(\sqrt{2T_{y,m}^c} \in [x, x+1]) \leq C m^C e^{-x^2/(2m)}. \quad (5.9)$$

On $\mathcal{G}_{y,L,z}^c$, $x \leq a := \alpha_{z,+}(m) = O(m)$. Apply (5.8) over the remaining N generations with this a and with $b = B_L + 1$. The needed deterministic barrier inclusion follows from

$$((m+j) \wedge (L-m-j))^{1/4} \leq (1-j/N)m^{1/4} + C(j \wedge (N-j))^{1/4}; \quad (5.10)$$

for $j \leq N/2$ use $(m+j)^{1/4} \leq m^{1/4} + j^{1/4}$ and $m^{1/4}j/N \leq j^{1/4}$, while for $j > N/2$ use the right-end distance directly. Thus every parameter in (5.8) has been specified, including $x \leq a$, $q \leq b$, and $x, q \leq \eta N$.

The sole positive start not covered by the hypothesis $x \geq 2$ is $t = 1$. Couple a process from two ancestors as the sum of two independent one-ancestor processes. On the event, of probability $1/2$, that the second ancestor has no child in the first generation, the total process agrees from generation 1 onward with the first process. The barrier in (5.8) is imposed precisely at generations $1, \dots, N-1$, so the $t = 1$ contribution is at most twice the same event for initial population two, to which (5.8) applies. The case $t = 0$ cannot have the positive endpoint $q \geq \epsilon_0 L$.

The source-height cost and the descendant cost combine through the identity

$$\frac{x^2}{m} + \frac{(q-x)^2}{N} = \frac{q^2}{L} + \frac{L(x-mq/L)^2}{mN}. \quad (5.11)$$

In particular, their product always contains $e^{-q^2/(2L)}$; the remaining exponential factor is at most one. Since $q \geq \epsilon_0 L$, $N \asymp L$, and $x \leq Cm$, the square-root factor in (5.8), together with its explicit N^{-1} , is at most $Cm^{1/2}L^{-2}$. Multiplying (5.9) by (5.8) and summing the $O(m)$ possible unit source-height bands therefore gives

$$P(\mathcal{G}_{y,L,z}^c, Y_y^c \in [q, q+1]) \leq Cm^C(2+B_L-q)L^{-2}e^{-q^2/(2L)},$$

which is (5.7). All source and band-counting factors are absorbed in m^C .

No estimate of the form (5.7) is asserted at $q = 0$: extinction there has order-one probability. In the lattice upper bound below, $q < \epsilon_0 L$ is instead disposed of by the saturated terminal Chernoff factor. We shall also use the following truncated version. Set $L = J - h$ and $N = L - m$, with $m \leq C_0 \log J$. For the barrier

$$B_l = \rho_J l + z + (l \wedge (J-l))^{1/4},$$

the estimate (5.7) holds using B_l in the no-crossing event and B_L on its right-hand side, for $\epsilon_0 L \leq q \leq B_L + 1$. To verify the only new point, put $\phi(l) = (l \wedge (J-l))^{1/4}$. Its two Hölder bounds from m and L give

$$\phi(m+j) \leq (1-j/N)\phi(m) + (j/N)\phi(L) + C(j \wedge (N-j))^{1/4}. \quad (5.12)$$

The affine part $\rho_J l + z$ interpolates exactly, so (5.8) applies with the arbitrary endpoints $a = B_m, b = B_L + 1$. In particular, no ρ_L -barrier theorem is being assumed.

5.1.2. A fixed number of source excursions. For the lower event, now choose r_0 sufficiently small that $4\vartheta_{-1} < r_*$, as in [13, Section 3]. The preceding upper estimates retain their separate choice of radii. Fix the source parameter $x_* > 0$ from that section so that

$$m_0 := x_*^2 \in \mathbb{N}. \quad (5.13)$$

(This x_* is not the variable called x in Appendix Theorem 9.1; the latter equals $\sqrt{2m_0}$ when that theorem is started with m_0 particles.) Put

$$F_L^0 = F_L \cap \{\vartheta_1/40 \leq d(v, y) \leq \vartheta_1/20\}.$$

This fixed annulus, instead of Rosen's full ball, retains a fixed positive area fraction; the one- and two-point sums in Rosen's proof are therefore unchanged up to constants, and every source index is $O(1)$.

We spell out the superscript in Rosen (3.3), because it is important for the stopping-time comparison below. Write $\Gamma_{y,j} = \partial B_d(y, \vartheta_j)$ and set

$$\begin{aligned} A_{y,1} &= H_{\Gamma_{y,1}}, & B_{y,i} &= \inf\{t > A_{y,i} : X_t \in \Gamma_{y,0}\}, \\ A_{y,i+1} &= \inf\{t > B_{y,i} : X_t \in \Gamma_{y,1}\}, & 1 \leq i < m_0. \end{aligned} \quad (5.14)$$

The intervals $[A_{y,i}, B_{y,i}]$, and only these intervals, are the first m_0 source excursions; the intervening intervals $[B_{y,i}, A_{y,i+1}]$ are connectors and are not counted. Set $T_{y,1}^{1,m_0} = m_0$. For $l \geq 2$, define inside each source interval

$$\begin{aligned} C_{i,l,1} &= \inf\{t \geq A_{y,i} : X_t \in \Gamma_{y,l-1}\}, \\ D_{i,l,q} &= \inf\{t > C_{i,l,q} : X_t \in \Gamma_{y,l}\}, & C_{i,l,q+1} &= \inf\{t > D_{i,l,q} : X_t \in \Gamma_{y,l-1}\}, \end{aligned}$$

and put

$$T_{y,l}^{1,m_0} = \sum_{i=1}^{m_0} \#\{q : D_{i,l,q} < B_{y,i}\}. \quad (5.15)$$

Thus Rosen's abbreviation is exactly $T_{y,l}^1 = T_{y,l}^{1,x_*^2}$, not a count in one source excursion. The strong Markov property and logarithmic gambler's ruin make $(T_{y,1+j}^{1,m_0})_{j \geq 0}$ a critical Galton–Watson chain with m_0 ancestors and offspring law $P(\xi = r) = 2^{-r-1}$, $r \geq 0$.

Define $\widehat{I}_{y,z}$ by

$$\sqrt{2T_{y,l}^{1,m_0}} \leq \alpha_{z,-}(l), \quad 1 \leq l < L, \quad \sqrt{2T_{y,L}^{1,m_0}} \geq \rho_L L + z. \quad (5.16)$$

For the outward excursions at scale k , let $\theta_{k,i}$ be their angular increments and let ν_k be the exit-angle law from radius r_k to r_{k-1} . If

$$N_{k,a} = \left\lfloor \frac{(\rho_L k + z - a + 1)^2}{2} \right\rfloor,$$

set $N_k = N_{k,a}$ on $\sqrt{2T_{y,k}^{1,m_0}} \in [\rho_L k + z - a, \rho_L k + z - a + 1]$, and

$$W_{y,k}(N_k) = \left\{ d_{W^1} \left(N_k^{-1} \sum_{i \leq N_k} \delta_{\theta_{k,i}}, \nu_k \right) \leq \frac{c_0 \log(L-k)}{2\sqrt{N_k}} \right\}. \quad (5.17)$$

With $L^+ = L - (500 \log L)^4$ and Rosen's fixed d_* , the event is

$$I_{y,z} = \widehat{I}_{y,z} \cap \bigcap_{k=L^+}^{L-d_*} W_{y,k}(N_k). \quad (5.18)$$

Rosen Lemmas 3.3–3.4 state completely that, uniformly for $0 \leq z \leq \log L$,

$$P(I_{y,z}) \asymp (1+z)e^{-2L}e^{-2z}, \quad (5.19)$$

and, for a pair in separation class k ,

$$P(I_{y,z} \cap I_{y',z}) \leq C(1+z)e^{-4L+2k}e^{-2z}e^{-c(k \wedge (L-k))^{1/4}}. \quad (5.20)$$

The convention for $L - d_* \leq k < L$ is the one-center bound from that lemma. Fix z_0 in Rosen's admissible range. After imposing the source-safety event below we will choose a fixed C_* and use the truncated events

$$I_y^\circ := I_{y,z_0} \cap \{\rho_L L + z_0 \leq \sqrt{2T_{y,L}^{1,m_0}} \leq \rho_L L + z_0 + C_*\} \quad (5.21)$$

The notation in (5.21) is provisional until C_* is chosen after safety has been proved. This order matters: because X was extended after τ , the inequality $T_{y,L}^{1,m_0} \leq T_{y,L}^\tau$ can fail before safety is imposed.

5.1.3. Keeping the source excursions before the stopping time. The lower event was defined using a Brownian continuation after τ . We must therefore force the entire source forest to finish before τ without losing its probability on the scale e^{-2L} . Let S_y require every connector $[B_{y,i}, A_{y,i+1}]$, $i < m_0$, to hit $\Gamma_{y,1}$ before the larger circle $\Gamma_{y,-1}$. In the centered logarithmic cylinder, the unrestricted connector endpoint subdensity and the safely killed one, relative to Haar measure, are respectively

$$q_\infty(u) = \frac{\sinh 1}{\cosh 1 - \cos u},$$

$$q_{\text{sf}}(u) = \frac{1}{2} + 2 \sum_{n \geq 1} \frac{\sinh n}{\sinh(2n)} \cos(nu) = \frac{\pi}{2} \sum_{k \in \mathbb{Z}} \operatorname{sech}\left(\frac{\pi}{2}(u + 2\pi k)\right). \quad (5.22)$$

Both are continuous and strictly positive, so $\kappa_* := \min_u q_{\text{sf}}(u)/q_\infty(u) > 0$. Thus the safe connector kernel dominates κ_* times the unrestricted kernel pointwise. Iterating the strong Markov factorization, with the whole later path functional left in the last factor, gives

$$P(I_{y,z} \cap S_y) \geq \kappa_*^{m_0-1} P(I_{y,z}). \quad (5.23)$$

The pointwise kernel bound can be integrated against every nonnegative function of the later path, so it preserves the barrier and angular conditions in $I_{y,z}$ together. Before endpoint truncation put

$$\tilde{Z}_L^{\text{sf}} := \sum_{y \in F_L^0} \mathbf{1}_{I_{y,z_0} \cap S_y}.$$

Equation (5.23) preserves its first moment, whereas $I_{y,z_0} \cap S_y \subset I_{y,z_0}$ preserves the pair upper bound (5.20). Since $|F_L^0| \asymp e^{2L}$, the one-center lower bound gives $E \tilde{Z}_L^{\text{sf}} \geq c$. For the second moment, group distinct pairs by separation shell k . There are at most Ce^{4L-2k} pairs in shell k , so (5.20) gives

$$E[(\tilde{Z}_L^{\text{sf}})^2] \leq C + C \sum_{k=0}^{L-d_*} e^{-c(k \wedge (L-k))^{1/4}} \leq C.$$

The last d_* shells contribute only $O(1)$ by the one-center bound. Paley–Zygmund now gives

$$E \tilde{Z}_L^{\text{sf}} \leq C, \quad P(\tilde{Z}_L^{\text{sf}} > 0) \geq c_1 > 0. \quad (5.24)$$

Rosen chooses r_0 so that $4\vartheta_{-1} < r_*$. Since $d(v, y) \leq \vartheta_1/20$, there is a fixed $\chi > 0$ such that

$$B_d(y, \vartheta_{-1}) \subset B_d(v, r_* - \chi). \quad (5.25)$$

Every source block and every connector on S_y lies in the left side, hence ends before τ . This is the exact stopped-source certificate used in the coupling below. In particular, on S_y ,

$$T_{y,L}^{1,m_0} \leq T_{y,L}^\tau. \quad (5.26)$$

Rosen's Theorem 1.6 gives the following upper tail on F_L ; extending its bound to bounded nonnegative z only enlarges the constant:

$$P\left(\sup_{y \in F_L} \sqrt{2T_{y,L}^\tau} \geq \rho_L L + z\right) \leq C(1+z)e^{-2z}, \quad 0 \leq z \leq \log L. \quad (5.27)$$

Choose C_* so large that the right side at $z = z_0 + C_*$ is below $c_1/2$, and now define I_y° by (5.21). Equations (5.24)–(5.27) give, with

$$Z_L^{\text{sf}} := \sum_{y \in F_L^0} \mathbf{1}_{I_y^\circ \cap S_y},$$

$$EZ_L^{\text{sf}} \leq C, \quad P(Z_L^{\text{sf}} > 0) \geq c_1/2. \quad (5.28)$$

The constant C_* and the safe count are now fixed.

LEMMA 5.1 (Continuum barrier inputs). *In the upper choice of radii, the first-crossing estimate (5.6) holds. For the complete centered traversal process, the endpoint estimate (5.7) holds, including the truncated barrier described in (5.12). In the lower choice of radii, on*

$I_y^\circ \cap S_y$ all first m_0 source excursions finish before τ , and the number of centers with this event satisfies (5.28).

Proof of Lemma 5.1. The representative estimates are the packing and one-center bounds of [13, Proposition 2.5, (2.34)–(2.35)], summed over the indicated source shells. The complete centered source calculation above proves (5.7), and (5.12) checks the change of barrier at a truncated horizon. The lower construction combines [13, Lemmas 3.3–3.4] with the connector-kernel domination (5.23) and the terminal truncation. All the first- and second-moment assertions (5.19)–(5.28) remain in Rosen’s original spherical geometry. In particular, we do *not* infer an event-level Euclidean two-center theorem from Rosen’s occupation-measure argument in Section 8. The next lemma instead maps the actual spherical interfaces into the plane, so the spherical event and its pair law are preserved exactly until the one-sided lattice coupling is made. ■

5.2. COMPARISON OF SMOOTH INTERFACE KERNELS. The word comparison requires relative errors for the probabilities of successive radial decisions. We obtain them from a one-interface estimate: compare harmonic boundary payoffs, take the payoff equal to one, and then multiply the resulting row-mass bounds along a complete word.

Fix $\alpha \in (0, 1)$. Let $(\Gamma_R^-, \Gamma_R^0, \Gamma_R^+)$ range over triples of disjoint Jordan curves which, after translation, rotation, and scaling by R^{-1} , together with their tubular-coordinate charts, belong to a bounded $C^{4,\alpha}$ family. Assume consecutive curves are separated by distances comparable to R , and that a specified conformal coordinate sends each triple to three concentric logarithmic levels. The interfaces used below are circles or uniformly Möbius-transformed circles, so this hypothesis entails no loss.

For a boundary function put

$$\|f\|_{4,R} := \sum_{j=0}^4 R^j \|\partial_\tau^j f\|_\infty + R^{4+\alpha} [\partial_\tau^4 f]_\alpha. \quad (5.29)$$

A lattice starting state is a directed edge whose endpoints lie on opposite sides of Γ_R^0 ; the lattice query starts from its post-crossing endpoint and the Brownian query from the normal projection of that endpoint onto Γ_R^0 . The side of a vertex is determined by membership in the bounded component of the complement (vertices lying on the curve are assigned to the exterior). Let Q_R^B be the Brownian selected-side subprobability kernel, stopped at first contact with either competing curve, and let Q_R^S be the lattice selected-side subprobability kernel, stopped at the first directed edge whose endpoint-side indicators differ for one of the two competing curves. A tangency, or any intersection of an interpolated edge whose two endpoints remain on the same side, is not a lattice crossing. Write π_0, π_+ for normal projection of the digital source and selected target states. For a function f on the selected target curve and a function g on the middle curve, pull them back to the corresponding digital states by

$$U_+ f = f \circ \pi_+, \quad U_0 g = g \circ \pi_0.$$

LEMMA 5.2 (Comparison of interface kernels). *As an estimate from functions on the selected target curve to functions on the digital source-edge space,*

$$\|Q_R^S U_+ f - U_0(Q_R^B f)\|_\infty \leq C R^{-1} \|f\|_{4,R}. \quad (5.30)$$

If the Brownian side probability $p_R = Q_R^B 1$ is constant on the middle curve and lies in $[c, 1 - c]$, then, uniformly in the starting angle,

$$Q_R^S 1 = p_R(1 + O(R^{-1})) \quad (5.31)$$

and the corresponding row-normalized angular kernels differ by $O(R^{-1})$ on C^4 tests.

Consequently, let a chronological radial word consist of N ordinary decisions between neighboring concentric interfaces at scales $R_i \geq M$. The next queried triple may depend on the already exposed radial prefix, but not on an unexposed angular mark. For a fixed word, let E_i be the directed-edge state space after its first i letters and let $K_{i,w_{i+1}}^S(e, de')$ be the subkernel realizing the next prescribed side. If p_i is the corresponding Brownian side probability, then, for every current edge,

$$e^{-C/R_i} p_i \leq K_{i,w_{i+1}}^S 1(e) \leq e^{C/R_i} p_i. \quad (5.32)$$

After summing all compatible directed-edge refinements,

$$e^{-C \sum_i R_i^{-1}} P^B(w) \leq P^S(w) \leq e^{C \sum_i R_i^{-1}} P^B(w). \quad (5.33)$$

The same inequality holds after integrating the initial edge against any entrance law and after summing a disjoint family of complete words. A radial-prefix stopping rule is padded by cemetery letters of row mass one. No assertion is made after conditioning on an edge list generated later by the word. Source-root and source-death letters are handled explicitly in Lemma 5.3.

Proof of Lemma 5.2. There are two errors to control: the discrete walk accumulates a small harmonicity defect before exit, and its crossing endpoints lie within one lattice unit of the smooth interfaces. The first error is $O(R^{-2})$; the second gives the stated $O(R^{-1})$ bound.

Let Ω_R be the annulus between the two competing curves and let u be the continuum Dirichlet solution with boundary data f on the selected component and zero on the other. Uniform rescaled boundary regularity gives

$$\|D^j u\|_{L^\infty(\overline{\Omega}_R)} \leq C R^{-j} \|f\|_{4,R}, \quad 0 \leq j \leq 4. \quad (5.34)$$

The compact rescaled family has a tubular radius ρR . In a signed normal chart $\Phi(s, t)$, with $t \geq 0$ on Ω_R , choose fixed numbers $c_1, \dots, c_5$ satisfying

$$\sum_{a=1}^5 c_a (-a)^q = 1, \quad q = 0, 1, 2, 3, 4,$$

and, for $-\rho R/5 < t < 0$, set

$$\tilde{u}(\Phi(s, t)) = \sum_{a=1}^5 c_a u(\Phi(s, -at)).$$

Do this in the two disjoint collars and retain $\tilde{u} = u$ on $\overline{\Omega}_R$. The five matching identities make $\tilde{u}$ a C^4 continuation through the boundary, with the same scaled derivative bounds as (5.34).

Let Ω_R be understood with the same vertex-side convention and set

$$T = \inf\{n \geq 1 : S_{n-1} \in \Omega_R, S_n \notin \Omega_R\}.$$

Equivalently, T is the first side-changing directed edge across one of the two competing components; same-side contacts of the interpolated edge are ignored. For $n < T$, S_n lies in the open annulus or on its inner boundary (with the exterior convention above); it cannot

lie on the outer boundary. At an open-annulus point $\Delta\tilde{u}(S_n) = \Delta u(S_n) = 0$, while at an inner-boundary lattice vertex the C^4 matching and continuity from the annulus give

$$\Delta\tilde{u}(S_n) = \lim_{\Omega_R \ni x \rightarrow S_n} \Delta u(x) = 0.$$

Thus Taylor expansion is legitimate on the full four-neighbor stencil and its quadratic term vanishes in either case. The linear and cubic terms cancel by symmetry. Thus

$$|(P - I)\tilde{u}(S_n)| \leq CR^{-4}\|f\|_{4,R}, \quad n < T. \quad (5.35)$$

For large R , the two boundary components are more than two lattice spacings apart and each has tubular radius ρR . Hence the exit edge crosses exactly one component, its post-crossing endpoint is within one lattice spacing of that component and has a unique normal projection. Moreover T is no larger than the exit time of a fixed CR -disk containing the annulus. Optional stopping of $\|S_n - z_*\|^2 - n$ gives $ET \leq CR^2$, so the accumulated defect in (5.35) is $O(R^{-2})\|f\|_{4,R}$. The terminal endpoint is within one lattice unit of the uniquely crossed component; its value under $\tilde{u}$ differs from the projected boundary payoff by $O(R^{-1})\|f\|_{4,R}$. The post-crossing starting endpoint and its projection onto Γ_R^0 have the same error. Optional stopping of $\tilde{u}(S_{n \wedge T})$ minus its defect sum proves (5.30). Taking $f = 1$, using $p_R \in [c, 1 - c]$, and dividing by the row mass proves (5.31). Notice that the continuation is used only to make a boundary-crossing stencil legal; harmonicity, or its boundary-continuity consequence, is invoked only at a live vertex.

For concentric interfaces, (5.31) also follows from the logarithmic annulus estimate in part (3) of Lemma 3.1, taken from [9, Lemma 2.1, (2.5)]. At fixed logarithmic gaps both side probabilities are bounded away from zero, so the additive error becomes a relative error.

Finally fix a prescribed radial word and let ν_i^S be the finite measure on its current directed-edge states after the first i letters, with every earlier edge integrated out. The radial prefix fixes the next interface and prescribed side before the current edge is integrated. Brownian conformal radial symmetry makes its row mass p_i independent of the angular endpoint, while (5.32) gives

$$e^{-C/R_i} p_i \|\nu_i^S\| \leq \|\nu_i^S K_{i,w_{i+1}}^S\| \leq e^{C/R_i} p_i \|\nu_i^S\|. \quad (5.36)$$

Forward induction proves (5.33). This remains valid for an adaptive exploration because the radial prefix determines its next query, and for a stopped exploration after adjoining the absorbing cemetery state. It is a forward marginal comparison, not a conditional law given future entrance points, and it never applies (5.30) to a rare future indicator. ■

5.3. TRANSFER OF TRAVERSAL EVENTS TO THE LATTICE. Fix $A, C_0 < \infty$ and a positive barrier intercept z in the admissible range above. The upper first-crossing estimate below is uniform for $0 \leq s \leq C_0 \log J$. Choose $h = h_J$ so that

$$\frac{h}{\log J} \longrightarrow \infty, \quad h = o(J), \quad M = e^h, \quad L = J - h + O(1). \quad (5.37)$$

For the lower comparison let v and r_* be Rosen's starting point and outer stopping radius. Fix stereographic coordinates σ with $\sigma(v) = 0$, put $R_* = 2 \tan(r_*/2)$, and dilate by $A_J = e^J/R_*$:

$$\Psi_J = A_J \sigma.$$

Thus the spherical stopped path becomes, up to an increasing time change, planar Brownian motion started at zero and stopped on $\partial B(0, e^J)$. For a spherical center y , let χ_y be the

rotated stereographic coordinate satisfying

$$\chi_y(y) = 0, \quad |\chi_y(x)| = 2 \tan(d(x, y)/2).$$

The lower-comparison physical interfaces are

$$\Gamma_{y,j} = \Psi_J\{x : |\chi_y(x)| = r_0 e^{-j}\}, \quad -1 \leq j \leq L. \quad (5.38)$$

They are Euclidean circles, although their Euclidean centers need not be the same. In the conformal coordinate $\chi_y \circ \Psi_J^{-1}$, they are exactly the concentric logarithmic levels used by Rosen. Their physical scale is comparable to e^{J-j} ; the $O(1)$ choice in $L = J - h + O(1)$ makes the deepest scale comparable to M .

The linearly interpolated lattice path identifies which smooth curve a lattice edge crosses, but every probabilistic cut is made at an integer edge-index stopping time. If $D_{y,j}$ is the bounded component inside $\Gamma_{y,j}$, then, for adjacent levels and after an outer hit η_i , set

$$\begin{aligned} \sigma_i &= \inf\{t > \eta_i : S_{t-1} \notin D_{y,j}, S_t \in D_{y,j}\}, \\ \eta_{i+1} &= \inf\{t > \sigma_i : S_{t-1} \in D_{y,j-1}, S_t \notin D_{y,j-1}\}. \end{aligned} \quad (5.39)$$

A state records the directed crossing edge; interpolation is used only to assign its projected χ_y -angle. Only portions belonging to the first m_0 source intervals $[A_{y,i}, B_{y,i}]$ in (5.14) contribute traversal counts or terminal bridges. Inter-source connectors contribute only their guarded coarse boundary decisions, never their interior occupation. The whole segment from σ_i to η_{i+1} , including further contacts with the inner interface, is one outward piece. Recursing inside each parent excursion gives the standard nested excursion tree. The lower *radial word* $W_y = (W_{y,1}, \dots, W_{y,m_0})$ is a fixed collection of m_0 centered words, one for each source interval, with all angles erased; the complete guard record additionally contains the finitely many connector boundary decisions. If $\ell_{y,i} \in \{1, -1\}$ records whether the connector after $B_{y,i}$ first hits $\Gamma_{y,1}$ or $\Gamma_{y,-1}$, write

$$\overline{W}_y = (W_{y,1}, \ell_{y,1}, W_{y,2}, \dots, \ell_{y,m_0-1}, W_{y,m_0}) \quad (5.40)$$

for the complete conditioned guard word. Its traversal coordinate $T_{y,j}^S$ is the sum over these words of completed crossings from $\Gamma_{y,j-1}$ to $\Gamma_{y,j}$. At the last fixed number of levels, the interfaces are replaced through one unit-width digital collar by concentric circles about the nearest lattice point to $\Psi_J(y)$, of radii M_y, eM_y with $M_y \in [cM, CM]$. We relabel M_y as M ; every later estimate is uniform under this fixed-factor change and $\log M_y = h + O(1)$. Define $\mathcal{R}_y$ by the chronological erasure map which replaces each completed terminal segment, from its inward directed crossing of radius M through its next outward directed crossing of radius eM , by the two endpoint edges. It retains $\overline{W}_y$, all radial letters and all crossing edges and projected angles outside the erased segments, but it does not retain an erased segment's duration, interior crossings, path, or occupation. Inward connector pieces outside these erased segments are exposed. This quotient sigma-field is the intrinsic coarse sigma-field.

The upper application is instead defined directly on the lattice and is entirely centered. Write $N = e^J$, $\tau_N = \tau_{D(0,N)}$, and, for each candidate cell center with $\|y\| < 2N$, set

$$\sigma_y = \tau_{D(y,3N)}. \quad (5.41)$$

The pathwise inclusions

$$D(0, N) \subset D(y, 3N), \quad \tau_N \leq \sigma_y, \quad L_{\tau_N}(x) \leq L_{\sigma_y}(x) \quad (5.42)$$

hold for every x . For each one-cell probability we decompose the extended path up to σ_y using only circles centered at y , and sum those probabilities only afterwards. In this upper

application, $T_{y,j}^S$ counts all completed level- j traversals before σ_y ; in the lower application it counts traversals within the first m_0 source intervals defined above. Replacing N by $3N$ changes every logarithmic index by $O(1)$. Use Rosen's formal horizon J , its barrier $\rho_J j + z + (j \wedge (J - j))^{1/4}$, and retain only interfaces through L . Let $\mathcal{U}_{J,s}$ be the union of these centered lattice upper first-crossing events, with source shell at least s , and let $\mathcal{G}_{y,L,z}^{S,J}$ be the corresponding no-crossing event through $L - 1$, and put

$$B_L^J = \rho_J L + z + (L \wedge (J - L))^{1/4}.$$

On the detailed range $\|y\| > M$, its exact source index is

$$u_y = \log \frac{3e^J}{\|y\|}, \quad k_y = \lceil u_y \rceil, \quad m_y = k_y + 1; \quad (5.43)$$

centers with $\|y\| \leq M$ belong to the separately treated near-start range. Thus $u_y \in (k_y - 1, k_y]$ and $m_y - u_y \in [1, 2)$. For the lower application, use the horizon- L slope ρ_L and its endpoint buffer. Let $I_y^{S,\circ}$ be the *radial* lattice version of (5.21): it retains the traversal barrier and the bounded terminal band but drops every angular screen. Angular screens are used only in the spherical Brownian certificate that supplies the second moment.

LEMMA 5.3 (Transfer of traversal events). *With the definitions above,*

$$P(\mathcal{U}_{J,s}) \leq J^{-2s/\log J + o(1)} + O(J^{-A}), \quad (5.44)$$

$$P(\mathcal{G}_{y,L,z}^{S,J}, \sqrt{2T_{y,L}^S} \in [a, a + 1]) \leq C(k_y + 1)^C (2 + B_L^J - a) L^{-2} e^{-a^2/(2L)}, \quad (5.45)$$

uniformly for $k_y \leq C_0 \log J$ and every fixed central range $\epsilon_0 L \leq a \leq B_L^J + 1$. No such assertion is made in the extinction range $a < \epsilon_0 L$. Most importantly, if

$$Z_L^S = \sum_{y \in F_L^0} \mathbf{1}_{I_y^{S,\circ}},$$

then

$$EZ_L^S \leq C, \quad P(Z_L^S > 0) \geq c. \quad (5.46)$$

Only the Brownian proof of (5.46) uses the two-center estimate (5.20). No two-center lattice estimate is claimed or used.

Proof of Lemma 5.3. There are two logically separate comparisons. Rare upper and end-point events are compared directly, word by word, using Lemma 5.2. The global strong coupling is used only in the safe direction “strict Brownian lower root implies loose lattice lower root.” At no point is a Brownian bad-subkernel estimate multiplied by an arbitrary future functional of the coupled lattice path. Throughout, L is the last retained logarithmic level. The upper barrier has formal horizon J and carries a superscript J ; the lower barrier has horizon L .

Step 1. Geometry and one global coupling.

The conformal image retains the original spherical excursion structure. Stereographic projection is conformal and sends spherical Brownian motion to planar Brownian motion under a continuous increasing time change. Moreover, by the definition of χ_y ,

$$\Psi_J(\partial B_d(y, \vartheta_j)) = \Gamma_{y,j}.$$

Thus hitting orders, the complete nested excursion tree, all traversal counts, and all χ_y -angular marks are literally Rosen's spherical objects. In particular, (5.20) will be applied only before any lattice comparison.

For the strong approximation we use Zaitsev, Estimates for the Strong Approximation in Multidimensional Central Limit Theorem, [14], Theorem 2 and Corollary 1. The form needed here is: for fixed dimension d , centered independent vectors with identity covariance whose laws lie in the analytic finite-exponential-moment class $\mathcal{A}_d(\tau)$ for a fixed $\tau \geq 1$, one can couple them with independent standard Gaussian vectors so that, for $\Delta_N = \max_{k \leq N} \|\sum_{i \leq k} (X_i - Y_i)\|$,

$$E \exp\{c\Delta_N/(\tau d^{7/2} \log^* d)\} \leq \exp\{Cd^{9/4+\alpha} \log^*(N/\tau^2)\},$$

and hence

$$P(\Delta_N > C \log N + x) \leq Ce^{-cx}, \quad x \geq 0, \quad (5.47)$$

for a fixed i.i.d. law. The planar walk increment multiplied by $\sqrt{2}$ has identity covariance and bounded support, so the hypotheses apply.

Take the deterministic time cap

$$T_J = e^{2J} J^{20}.$$

Lemma 3.1 and its Brownian analogue show that the walk or Brownian motion remains inside the outer disk beyond T_J with probability $e^{-cJ^{10}}$. By (5.47), after changing Brownian time by the covariance factor $1/2$, there is a coupling for which

$$\sup_{t \leq T_J} \|S_{2t}^{\text{lin}} - B_t\| \leq b_J := CJ^8 \quad (5.48)$$

except on an event of probability e^{-cJ^8} . Brownian interpolation between integer times has a smaller Gaussian error.

The crossing convention is an intrinsic deterministic function of the finite lattice path. For a curve with bounded component D , a lattice crossing is recorded exactly when $\mathbf{1}_{\{S_{t-1} \in D\}} \neq \mathbf{1}_{\{S_t \in D\}}$; a curve vertex is assigned to the exterior, and same-side tangencies or chord contacts are ignored. The crossing angle is the normal projection of the post-crossing endpoint. At all used scales the endpoint lies in the uniform tubular collar, so this projection is unique. On the compact family of centers under consideration, the maps $\chi_y \circ \Psi_J^{-1}$ and their inverses have uniformly bounded scaled first two derivatives. Consequently, if two physical paths are within b , then every crossing of a nominal interface by one path is sandwiched between the crossings by the other path of the two interfaces obtained by changing its $\log |\chi_y|$ -level by Cb/R , where R is the physical scale. This is just the deterministic inclusion of the two b -neighborhoods of a smooth separating curve; applying it successively preserves chronological order. The crossed directed edge retains the digital-boundary overshoot, which costs at most two extra lattice units. Thus the only difference between the nominal Brownian word and the lattice word comes from collars of relative logarithmic width

$$\delta_J = C \frac{b_J + 1}{M} = O\left(\frac{J^8}{M}\right). \quad (5.49)$$

Step 2. Preserving the lower radial word.

The target is the pathwise inclusion (5.61). It suffices to exclude a set $\mathcal{D}_y$ of collar failures whose conditional probability is small for each complete capped radial word. Outside $\mathcal{D}_y$, collar closeness must give the same ordered traversal forest, and hence the same counts $T_{y,j}^{1,m_0}$, for the two paths. The three guards below prevent a wrong neighboring hit, a retreat before the intended hit, and a reversal just after it, respectively.

Uniform path closeness alone does not preserve an excursion tree: a path may enter a narrow collar, retreat, and thereby insert or delete a traversal. We exclude precisely these retreats and wrong-side approaches. Their total conditional probability will be small for every prescribed radial word. Work in the centered cylinder coordinate

$$U_t = -\log(|\chi_y(X_t)|/r_0),$$

after its conformal time change, so that U is one-dimensional Brownian motion and $\Gamma_{y,j} = \{U = j\}$. For a stopping time s , write $H_a[s] = \inf\{t \geq s : U_t = a\}$, with $\inf \emptyset = \infty$. Put

$$\eta_J = 8\delta_J, \quad N_0 = J^4, \quad (5.50)$$

and take J large enough that $0 < \delta_J < \eta_J/4$ and $\eta_J \leq 1/4$.

The truncated word. Here is the finite depth- L word belonging to (5.14). From $S_{i,0} = A_{y,i}$, $K_{i,0} = 1$, recursively set, when $K_{i,r} < L$,

$$S_{i,r+1} = \inf\{t > S_{i,r} : U_t \in \{K_{i,r} - 1, K_{i,r} + 1\}\}, \quad K_{i,r+1} = U_{S_{i,r+1}}.$$

When $K_{i,r} = L$, instead set

$$S_{i,r+1} = H_{L-1}[S_{i,r}], \quad K_{i,r+1} = L - 1; \quad (5.51)$$

this is a forced terminal return, and the motion below $\Gamma_{y,L}$ is deliberately not resolved. Stop at $N_i = \inf\{r \geq 1 : K_{i,r} = 0\}$, so $B_{y,i} = S_{i,N_i}$. For $i < m_0$ put

$$C_i = \inf\{t > B_{y,i} : U_t \in \{-1, 1\}\}.$$

On S_y , $U_{C_i} = 1$, $C_i = A_{y,i+1}$, and the connector is the single prescribed sign $0 \rightarrow 1$. Concatenating the m_0 block words and these $m_0 - 1$ connector signs gives exactly $\overline{W}_y$ in (5.40). A block letter $j - 1 \rightarrow j$ pushes the next Ulam–Harris child, a letter $j \rightarrow j - 1$ pops it, and (5.51) closes a terminal leaf. Hence

$$T_{y,1}^{1,m_0} = m_0, \quad T_{y,j}^{1,m_0} = \#\{j - 1 \rightarrow j \text{ block letters}\}, \quad 2 \leq j \leq L. \quad (5.52)$$

All these times are recursively defined hitting times of closed sets; the completed m_0 -block word and the cap $|\overline{W}_y| \leq N_0$ are therefore stopping-line measurable.

The collar conditions. Consider first a competitive letter beginning at a stopping time s , at level $k < L$, whose prescribed sign is $\epsilon \in \{-1, 1\}$. Its end is

$$t = \inf\{u > s : U_u \in \{k - 1, k + 1\}\}, \quad U_t = k + \epsilon.$$

Define

$$\begin{aligned} p &= H_{k+\epsilon(1-\eta_J)}[s], & w &= H_{k-\epsilon(1-\eta_J)}[s], & r &= H_{k+\epsilon\eta_J}[p], \\ e &= H_{k+\epsilon(1+\eta_J)}[t], & o &= H_{k+\epsilon\eta_J}[t]. \end{aligned}$$

The bad events for this letter are

$$D^{\text{wrong}} = \{w < t\}, \quad D^{\text{ret}} = \{r < t\}, \quad D^{\text{post}} = \{o < e\}. \quad (5.53)$$

The last comparison is with the old-side collar $k + \epsilon\eta_J$, not merely with the old nominal line k . For a forced terminal return $L \rightarrow L - 1$ beginning at s , set

$$t = H_{L-1}[s], \quad p = H_{L-1+\eta_J}[s], \quad r = H_{L-\eta_J}[p], \quad e = H_{L-1-\eta_J}[t], \quad o = H_{L-\eta_J}[t],$$

and use $D^{\text{ret}} = \{r < t\}$ and $D^{\text{post}} = \{o < e\}$. There is no wrong-side guard because the path below level L is not resolved.

The first source opening needs its own two guards. The fixed annulus defining F_L^0 gives a uniform initial margin. Indeed, $d(v, y) \leq \vartheta_1/20$, convexity of $\tan$ on the relevant fixed small interval, and $r_1 = r_0 e^{-1}$ imply

$$|\chi_y(v)| = 2 \tan\left(\frac{d(v, y)}{2}\right) \leq 2 \tan\left(\frac{\vartheta_1}{40}\right) \leq \frac{r_1}{20}, \quad U_0 \geq 1 + \log 20.$$

(The lower annulus bound keeps U_0 finite.) Thus, with $c_{\text{src}} := \log 20$, one has $U_0 \geq 1 + c_{\text{src}}$, and for $\eta_J < c_{\text{src}}$ the stopping time $c_1 = H_{1+\eta_J}[0]$ satisfies $c_1 < A_{y,1}$. Since the first opening approaches level 1 from the inner side, define

$$D^{A_1, \text{pre}} = \{H_{2-\eta_J}[c_1] < A_{y,1}\}, \quad D^{A_1, \text{post}} = \{H_{2-\eta_J}[A_{y,1}] < H_{1-\eta_J}[A_{y,1}]\}. \quad (5.54)$$

For $i \geq 2$, the pre-opening guard at $A_{y,i}$ is exactly D^{ret} for connector $i - 1$, and its post-opening guard is exactly D^{post} for that connector; they are named both ways but counted once. At every $B_{y,i}$, the last block symbol is the competitive closing letter $1 \rightarrow 0$; its wrong, retreat, and post events in (5.53) are respectively the pre-closing and closing-recognition guards. For $i < m_0$ the following symbol is the connector $0 \rightarrow 1$. At the last close B_{y,m_0} , retain its post-guard and expose it without any following letter. Let $\mathcal{D}_y$ be the union of these events through the last post-closing collar hit. Every member of this union has the form $\{H_a[S] < H_b[S]\}$ for a named stopping time S , so $\mathcal{D}_y$ is measurable at that finite stopping line.

The conditional failure bound. Each guard is decided at a named stopping line. Strong Markov cancels the likelihood of all later prescribed letters in the numerator and denominator of its conditional ratio; for a post-guard the next prescribed sign is kept in the calculation. We record these ratios for an admissible complete word w . Translation, reflection, gambler's ruin, and strong Markov at the integer-line hits give, for every competitive letter,

$$\frac{P(D^{\text{wrong}}, \overline{W}_y = w)}{P(\overline{W}_y = w)} = \frac{\eta_J}{2 - \eta_J}, \quad \frac{P(D^{\text{ret}}, \overline{W}_y = w)}{P(\overline{W}_y = w)} = \frac{\eta_J(1 + \eta_J)}{(1 - \eta_J)(2 - \eta_J)}. \quad (5.55)$$

If the next symbol is competitive, then

$$\frac{P(D^{\text{post}}, \overline{W}_y = w)}{P(\overline{W}_y = w)} = \begin{cases} \eta_J^2, & \text{if the next sign continues the present direction,} \\ 2\eta_J - \eta_J^2, & \text{if it reverses it.} \end{cases} \quad (5.56)$$

Indeed, after translating the new level to 1, $P_1(H_{\eta_J} < H_{1+\eta_J}) = \eta_J$; from η_J , the probabilities of the next exits at 2 and 0 are respectively $\eta_J/2$ and $1 - \eta_J/2$, while either prescribed next sign has probability $1/2$. If the successor is the forced return in (5.51), or there is no successor, the post ratio is simply η_J . For a forced return $L \rightarrow L - 1$,

$$\frac{P(D^{\text{ret}}, \overline{W}_y = w)}{P(\overline{W}_y = w)} = \frac{\eta_J}{1 - \eta_J}. \quad (5.57)$$

The first-opening pre ratio in (5.54) is also $\eta_J/(1 - \eta_J)$. If $d_1 = -1$ is its crossing direction and ϵ_1 is the first block sign, its post ratio is η_J^2 when $\epsilon_1 = d_1$, and $2\eta_J - \eta_J^2$ otherwise. The same table applies at every later opening through its incoming connector. In particular, at $B_{y,i}$, $i < m_0$, the close has direction -1 and the next connector has sign $+1$, so its post ratio is $2\eta_J - \eta_J^2$. At the final close B_{y,m_0} it is exactly η_J , with no fictitious next sign. No angular or future functional is inserted into these ratios. A union bound, without any

independence assertion among guards, gives

$$P(\mathcal{D}_y \mid \overline{W}_y = w) \leq 8(|w| + 2m_0 + 1)\eta_J \leq \varepsilon_J, \quad \varepsilon_J := CJ^4\delta_J = o(J^{-A}) \quad (5.58)$$

for every $|w| \leq N_0$ and every fixed A . The estimate remains valid after conditioning on barriers, traversal counts, or any other function of w , but is not claimed after conditioning on Rosen's angular screens.

Preserving the traversal counts. We now check the deterministic implication used below. Let $\mathcal{A}_J$ be the finite set of all nominal integer levels and all guard-collar levels occurring in (5.53)–(5.54), including the forced-return collars. Let V be the corresponding log-radius of the linearly interpolated lattice path, in the coupling time parametrization. Its nominal source blocks and truncated word are defined at integer edge indices by the endpoint-side-change convention above: a letter is recorded only when an edge's endpoints change sides, and interpolation along such an edge is used only to locate its crossing and evaluate collar separation. Same-side tangencies and chord contacts record no letter. Assume the following symmetric collar condition: whenever either U_t or V_t is within $2\delta_J$ of $\mathcal{A}_J$, both log-radii are finite and $|U_t - V_t| \leq \delta_J$. No comparison is required while both paths lie strictly on the unresolved inner side of the deepest collars. Suppose $S_y, |\overline{W}_y| \leq N_0$, and $\mathcal{D}_y^c$ occur. Before the target collar of a competitive Brownian letter, V cannot make a side-changing crossing of the target nominal line, while $D^{\text{wrong},c}$ prevents it from hitting the other neighbor. From the first target-collar hit until the nominal target, $D^{\text{ret},c}$ keeps U beyond the old-side collar; hence an early side-changing target crossing by V cannot be followed by a reversal. From the nominal hit to the far collar, $D^{\text{post},c}$ gives the same separation, and at the far collar

$$\epsilon(V - (k + \epsilon)) \geq \eta_J - \delta_J > 0.$$

Thus V 's next nominal neighboring line is exactly $k + \epsilon$, and it has made no intervening push or pop. The forced return (5.51) uses the retreat and post guards only; deeper visits are ignored by both truncated words.

At $A_{y,1}$, (5.54) first prevents a premature completed neighboring letter and then forces V across the source line. The preceding letter argument initializes every later $A_{y,i}$ through its connector. Induction in chronological order on the active Ulam–Harris stack now gives the same push/pop operation at every symbol. At $B_{y,i}$ the same stack empties; for $i < m_0$ the guarded connector has sign $0 \rightarrow 1$ and starts the next stack, while the unconditional final post-guard forces V across level 0 at B_{y,m_0} . Hence the two stopped ordered forests are canonically isomorphic and

$$T_{y,j}^{1,m_0}(V) = T_{y,j}^{1,m_0}(U), \quad 1 \leq j \leq L. \quad (5.59)$$

Moreover the exact word length is

$$|\overline{W}_y| = 2 \sum_{j=2}^L T_{y,j}^{1,m_0} + 2m_0 - 1 \leq CL^3 < N_0, \quad (5.60)$$

on the lower barrier and terminal band. On the single global coupling event (5.48), the symmetric collar condition holds. Indeed, every $2\delta_J$ -neighborhood of a nominal or guard collar used by the truncated word has physical radius at least cM . If either path is in such a neighborhood, physical b_J -closeness, the radial Lipschitz inequality, chart distortion, and the $O(1)$ terminal-interface rounding give

$$|U_t - V_t| \leq C(b_J + 1)/M \leq \delta_J$$

after fixing the constant in (5.49). No log-radius comparison is asserted or used while both paths are deeper than the terminal collars. Source safety keeps every collar test before global killing. This condition suffices: any premature target, wrong-neighbor, or reversal crossing occurs at a member of $\mathcal{A}_J$, where collar closeness and $\delta_J < \eta_J/4$ contradict the corresponding safe guard. This also covers a premature $L - 1$ hit during a forced $L \rightarrow L - 1$ return. Consequently

$$I_y^\circ \cap S_y \cap \mathcal{D}_y^c \cap \{\text{global coupling (5.48)}\} \subseteq I_y^{S,\circ}. \quad (5.61)$$

Only the radial consequence is transferred: Rosen's original angular screens remain solely on the Brownian certificate.

Step 3. The upper first-crossing probability.

We now prove the rare upper estimates by direct lattice comparison. Fix a target cell and use the centered continuation (5.41)–(5.42). Put

$$R_j = R_0 e^{-j}, \quad R_0 = 3e^J,$$

rounding every radius by at most two lattice units. On the detailed-source range $\|y\| > M$, define from the actual starting radius

$$u_y = \log \frac{R_0}{\|y\|}, \quad k_y = \lceil u_y \rceil, \quad m = k_y + 1. \quad (5.62)$$

Then $u_y \in (k_y - 1, k_y]$, $m - u_y \in [1, 2)$, and the deterministic start lies outside $D(y, R_{m-1})$. This index differs by only $O(1)$ from $\lfloor \log(e^J / \|y\|) \rfloor$, so it changes no shell exponent. Write

$$\begin{aligned} E_j^{\text{in}} &= \{(a, b) : a \notin D(y, R_j), b \in D(y, R_j), a \sim b\}, \\ E_j^{\text{out}} &= \{(a, b) : a \in D(y, R_j), b \notin D(y, R_j), a \sim b\}. \end{aligned} \quad (5.63)$$

Every radial query retains its phase, level, Ulam–Harris stack and child counters, population vector, and current directed crossing edge.

Potential-kernel optional stopping, uniformly in that edge, gives for $k < l$

$$q_{k,l}^S \leq C \frac{k+1}{l}, \quad p_l^S(e) = \frac{l-1 + O(R_l^{-1})}{l + O(R_l^{-1})}, \quad d_l^S(e) = \frac{1 + O(R_l^{-1})}{l + O(R_l^{-1})}. \quad (5.64)$$

Here $q_{k,l}^S$ includes the first completion at level l , while p_l^S and $d_l^S = 1 - p_l^S$ are the continuation and death rows after return to level $l - 1$. Indeed, the two complete formulas from Hao–Li–Okada–Zheng, [9, Lemma 2.1, (2.5)], recalled in part (3) of Lemma 3.1, have respective logarithmic gaps $l - 1 + O(R_l^{-1})$, $1 + O(R_l^{-1})$, and $l + O(R_l^{-1})$. Thus both the continuation and the order-1/ l death probability are relative estimates. Strong Markov at every successive directed hit yields, for $N \geq 1$,

$$P(T_{y,l}^{S,k \rightarrow 0} \geq N \mid \text{initial directed edge}) \leq C \frac{k+1}{l} \exp \left\{ -\frac{N}{l} + \frac{CN}{lM} \right\}. \quad (5.65)$$

In particular, for $X = \sqrt{2T_{y,m}}$ and $x \leq Cm$,

$$P(X \in [x, x+1]) \leq Cm^C e^{-x^2/(2m)}. \quad (5.66)$$

No independence of directed entrance edges is used in these scalar bounds.

A union bound over all fine centers would count many copies of the same crossing. We replace each center by a deterministic nearby representative and enlarge its target disk so that the crossing count can only increase. Put

$$\phi(l) = \exp\{\tfrac{1}{4}(l \wedge (J - l))^{1/4}\}.$$

Fix constants

$$0 < c_{\text{sep}} \leq c_{\text{mesh}} < C_\delta < \frac{e-1}{e+1}.$$

For each l , choose a deterministic maximal $c_{\text{sep}}R_l/\phi(l)$ -separated subset of $\mathbb{Z}^2$, with c_{sep} chosen so that its covering radius (for lattice points) is at most $c_{\text{mesh}}R_l/\phi(l)$. Given a candidate lattice center y in source shell k , choose a mesh point $\hat{y} = \hat{y}(y, l) \in \mathbb{Z}^2$ with

$$|y - \hat{y}| \leq c_{\text{mesh}}R_l/\phi(l), \quad \delta_l = C_\delta/\phi(l), \quad R_j^\pm = (1 \pm \delta_l)R_j.$$

For all sufficiently large J , the gap $(C_\delta - c_{\text{mesh}})R_l/\phi(l)$ absorbs the at-most-two-unit rounding of every radius on the used range. Moreover $\delta_l < (e-1)/(e+1)$, and hence $R_{l-1}^- > R_l^+$, so every buffered annulus below is nonempty. Starting with the first entrance into $D(\hat{y}, R_k^+)$, and stopping on exiting $D(\hat{y}, R_0^+)$, let $\tilde{T}_{\hat{y},l}^{S,k \rightarrow 0}$ count completed crossings from $\partial D(\hat{y}, R_{l-1}^-)$ to $\partial D(\hat{y}, R_l^+)$. The inclusions

$$\begin{aligned} D(y, R_k) &\subset D(\hat{y}, R_k^+), \\ D(y, R_l) &\subset D(\hat{y}, R_l^+), \\ D(\hat{y}, R_{l-1}^-) &\subset D(y, R_{l-1}), \\ D(y, R_0) &\subset D(\hat{y}, R_0^+) \end{aligned} \tag{5.67}$$

show pathwise that

$$T_{y,l}^{S,k \rightarrow 0} \leq \tilde{T}_{\hat{y},l}^{S,k \rightarrow 0}. \tag{5.68}$$

Indeed every y -crossing begins outside the representative's buffered outer circle, ends inside its buffered inner circle, and occurs after the representative start and before its stop. The same inclusions cover $k = l-1$. Area comparison for the mesh in a shell of diameter $O(R_k)$ gives, for every $k \leq l-1$,

$$\#\{\text{representatives at } (k, l)\} \leq Ce^{2(l-k)}\phi(l)^2. \tag{5.69}$$

The buffered inclusions in Rosen (2.26)–(2.33) do not impose $k \leq \log J$. For the four modified radii $(R_k^+, R_{l-1}^-, R_l^+, R_0^+)$, the same potential-kernel calculation gives, uniformly in all directed entrance edges,

$$\tilde{q}_{k,l}^S \leq C \frac{k+1}{l}, \quad \tilde{p}_l^S \leq 1 - \frac{1 - C/\phi(l) - C/M}{l}. \tag{5.70}$$

Consequently (5.65) holds with T replaced by $\tilde{T}_{\hat{y},l}^{S,k \rightarrow 0}$, the additional exponent being $CN/(l\phi(l)) + CN/(lM)$. Taking the upper horizon- J barrier level

$$N = \frac{1}{2}(\rho_J l + z + (l \wedge (J-l))^{1/4})^2$$

there gives

$$P(\tilde{T}_{\hat{y},l}^{S,k \rightarrow 0} \geq N) \leq Ce^{-2l - (l \wedge (J-l))^{1/4}} e^{-2z}. \tag{5.71}$$

On the used range $l \leq J - (4 \log J)^4$, the elementary exponent calculation gives $l/\phi(l) = O(1)$ (the bounded early values are absorbed into C), and $N/(lM) = o(1)$. Multiplying (5.69) and (5.71) and summing first in l , then in $k \geq s$, proves (5.44), including the range $\log J < k \leq C_0 \log J$.

Step 4. The centered endpoint probability.

The source population and its descendants must be compared together. Conditioning on a later list of entrance edges would change the law of the current radial decision. We therefore compare the entire chronological word, integrating each edge when it is first

produced, and only then condition under the ideal Galton–Watson law. In addition to the directed-edge/stack states in (5.63), introduce a special deterministic initial atom $\star_y$, representing $S_0 = 0$. The first query is ROOT (hit D_m before leaving D_0), whose kernel maps $\star_y$ to E_m^{in} , or FAILURE, whose kernel maps $\star_y$ to the cemetery state. After every root, its descendants are explored depth first: at an active node of level $j < L$, BIRTH is the first hit of D_{j+1} and pushes the next child, while CLOSE is the first exit from D_{j-1} and pops the node. A level- L leaf has a forced return to level $L - 1$, of row mass one. After a root closes, the current state lies in E_{m-1}^{out} ; the next source query is another ROOT, mapping to E_m^{in} , or the final DEATH, mapping through E_0^{out} to the cemetery state. Thus a word with $t \geq 1$ roots has the physical order

$$\text{ROOT, DFS}(1), \text{ROOT, DFS}(2), \dots, \text{ROOT, DFS}(t), \text{DEATH}. \quad (5.72)$$

The word stops at the first barrier violation, endpoint decision, death, or a fixed deterministic cap $Q_* = C_* J^3$, chosen above $3 + 2 \sum_{j=m}^L (T_{y,j} + 1)$ on the coefficient-two barrier, and is then padded by cemetery letters. It contains every terminating close and the final source death, so there is no residual future condition.

Let P_* be the angle-free ideal law. Its first root and failure probabilities are u_y/m and $(m - u_y)/m$; subsequent root and death probabilities are $(m - 1)/m$ and $1/m$; every descendant birth/close decision has probability $1/2$. Potential-kernel optional stopping, with every output edge summed but every current directed edge retained, gives the four lattice source rows

$$\begin{aligned} r_{0,y}^S &= \frac{u_y + O(R_m^{-1})}{m + O(R_m^{-1})}, & f_{0,y}^S &= \frac{m - u_y + O(R_m^{-1})}{m + O(R_m^{-1})}, \\ r_m^S(e) &= \frac{m - 1 + O(R_m^{-1})}{m + O(R_m^{-1})}, & d_m^S(e) &= \frac{1 + O(R_m^{-1})}{m + O(R_m^{-1})}. \end{aligned} \quad (5.73)$$

Here the first pair starts from $\star_y$, while the second pair is uniform in $e \in E_{m-1}^{\text{out}}$. Since every candidate satisfies $\|y\| < 2e^J$,

$$u_y \geq \log(3/2), \quad m - u_y \in [1, 2), \quad m - 1 \geq 1.$$

Also $R_m = e^{u_y - m} \|y\| \geq e^{-2} \|y\| \geq e^{-2} M$ on the detailed range. Consequently each of the four displayed rows, including the two order- $1/m$ FAILURE and DEATH rows, equals its ideal row times $1 + O(M^{-1})$. Ordinary birth/close rows are $(1/2)(1 + O(R_{j+1}^{-1}))$.

Fix a complete radial word w , but no directed-edge refinement. If ν_r is the edge-valued finite measure after its first r letters, the next ideal row p_r is determined by the radial prefix, and the uniform row estimates give

$$e^{-C/M} p_r \|\nu_r\| \leq \|\nu_{r+1}\| \leq e^{C/M} p_r \|\nu_r\|.$$

Forward induction, with all compatible entrance and exit edges integrated, therefore gives

$$e^{-C|w|/M} P_*(w) \leq P_S(w) \leq e^{C|w|/M} P_*(w). \quad (5.74)$$

This is not a conditional comparison given a future edge list.

Under P_* , the source population has the exact law

$$\pi_{m,u_y}(0) = 1 - \frac{u_y}{m}, \quad \pi_{m,u_y}(t) = \frac{u_y}{m} \left(\frac{m-1}{m} \right)^{t-1} \frac{1}{m}, \quad t \geq 1, \quad (5.75)$$

and, conditional on t , the t descendant trees are independent critical Galton–Watson trees with $P(\xi = r) = 2^{-r-1}$. Hence, for the disjoint union $\mathcal{U}(q)$ of complete barrier-respecting

words ending in $\sqrt{2T_L} \in [q, q+1]$,

$$P_S(\mathcal{W}(q)) \leq e^{CJ^3/M} \sum_{t \geq 1} \pi_{m, u_y}(t) P_t^{\text{GW}}(\mathcal{A}_t^{\text{GW}}(q)). \quad (5.76)$$

Only the ideal law on the right has been disintegrated. Moreover, for $x \leq Cm$,

$$\sum_{\sqrt{2t} \in [x, x+1]} \pi_{m, u_y}(t) \leq C(1+x)m^{-1}e^{-x^2/(2m)+O(1)} \leq Ce^{-x^2/(2m)}. \quad (5.77)$$

Apply Rosen Appendix Theorem 9.1(a), quoted in (5.8), to the ideal Galton–Watson factor in (5.76), group $\sqrt{2t}$ into unit bands, and use (5.11)–(5.12). This proves (5.45). The $t = 1$ term is covered by the extra-ancestor argument following (5.8), while $t = 0$ cannot reach a positive central endpoint. Since $J^3/M = o(J^{-A})$, the comparison is relative even for the rare endpoint event.

Step 5. The lower count and the terminal interfaces.

We apply the Brownian second moment before transferring any event to the lattice. We retain Rosen’s original event I_y° and its original angular screens exactly as written in (5.17)–(5.21), and intersect it with the source-safety event S_y . No tightened or halved screen is used. Put $I_y^{\text{sf}} = I_y^\circ \cap S_y$. By (5.28),

$$P\left(\sum_y 1_{I_y^{\text{sf}}} > 0\right) \geq c_*. \quad (5.78)$$

The screens occur only in this Brownian second-moment certificate; the lattice target discards all angular screens.

Let $\hat{I}_y^B$ be the Brownian event obtained from I_y° by deleting every angular screen and retaining only the radial barrier, the bounded endpoint band, and the fixed m_0 -source definition. Rosen Appendix Theorem 9.1(a), followed by the same source and barrier calculation as (5.7), gives

$$P(\hat{I}_y^B) \leq Ce^{-2L}, \quad \sum_y P(\hat{I}_y^B) \leq C. \quad (5.79)$$

Since $I_y^{\text{sf}} \cap \mathcal{D}_y \subseteq \hat{I}_y^B \cap S_y \cap \mathcal{D}_y$, estimate (5.58) may be summed over complete safe radial words before any angular screen is imposed. Hence

$$\sum_y P(I_y^{\text{sf}} \cap \mathcal{D}_y) \leq \varepsilon_J \sum_y P(\hat{I}_y^B \cap S_y) \leq \varepsilon_J \sum_y P(\hat{I}_y^B) = o(1). \quad (5.80)$$

Let $\mathcal{K}_J$ be the single global coupling/time-cap event (5.48). The pathwise implication (5.61) gives

$$\{Z_L^{\text{sf}} > 0\} \cap \mathcal{K}_J \setminus \bigcup_y (I_y^{\text{sf}} \cap \mathcal{D}_y) \subseteq \{Z_L^S > 0\}.$$

Consequently, paying the coupling failure only once,

$$P(Z_L^S > 0) \geq P(Z_L^{\text{sf}} > 0) - \sum_y P(I_y^{\text{sf}} \cap \mathcal{D}_y) - P(\mathcal{K}_J^c) \geq c_* - o(1). \quad (5.81)$$

Source safety and (5.25) keep every block, connector, and final exposed post-closing collar a fixed positive distance inside the stopping boundary, so the close lattice path is not killed before the transferred blocks finish. No Brownian angular screen is transferred. This is the only use of the two-center estimate (5.20), and it occurs entirely on the spherical Brownian side.

For the lattice first moment, extend the path if necessary and decompose its first m_0 source blocks, treating every connector as an integrated subkernel of row mass at most one. For a fixed collection of block words, apply the edge-marginal induction (5.36) to all competitive letters, uniformly over the entrance law supplied by each preceding connector. Forced terminal returns have row mass one. Thus, if q is the number of competitive decisions,

$$P_S(\text{these block words}) \leq e^{CJ^3/M} 2^{-q}.$$

Summing the disjoint words obeying the lower barrier and terminal band gives the critical geometric Galton–Watson probability with exactly m_0 ancestors. Rosen Appendix Theorem 9.1(a), followed by the fixed-source calculation already used in (5.79), gives

$$P(I_y^{S,\circ}) \leq C e^{-2L}, \quad EZ_L^S \leq C. \quad (5.82)$$

This proves (5.46); in particular, the lower first moment does not use the upper source-population estimate (5.66).

Finally we justify the terminal switch used in the definition of $\mathcal{R}_y$. On the compact center family, with $F_y = \sigma \circ \chi_y^{-1}$,

$$F_y(w) = F_y(0) + a_y w + b_y w^2 + O(|w|^3), \quad 0 < c \leq |a_y| \leq C.$$

At $r_L \asymp e^{-J+h}$, dilation by $A_J \asymp e^J$ shows that the actual deepest image circle is at Hausdorff distance

$$O(A_J r_L^2) = O(e^{-J+2h}) = o(1) \quad (5.83)$$

from its affine concentric circle. Rounding the center and the digital boundary costs $O(1)$, or relative width $O(M^{-1})$, and is one additional collar in the lower guard construction (5.50)–(5.59). The induced angular conjugacy, after removing the fixed rotation $a_y/|a_y|$, is C^4 -close to the identity by $O(e^{-J+h})$; its Haar-density defect is therefore absorbed by $M^{-7/24} \log M + M^{-1}$. Thus the outer χ_y -rotation kernels and the terminal Euclidean rotation kernels use the same common center up to the stated error. All lattice events were defined before introducing the auxiliary coupling and are measurable in $\mathcal{R}_y$. We may therefore discard the Brownian path after (5.81); every later conditional law is the original simple-random-walk law given $\mathcal{R}_y$. ■

6. Terminal occupation and the localized upper bound

The traversal estimates control how often the walk reaches a terminal disk. We now control the occupation accumulated during those visits. The first subsection compares the endpoint laws needed for the first two bridge moments. The second turns this comparison into an exponential occupation moment, including the likelihood of the prescribed radial word. The last subsection combines the traversal and occupation costs: their Gaussian factors cancel, leaving the ballot prefactor that gives the localized upper bound.

6.1. ENDPOINT PRODUCTS IN ONE CENTERED BLOCK. For a killed walk with prescribed entrance and exit, the observables m_x and $q_{x,x'}$ below are its first and second occupation moments. Their dependence on the lattice exit edge can be rough, but averaging that edge gives Green functions whose poles stay away from the entrance circle. The next lemma compares products only for these averaged observables; it does not require comparison of arbitrary endpoint functions.

LEMMA 6.1. *Let $D = D(y, eM)$, let A be the inner vertex boundary of $D(y, M)$, and let E be the directed edges leaving D . Put*

$$H^S(a, e) = P_a((S_{\tau_D-1}, S_{\tau_D}) = e), \quad (\mathcal{H}^S f)(a) = \sum_e H^S(a, e) f(a, e).$$

Let $G^S = G_D$. For $x, x' \in V := D(y, M/2)$, define on $A \times E$

$$\begin{aligned} m_x(a, e) &= \frac{G^S(a, x) H^S(x, e)}{H^S(a, e)}, \\ q_{x, x'}(a, e) &= \frac{1}{H^S(a, e)} (G^S(a, x) G^S(x, x') H^S(x', e) \\ &\quad + G^S(a, x') G^S(x', x) H^S(x, e) \\ &\quad - \mathbf{1}_{\{x=x'\}} G^S(a, x) H^S(x, e)), \end{aligned}$$

and let

$$\mathcal{F}_M = \{m_x/C : x \in V\} \cup \{q_{x, x'}/[C(1 + G^S(x, x'))] : x, x' \in V\}, \quad (6.1)$$

where the fixed C is large enough that all functions have sup norm at most one. Let $R_i^S(e, da')$ be a fixed radial-only centered return word from a previous exit edge to the next inward entrance, let $\sigma_i = R_i^S 1$, $J_i^S = R_i^S / \sigma_i$, and set

$$P_i^S((a, e), d(a', e')) = J_i^S(e, da') H^S(a', de'). \quad (6.2)$$

The word may be selected adaptively from its radial prefix, but not from an angular endpoint. If it contains q_i elementary letters, all at scales at least M , and λ_A^S is the center-to- A first-hit law, put $\Lambda^S = \lambda_A^S H^S$. For $f \in \mathcal{F}_M$ and every consecutive portion $r \leq s$ of one centered block,

$$|\Lambda^S P_r^S \cdots P_s^S f - \Lambda^S f| \leq C \left\{ M^{-7/24} \log M + \frac{1}{M} \sum_{i=r}^s (q_i + 1) \right\}. \quad (6.3)$$

The bound is uniform over the fixed realized radial word and applies only within one centered block. An arbitrary inter-block reset ends this comparison; the effect of the next block's initial law will be controlled by contraction in Lemma 6.2.

Proof of Lemma 6.1. The comparison is applied after averaging the exit edge, which removes the roughness of the digital boundary. We then normalize each complete return word and compare the resulting product with rotation-invariant Brownian kernels.

Step 1. Smooth the endpoint observables. A discrete pair is $(a, e) \in A \times E$. Every exterior kernel starts at the head $\text{out}(e)$ of the preceding exit edge. With $P_i^S = J_i^S H^S$, exact operator composition is

$$\Lambda^S P_r^S \cdots P_s^S f = \lambda_A^S H^S J_r^S H^S \cdots J_s^S (\mathcal{H}^S f). \quad (6.4)$$

Thus the first operation on the terminal observable is the exact exit-edge average $\mathcal{H}^S$. This is the operation that makes the smooth angular comparison applicable.

The continuum entrance and exit state spaces are separate copies of $\mathbb{T}$, in the common angular coordinate of the centered block. Let $\bar{H}$ be Brownian exit from radii M to eM , and let $\bar{J}_i$ be the normalized Brownian return word. Their densities are rotation convolutions. Explicitly,

$$\frac{d\bar{H}(\alpha, \cdot)}{d\theta/(2\pi)}(\beta) = \frac{e^2 - 1}{e^2 + 1 - 2e \cos(\beta - \alpha)}, \quad (6.5)$$

and an elementary side kernel from logarithmic level x to b before a has unnormalized density

$$\frac{1}{2\pi} \left\{ \frac{x-a}{b-a} + 2 \sum_{n \geq 1} \frac{\sinh(n(x-a))}{\sinh(n(b-a))} \cos n(\beta - \alpha) \right\}. \quad (6.6)$$

Consequently Haar measure $m(d\alpha) = d\alpha/(2\pi)$ is preserved by $\bar{H}$ and every $\bar{J}_i$. The continuum pair transition $\bar{J}_i \bar{H}$ therefore preserves $\bar{\Lambda}(d\alpha, d\beta) = m(d\alpha) \bar{H}(\alpha, d\beta)$.

We next construct the continuum test functions. Write $G^S = G_D$. For $x, x' \in D(y, M/2)$, the exact bridge identities are

$$\mathcal{H}^S m_x(a) = G^S(a, x), \quad (6.7)$$

$$\begin{aligned} \mathcal{H}^S q_{x,x'}(a) &= G^S(a, x) G^S(x, x') + G^S(a, x') G^S(x', x) \\ &\quad - \mathbf{1}_{\{x=x'\}} G^S(a, x). \end{aligned} \quad (6.8)$$

Let $a_\alpha = y + M e^{i\alpha}$ and

$$\mathcal{G}^B(a_\alpha, x) = \frac{2}{\pi} g_{B(y, eM)}^B(a_\alpha, x).$$

For $f = m_x/C$, define $\varphi_f(\alpha) = \mathcal{G}^B(a_\alpha, x)/C$. For $f = q_{x,x'}/[C(1 + G^S(x, x'))]$, define φ_f by the right side of (6.8), divided by $C(1 + G^S(x, x'))$, and replace only the boundary-variable factors $G^S(a, x), G^S(a, x')$ by $\mathcal{G}^B(a_\alpha, x)$ and $\mathcal{G}^B(a_\alpha, x')$. Only the dependence on the entrance angle is compared with Brownian motion. The short-distance lattice coefficients $G^S(x, x')$ are retained, including on the diagonal, so the comparison does not have to approximate a lattice singularity.

After scaling by M ,

$$g_{B(0,e)}^B(z, \xi) = \log \left| \frac{e^2 - z\bar{\xi}}{e(z - \xi)} \right|, \quad |z| = 1, \quad |\xi| \leq \frac{1}{2}, \quad (6.9)$$

where the logarithmic Green function is normalized as in the preceding display. The denominators stay uniformly away from zero, and symmetry of G^S bounds all retained normalized coefficients. Hence

$$\sup_f \|\varphi_f\|_{C^4(\mathbb{T})} \leq C. \quad (6.10)$$

We use the Green-function comparison of Kozdron–Lawler [10, Corollary 3.5, equation (30)]. For a simply connected lattice domain A of inradius in $[n, 2n]$, let $\tilde{A}$ be its union-of-squares domain. If $u \in A_n^* := \{z : g_{\tilde{A}}(0, z) \geq n^{-1/16}\}$, the comparison, away from the diagonal, is

$$G_A(u, z) = \frac{2}{\pi} g_{\tilde{A}}(u, z) + k_{z-u} + O(n^{-7/24} \log n), \quad (6.11)$$

where a is the lattice potential kernel and $k_w = k_0 + (2/\pi) \log |w| - a(w)$ for $w \neq 0$. Apply this with the core pole in $D(y, M/2)$. This pole is a fixed fraction of the radius from the boundary and belongs to A_n^* . The boundary-variable point is at distance at least $M/2 - O(1)$ from the pole, so $k_{a-x} = O(M^{-2})$. Sandwiching the union-of-squares disk between disks whose radii differ by $O(1)$, and projecting radially to the continuum circle, each costs $O(M^{-1})$. Equations (6.7)–(6.11) therefore give, with $U_A F(a) = F(\arg(a - y))$,

$$\|\mathcal{H}^S f - U_A \varphi_f\|_\infty \leq C M^{-7/24} \log M. \quad (6.12)$$

For an exit edge put $U_E F(e) = F(\arg(\text{out}(e) - y))$. Lemma 5.2, with the same tubular continuation for a disk having only its exit boundary, gives for every smooth angular test F

$$\|H^S U_E F - U_A \bar{H} F\|_\infty + |\lambda_A^S U_A F - mF| \leq \frac{C}{M} \|F\|_{C^4}. \quad (6.13)$$

For the second term, solve the disk Dirichlet problem with boundary data F , start at its center, and apply the tubular continuation argument of (5.34)–(5.35). For the first term, solve the exit-disk problem and start on the inner circle. This proves the comparison for the digital circles directly.

Step 2. Normalize each complete return word. A prescribed return word may have very small probability. An absolute error in its probability would therefore be insufficient after conditioning. We first multiply the relative estimates for its elementary letters, then normalize the complete word. Write its elementary kernels as

$$T_{i,l}^S(z, dz') = p_{i,l}^S(z) K_{i,l}^S(z, dz'), \quad \bar{T}_{i,l} = \bar{p}_{i,l} \bar{K}_{i,l}, \quad 1 \leq l \leq q_i,$$

and let $Q_i^S = K_{i,1}^S \cdots K_{i,q_i}^S$. Under this normalized digital chain put

$$W_i = \prod_{l=1}^{q_i} \frac{p_{i,l}^S(Z_{l-1})}{\bar{p}_{i,l}}.$$

By (5.31), if $\eta_i = C \sum_l R_{i,l}^{-1} \leq C q_i / M$, then $e^{-\eta_i} \leq W_i \leq e^{\eta_i}$. With $\bar{s}_i = \prod_l \bar{p}_{i,l}$, the identities

$$\frac{R_i^S 1(e)}{\bar{s}_i} = E_{Q_{i,e}^S} W_i, \quad J_i^S F(e) = \frac{E_{Q_{i,e}^S} [W_i F(Z_{q_i})]}{E_{Q_{i,e}^S} W_i} \quad (6.14)$$

are exact. Hence

$$\|J_i^S F - Q_i^S F\|_\infty \leq C \eta_i \|F\|_\infty. \quad (6.15)$$

Apply the normalized smooth-test estimate following (5.31) to the $K_{i,l}^S$ and telescope from the right. If $\Phi_{l,F} = \bar{K}_{i,l+1} \cdots \bar{K}_{i,q_i} F$, convolution gives $\|\Phi_{l,F}\|_{C^4} \leq \|F\|_{C^4}$. Let $U_{i,l}$ pull an angular test back to the state after letter l , so that $U_{i,0} = U_E$ and $U_{i,q_i} = U_A$. The indexed telescoping identity is

$$Q_i^S U_A F - U_E \bar{J}_i F = \sum_{l=1}^{q_i} K_{i,1}^S \cdots K_{i,l-1}^S (K_{i,l}^S U_{i,l} \Phi_{l,F} - U_{i,l-1} \bar{K}_{i,l} \Phi_{l,F}).$$

Each Markov prefix contracts the sup norm, and the normalized smooth-test estimate bounds the l th bracket by $C R_{i,l}^{-1} \|F\|_{C^4}$. Hence

$$\|Q_i^S U_A F - U_E \bar{J}_i F\|_\infty \leq C \eta_i \|F\|_{C^4}.$$

Together with (6.15),

$$\|J_i^S U_A F - U_E \bar{J}_i F\|_\infty \leq C \eta_i \|F\|_{C^4}. \quad (6.16)$$

The weight in (6.14) contains the row masses of every letter of the fixed word, including letters after the state currently being compared.

Step 3. Compare the complete product.

Write (6.4) as alternating operators $H^S, J_r^S, H^S, \dots, J_s^S$, with terminal test $g_f^S = \mathcal{H}^S f$. Let $L_j^S, \bar{L}_j$ be the corresponding discrete and Brownian operators. Write U_j for the angular

pullback on the state space after the j -th operator, and put $\Phi_{j,f} = \bar{L}_{j+1} \cdots \bar{L}_N \varphi_f$. The operator identity, with every pullback on its corresponding state space, is

$$\begin{aligned} L_1^S \cdots L_N^S g_f^S - U_0 \bar{L}_1 \cdots \bar{L}_N \varphi_f &= L_1^S \cdots L_N^S (g_f^S - U_N \varphi_f) \\ &+ \sum_{j=1}^N L_1^S \cdots L_{j-1}^S [L_j^S U_j \Phi_{j,f} - U_{j-1} \bar{L}_j \Phi_{j,f}]. \end{aligned} \quad (6.17)$$

All $\Phi_{j,f}$ have uniformly bounded C^4 -norm by (6.10) and convolution contraction. The terminal mismatch in the first line of (6.17) costs $CM^{-7/24} \log M$ by (6.12). Each exit operator costs C/M by (6.13), and each return word costs Cq_i/M by (6.16). Sup-norm contraction of the discrete prefixes therefore bounds the whole product by the right side of (6.3). In the continuum the alternating product preserves Haar measure, so it has integral $m\varphi_f$. A separate zero-length use of (6.12)–(6.13) gives $|\Lambda^S f - m\varphi_f| \leq C(M^{-7/24} \log M + M^{-1})$. Subtracting these two comparisons proves (6.3).

For a lower conformal block the same proof uses its one common χ_y -angle. Each occurrence of H^S has angular conjugacy error $O(M^{-1} + e^{-J+h})$. Since $2h \leq J$, $e^{-J+h} \leq M^{-1}$, and paying this error once per occurrence is already contained in $M^{-1} \sum_i (q_i + 1)$. A decision depending on an angular endpoint can change Haar measure. Such decisions are therefore included in the reset density constructed below, at the boundary between centered blocks. ■

6.2. OCCUPATION OF COMPLETED TERMINAL EXCURSIONS. We now apply the endpoint comparison to completed terminal excursions. Fix a center y , and omit the superscript S from the lattice kernels in this subsection. Let

$$\begin{aligned} D_y &= \{z \in \mathbb{Z}^2 : \|z - y\| < eM\}, \\ A_y &= \partial_{\text{in}} \{z : \|z - y\| < M\}, \\ V_y &= \{z : \|z - y\| < M/2\}, \end{aligned}$$

where $\partial_{\text{in}} D = \{a \in D : \exists a' \notin D, a' \sim a\}$, and let $E_y = \{(u, v) : u \in D_y, v \notin D_y, u \sim v\}$ be the directed exit edges. A completed terminal piece begins at the first lattice vertex a after an inward crossing of radius M , and is killed on the directed edge $e \in E_y$ that first crosses radius eM . Erasing the interiors of all such outward pieces gives exactly the intrinsic sigma-field $\mathcal{R}_y$ of Lemma 5.3. Conditional on their endpoint pairs $\omega_i = (a_i, e_i)$, their occupation vectors $X_i = (X_i(x) : x \in V_y)$ are independent killed lattice bridges. Write $G = G_{D_y}$. Thus the endpoint observables in (6.24)–(6.25) below satisfy $m_x(a, e) = E^{a,e} X(x)$ and $q_{x,x'}(a, e) = E^{a,e} [X(x)X(x')]$.

An *admissible centered block word* w is a positive-probability radial-only cylinder. It has $b = O(J^2)$ completed pieces, at most $Q \leq J^3$ adjacent-shell decisions, and a fixed number G_0 of blocks. Within each block every decision is between circles centered at y , or, in the lower spherical transfer, between successive level sets of the single conformal coordinate χ_y . In either case the normalized Brownian kernels are rotation convolutions in one common angular coordinate; this is what “centered block” means here.

Between two lower blocks the safe connector contributes no core occupation and its normalized angular kernel is treated only as an arbitrary reset. Its scalar prescribed-side row mass is retained in the exact word factorization. We use no invariant law for this connector.

Before factorization, retain only source intervals containing at least one completed terminal piece as blocks. Fold any initial empty intervals and connectors into the first chronological law, every intervening string of empty intervals and connectors into the preceding reset, and every terminal such string into the final reset. Hence $b_g \geq 1$, and in the lower application $G_0 \leq m_0$; the inter-source connectors and empty intervals contribute no core occupation. In the upper application $G_0 = 1$ when $b > 0$, after the centered extension (5.41). The case $b = 0$ is trivial and is henceforth excluded. The reset construction below assigns every initial/final suffix and connector exactly once, so there is no unrecorded future weight.

We separate a subkernel into its row mass and its normalized transition. The row masses will retain the probability of the complete word; the normalized transitions will provide contraction of endpoint observables. Put

$$H(a, e) = P_a((S_{\tau_{D_y^c}-1}, S_{\tau_{D_y^c}}) = e).$$

Write b_g for the number of completed pieces in block g , so $\sum_g b_g = b$. For $i < b_g$, let $R_{g,i}^w(e, da')$ be the sub-Markov kernel of the prescribed exterior return word, started at $\text{out}(e_{g,i})$, from $e_{g,i} = e$ to the next entrance $a_{g,i+1} = a'$. The pair subkernel factors exactly as

$$Q_{g,i}^w((a, e), d(a', e')) = R_{g,i}^w(e, da')H(a', de'). \quad (6.18)$$

Write

$$\sigma_{g,i}(e) = R_{g,i}^w 1(e), \quad J_{g,i}^w = \frac{R_{g,i}^w}{\sigma_{g,i}}, \quad \tilde{K}_{g,i}^w = J_{g,i}^w H. \quad (6.19)$$

For $g < G_0$, let

$$C_g^w(e, d\zeta, d\omega') \quad (6.20)$$

be the genuine sub-Markov reset kernel which, after the last endpoint of block g , contains the prescribed final no-more-piece suffix, the safe connector and its data ζ , and the prefix of block $g + 1$ through its first endpoint pair ω' . For $g = G_0$, the same notation denotes the final suffix to a cemetery state. Put

$$\sigma_{g,b_g}(e) = C_g^w 1(e), \quad J_{g,b_g}^w = C_g^w / \sigma_{g,b_g}. \quad (6.21)$$

In a centered suffix every adjacent-shell letter is included in Q . A lower inter-source connector is not declared centered; its normalized kernel J_{g,b_g}^w is used as an arbitrary reset.

Let $\mu_{1,w}$ be the law of the first endpoint pair conditioned only on the word prefix through that piece; this prefix has positive probability by admissibility. Define $\tilde{P}_w$ to be the joint law of all endpoint and reset variables that starts with $\mu_{1,w}$, uses $\tilde{K}_{g,i}^w$ internally, and uses the normalized reset kernels J_{g,b_g}^w between blocks and at the final cemetery transition. Thus the first law of every later block is generated chronologically by the preceding reset, not conditioned on its future output. Write P_w for the true endpoint-and-reset law conditioned on the complete word w . It is obtained from $\tilde{P}_w$ by weighting endpoint sequences by the product of their subsequent row masses. Formula (6.32) will express this weighting exactly. All estimates are uniform over the arbitrary normalized reset kernels, which are marginalized at the end.

Let λ_A be the law of the first hit of A_y by a walk started at y , and set

$$\Lambda_M(da, de) = \lambda_A(da)H(a, de). \quad (6.22)$$

For a pair observable f , write

$$(\mathcal{H}f)(a) = \sum_e H(a, e) f(a, e). \quad (6.23)$$

For $x, x' \in V_y$, define

$$m_x(a, e) = \frac{G(a, x)H(x, e)}{H(a, e)}, \quad (6.24)$$

$$\begin{aligned} q_{x,x'}(a, e) = \frac{1}{H(a, e)} & \left(G(a, x)G(x, x')H(x', e) \right. \\ & + G(a, x')G(x', x)H(x, e) \\ & \left. - \mathbf{1}_{\{x=x'\}}G(a, x)H(x, e) \right). \end{aligned} \quad (6.25)$$

Uniformly,

$$0 \leq m_x \leq C, \quad 0 \leq q_{x,x'} \leq C(1 + G(x, x')), \quad (6.26)$$

and the exit-edge averages are exactly

$$\mathcal{H}m_x(a) = G(a, x),$$

$$\mathcal{H}q_{x,x'}(a) = G(a, x)G(x, x') + G(a, x')G(x', x) - \mathbf{1}_{\{x=x'\}}G(a, x). \quad (6.27)$$

The finite normalized observable family is

$$\mathcal{F}_M = \{m_x/C : x \in V_y\} \cup \left\{ \frac{q_{x,x'}}{C(1 + G(x, x'))} : x, x' \in V_y \right\}; \quad (6.28)$$

it has at most CM^4 members. Put

$$\varepsilon_M = CM^{-7/24} \log M, \quad \eta_w^* := CQ/M + CG_{\text{nc}}, \quad (6.29)$$

where $G_{\text{nc}} \leq G_0 \leq m_0$ counts noncentered whole resets; each is factored once through the separator (6.49) below. Thus $G_{\text{nc}} = 0$ in the upper application and is fixed in the lower one. The reference expectation of an observable is $c_f = \Lambda_M f$. We use the same notation for the unnormalized observables m_x and $q_{x,x'}$; their reference expectations satisfy

$$c_{m_x} = g + O(\varepsilon_M), \quad (6.30)$$

$$c_{q_{x,x'}} = gG(x, x') + gG(x', x) - \mathbf{1}_{\{x=x'\}}g + O(\varepsilon_M(1 + G(x, x'))). \quad (6.31)$$

LEMMA 6.2. *Let w be an admissible word with $b > 0$ completed pieces as above. The following estimates compare its endpoint law, concentrate its first two bridge moments, and control its occupation.*

The exact density of the complete word is

$$\frac{dP_w}{d\tilde{P}_w} = \frac{\prod_{g=1}^{G_0} \prod_{i=1}^{b_g} \sigma_{g,i}(e_{g,i})}{E_{\tilde{P}_w} \prod_{g=1}^{G_0} \prod_{i=1}^{b_g} \sigma_{g,i}(e_{g,i})}, \quad e^{-2\eta_w^*} \leq \frac{dP_w}{d\tilde{P}_w} \leq e^{2\eta_w^*}. \quad (6.32)$$

The product includes the final no-more-piece suffix and every connector likelihood. The endpoint averages concentrate about their reference expectations: for every $t > 0$,

$$P_w \left(\max_{f \in \mathcal{F}_M} \left| b^{-1} \sum_{i \leq b} f(\omega_i) - c_f \right| > t + C\{G_0/b + \varepsilon_M + Q/M\} \right) \leq 2|\mathcal{F}_M| \exp\{2\eta_w^* - cbt^2\}. \quad (6.33)$$

For $x \in V_y$, $G = G_{D_y}$, and

$$r_\theta(x) := \frac{e^\theta - 1}{1 - (G(x, x) - 1)(e^\theta - 1)}, \quad (6.34)$$

the logarithmic occupation moment satisfies, uniformly for $0 \leq r_\theta \leq c/h$, $h = \log M$,

$$\log E \left[\exp \left\{ \theta \sum_{i=1}^b X_i(x) \right\} \middle| w \right] = bg r_\theta + O(b r_\theta^2 + r_\theta\{G_0 + b\varepsilon_M + bQ/M\} + \eta_w^*). \quad (6.35)$$

The leading term is the number of pieces times the reference mean g , expressed in the tilt parameter r_θ . The errors record the quadratic expansion, the initial laws of the G_0 blocks, the Green and return-word comparisons, and the full-word likelihood. The estimate is uniform in x, w, b and concerns completed pieces. In both applications below every visit to V_y lies in a completed piece, as verified at the end of the proof.

For the upper-bound choice from Lemma 5.3,

$$h = (C_h \log J)^4, \quad L = J - h + O(1), \quad U = u_{J,c}, \quad c > 2,$$

put $Y = \sqrt{2b}$. On a radial word satisfying the coefficient-two coarse barrier below, the upper occupation tail satisfies

$$P \left(\max_{x \in V_y} \sum_i X_i(x) \geq U \mid w, b \right) \leq C e^{-2L} J^c e^{b/L} \times \exp \left\{ -\kappa \min \left(\frac{(p - p_*)^2}{h}, \frac{J^2}{h} \right) \right\}, \quad (6.36)$$

in the central endpoint range $\epsilon L \leq Y \leq B_L + 1$, for one fixed small $\epsilon > 0$. Here

$$p = B_L + 1 - Y, \quad B_L = \rho_J L + z_0 + h^{1/4}, \quad p_* = O(\log J).$$

The gap p measures the distance of the endpoint below the barrier, and p_* is the gap at the preferred endpoint of the terminal tilt. The coefficient-two barrier is

$$\sqrt{2T_{y,j}^S} \leq B_j := \rho_J j + z_0 + (j \wedge (J - j))^{1/4} \quad (k_y < j \leq L), \quad (6.37)$$

after the fixed $O(1)$ relaxation of Lemma 5.3. For $Y < \epsilon L$, (6.36) holds with its last factor replaced by $\exp(-\kappa J^2/h)$.

Proof of Lemma 6.2. The bridge formula reduces the occupation moment to $\prod_i (1 + r_\theta m_x(\omega_i))$. We first identify the reference expectations and contract the centered endpoint transitions. This gives concentration and the logarithmic moment under $\tilde{P}_w$. We then compare this law with P_w , retaining the last suffix and every reset in the likelihood. Finally we optimize the tilt to obtain the tail bound (6.36).

Step 1. Compute bridge moments and their centers. Let

$$H(a, e) = P_a((S_{\tau-1}, S_\tau) = e), \quad \tau = \tau_{D_y^c}.$$

For the bridge from a to e , write $X(x)$ for its occupation of x . Splitting at the first visit to x , and then summing the geometric number of returns to x , gives the exact rank-one identity

$$\begin{aligned} H_\theta(a, e) &:= E_a[e^{\theta X(x)}; (S_{\tau-1}, S_\tau) = e] \\ &= H(a, e) + r_\theta G(a, x)H(x, e). \end{aligned} \quad (6.38)$$

Thus

$$\frac{H_\theta(a, e)}{H(a, e)} = 1 + r_\theta m_x(a, e), \quad m_x(a, e) := \frac{G(a, x)H(x, e)}{H(a, e)}. \quad (6.39)$$

A Harnack chain inside the fixed-ratio disk gives $0 \leq m_x \leq C$, uniformly in $x \in V_y$. Although $m_x(a, e)$ is indexed by the rough digital exit edges, the transition (6.18) first averages over that edge. From (6.39),

$$(\mathcal{H}m_x)(a) = \sum_e H(a, e)m_x(a, e) = G(a, x). \quad (6.40)$$

More generally, if $q_{x,x'}(a, e) = E^{a,e}[X(x)X(x')]$, the two ordered visit-time decompositions give

$$\begin{aligned} (\mathcal{H}q_{x,x'})(a) &= G(a, x)G(x, x') + G(a, x')G(x', x) \\ &\quad - \mathbf{1}_{\{x=x'\}}G(a, x). \end{aligned} \quad (6.41)$$

Thus the only functions subsequently fed into a return-word kernel are Green functions with their singularities a distance at least $M/2$ from A_y .

We next identify the common center and the only observables that require comparison. With λ_A defined as the first-hit law of the inner vertex boundary, the strong Markov property gives

$$d_x := \sum_a \lambda_A(a)G_{D_y}(a, x) = G_{D(y, eM)}(y, x) - G_{D(y, M)}(y, x) + O(M^{-1} \log M). \quad (6.42)$$

Indeed the one-edge collar can return from A_y to $D(y, M/2)$ before exiting $D(y, M)$ with probability $O(M^{-1})$, while the Green function is at most $C \log M$. This remainder is $o(\varepsilon_M)$.

Apply the Green comparison (6.11) to both disks in (6.42). For distinct arguments, their inradii are comparable to M , so it reads

$$G_A(u, z) = \frac{2}{\pi} g_A(u, z) + k_{z-u} + O(M^{-7/24} \log M), \quad (6.43)$$

with $k_w = k_0 + (2/\pi) \log |w| - a(w)$ for $w \neq 0$. The lattice correction is identical in the two domains and cancels. When $x = y$, use instead the diagonal formula in [10, Theorem 1.2]; its additive potential-kernel constant also cancels. The core points lie a fixed fraction of the radius from both boundaries, and the union-of-squares domains are between disks whose radii differ by $O(1)$. The Brownian Green-function difference for the concentric disks is exactly $\log e = 1$. Consequently, uniformly in $x \in V_y$,

$$d_x = g + O(\varepsilon_M), \quad \varepsilon_M = CM^{-7/24} \log M. \quad (6.44)$$

The identities (6.40)–(6.41) now give the centers (6.30)–(6.31).

Step 2. Compare and contract centered transitions. The state of $R_{g,i}^w$ is the preceding directed exit edge, and the walk restarts at its head; its target is the next inward directed edge. A word is called admissible only when its cylinder has positive probability, so every row normalization below is defined. Brownian row masses of its elementary letters are scalar, while Lemma 5.2 gives their relative lattice comparison. The exact whole-word weighting formula (6.14) therefore yields, for a scalar $s_{g,i} > 0$,

$$\sup_e \left| \frac{\sigma_{g,i}(e)}{s_{g,i}} - 1 \right| \leq \frac{Cq_i}{M}. \quad (6.45)$$

This estimate holds for the complete word, since (6.14) retains the row masses of all its letters.

In the common logarithmic conformal coordinate, every normalized Brownian letter is the rotation convolution (6.6), and the terminal disk exit is (6.5). Apply Lemma 6.1 to the exact exit-edge averages (6.40)–(6.41). For consecutive internal transitions $r \leq s$ in one block and every $f \in \mathcal{F}_M$, it gives

$$\left| \Lambda_M \tilde{K}_{g,r}^w \tilde{K}_{g,r+1}^w \cdots \tilde{K}_{g,s}^w f - c_f \right| \leq C \left\{ \varepsilon_M + \frac{1}{M} \sum_{i=r}^s (q_i + 1) \right\}. \quad (6.46)$$

The two terms are compared separately with the same Haar integral of the disk-Green traces (6.7)–(6.10). The estimate therefore controls complete prefixes even though Λ_M need not be invariant for a single discrete transition. In a lower source interval the common coordinate is χ_y ; its terminal chart error is included in the displayed sum. An inter-source connector ends this comparison.

To remove dependence on the initial law of each block, we also need a uniform contraction in total variation. The elliptic Harnack inequality [1, Theorem 1.1] states that for every $d \geq 1$ there is C_d such that a nonnegative function harmonic in $B(x_0, R)$ satisfies

$$\max_{B(x_0, R/2)} u \leq C_d \min_{B(x_0, R/2)} u. \quad (6.47)$$

Starting one step inside the outer boundary is too close to use (6.47) directly. Insert instead the deterministic intermediate circle

$$C_M = \partial D(y, e^{1/2}M),$$

which every successful outer-to-inner crossing must hit. For a target atom $a \in A_y$, put

$$h_a(z) = P_z(S_{H_{A_y}} = a, H_{A_y} < \tau_{D(y, eM)}), \quad s(z) = \sum_{a \in A_y} h_a(z), \quad z \in C_M.$$

A bounded chain of lattice balls of radius cM , all a fixed positive fraction from both boundary components, applies (6.47) to h_a and s . Uniformly in $a, z, z_0 \in C_M$,

$$ch_a(z_0) \leq h_a(z) \leq Ch_a(z_0), \quad cs(z_0) \leq s(z) \leq Cs(z_0).$$

After division by the total successful-leg probability this gives

$$J_{\text{in}}(z, \cdot) \geq \alpha J_{\text{in}}(z_0, \cdot) \quad (6.48)$$

where $J_{\text{in}}(z, \cdot) = h_a(z)/s(z)$ is the entrance law conditional on reaching the inner boundary, and $\alpha > 0$ is uniform in M, z, z_0 . Precomposition by all earlier within-block word factors and postcomposition by H preserve this row-independent minorization. Writing $\text{osc}(f) = \sup f - \inf f$, we obtain $\text{osc}(\tilde{K}_{g,i}^w f) \leq (1 - \alpha) \text{osc}(f)$ for every bounded f at

every within-block transition. Write $\rho = 1 - \alpha < 1$. A reset transition is used exactly and has Dobrushin coefficient at most one; there are only $G_0 - 1$ of them.

Step 3. Retain the full likelihood at resets. A connector can have an arbitrary normalized angular kernel. What is needed is a bound on the oscillation of its row mass, uniform even when the prescribed future event is rare. To obtain it, factor each reset through a deterministic separator. Fix

$$\lambda_* = e^{3/2} \in (e, e^2), \quad \Sigma_y = \partial D(y, \lambda_* M). \quad (6.49)$$

The next active outer interface after the terminal eM -circle has scale $e^2 M$. The fixed-gap geometry and (6.18) put Σ_y strictly between them and keep $D(y, \lambda_* M)$ inside the applicable source interval and global killing domain.

Let $e = (u, v) \in E_y$ be the last exit edge of a nonempty block, so the exterior kernel starts at $v = \text{out}(e)$ and $|v - y| = eM + O(1)$. Put

$$\theta_- = H_{D(y, M)}, \quad \theta_+ = \tau_{D(y, \lambda_* M)},$$

and on $\{\theta_+ < \theta_-\}$ let F_+ be the directed outward edge at θ_+ . For $g < G_0$, let $B_g^w(f, d\zeta, d\omega')$ be the genuine post-separator subkernel which realizes the rest of the no-more-piece suffix, all folded empty source intervals and safe connector signs, and the next nonempty block prefix through its first endpoint pair. For $g = G_0$ it instead contains the remaining suffix and final cemetery transition. Durations, parities, angles, and already visible edge data are outputs of this kernel, never extra scalar conditions. Set $r_g^w(f) = B_g^w(f, \text{all outputs})$. The strong Markov property gives the exact factorization

$$C_g^w(e, d\gamma) = \sum_f P_v(\theta_+ < \theta_-, F_+ = f) B_g^w(f, d\gamma). \quad (6.50)$$

Indeed, a return to $D(y, M)$ before Σ_y after the alleged last completed piece would, on the subsequent trip to the block's outer endpoint, force another exit of D_y , hence another completed terminal piece. Conversely, no reset restriction other than avoiding the inner circle is tested before Σ_y ; all remaining restrictions occur in B_g^w .

Define on the digital annulus

$$\Omega_y = D(y, \lambda_* M) \setminus D(y, M)$$

$$h_g^w(z) = E_z[r_g^w(F_+); \theta_+ < \theta_-]. \quad (6.51)$$

This is a nonnegative discrete harmonic function at every interior vertex, with exact Dirichlet value zero on the inner boundary and the possibly rough edge data r_g^w on the outer boundary. Global killing and the entire later connector/prefix occur only in that outer boundary datum. Moreover,

$$\sigma_{g, b_g}(e) = C_g^w 1(e) = h_g^w(v). \quad (6.52)$$

All such v 's lie in the band $||v - y| - eM| \leq 2$, a distance comparable to M from both components of $\partial\Omega_y$. A fixed number of overlapping lattice balls of radius cM , whose doubled balls remain in Ω_y , connects any two points of this band. Applying (6.47) along this chain gives

$$\sup_e \sigma_{g, b_g}(e) \leq C_H \inf_e \sigma_{g, b_g}(e). \quad (6.53)$$

If h_g^w vanishes at one interior start, the mean-value property and connectedness make it identically zero, and the complete word cylinder has zero mass and is discarded. Otherwise

every row is positive and the normalization $J_{g,b_g}^w = C_g^w / \sigma_{g,b_g}$ is defined. This proves (6.53) independently of how rare or rough the later event is.

Start the first block with its actual chronological law through the first endpoint. Use every normalized internal kernel and every normalized reset in chronological order, including the final cemetery output, to define $\tilde{P}_w$. Direct disintegration of the unnormalized cylinder measure gives

$$\frac{dP_w}{d\tilde{P}_w} = \frac{\prod_{g=1}^{G_0} \prod_{i=1}^{b_g} \sigma_{g,i}(e_{g,i})}{E_{\tilde{P}_w} \prod_{g=1}^{G_0} \prod_{i=1}^{b_g} \sigma_{g,i}(e_{g,i})}. \quad (6.54)$$

The initial prefix is absorbed into $\mu_{1,w}$, and the product includes the final suffix. Equation (6.45), applied also to a centered final suffix, bounds the logarithmic row oscillation of centered factors by Cq_i/M ; (6.53) costs $\log C_H$ once per genuinely noncentered reset. Thus in the upper centered extension $G_{\text{nc}} = 0$, while in the lower construction $G_{\text{nc}} \leq G_0 \leq m_0$. Scalar constants cancel between numerator and denominator, so, after absorbing $\log C_H$ into C ,

$$\left| \log \frac{dP_w}{d\tilde{P}_w} \right| \leq 2\{CQ/M + CG_{\text{nc}}\} = 2\eta_w^*. \quad (6.55)$$

Step 4. Concentrate the endpoint averages. Fix w and a normalized $f \in \mathcal{F}_M$, expose the sequential joint chain under $\tilde{P}_w$, and use the Doob martingale of the whole sum $\sum_{i \leq b} f(\omega_i)$. Changing ω_i changes the conditional expectation of the future sum by at most

$$2G_0 \sum_{j \geq 0} \rho^j \leq \frac{2G_0}{1 - \rho};$$

the factor G_0 accounts for the possibility that a reset preserves all the remaining variation. Azuma–Hoeffding consequently gives

$$\tilde{P}_w \left(\left| \sum_{i \leq b} f(\omega_i) - E_{\tilde{P}_w} \sum_{i \leq b} f(\omega_i) \right| > bt \right) \leq 2e^{-cbt^2}. \quad (6.56)$$

To center this concentration bound at bc_f , let μ be the law of the first endpoint of a block, and consider its k -th endpoint. Comparing the same transition product started from μ and from Λ_M costs $2\rho^{k-1}$ by contraction. The reference product is controlled by (6.46), so

$$\begin{aligned} \left| \mu \tilde{K}_{g,1}^w \cdots \tilde{K}_{g,k-1}^w f - c_f \right| &\leq 2\rho^{k-1} \\ &+ C \left\{ \varepsilon_M + \frac{1}{M} \sum_{j < k} (q_j + 1) \right\}. \end{aligned} \quad (6.57)$$

For $k = 1$ this just uses $\|f\|_\infty \leq 1$. Summing (6.57) over all endpoints and blocks, and using $\sum_j (q_j + 1) \leq CQ$, gives

$$\left| E_{\tilde{P}_w} \sum_{i \leq b} f(\omega_i) - bc_f \right| \leq C\{G_0 + b\varepsilon_M + bQ/M\}. \quad (6.58)$$

A union bound under $\tilde{P}_w$, followed by (6.32), proves (6.33).

Step 5. Bound the exponential occupation moment. Conditional on the endpoints, (6.38)–(6.39) makes the exponential moment $\prod_i (1 + rm_x(\omega_i))$. Under $\tilde{P}_w$, put

$$\psi_i = \log(1 + rm_x(\omega_i)). \quad (6.59)$$

Since $0 \leq m_x \leq C$, $\psi_i = rm_x(\omega_i) + O(r^2)$ and $\text{osc}(\psi_i) \leq Cr$. The same influence calculation as for (6.56) shows that every Doob martingale difference for $\sum_i \psi_i$ is at most $CG_0 r/(1 - \rho)$. Conditional Hoeffding and Jensen therefore give

$$0 \leq \log E_{\tilde{P}_w} e^{\sum_i \psi_i} - E_{\tilde{P}_w} \sum_i \psi_i \leq Cbr^2. \quad (6.60)$$

The exact exit-edge average and its Taylor remainder are

$$\sum_e H(a, e) \log(1 + rm_x(a, e)) = rG(a, x) + O(r^2), \quad (6.61)$$

because $m_x \leq C$ and $\sum_e H(a, e)m_x(a, e) = G(a, x) \leq C$. The endpoint mean estimate (6.58), applied to m_x/C , and the reference mean (6.44) give

$$\sum_{i \leq b} E_{\tilde{P}_w} m_x(\omega_i) = bg + O\{G_0 + b\varepsilon_M + bQ/M\}. \quad (6.62)$$

Combining (6.59)–(6.62) proves the completed-piece version of (6.35) under $\tilde{P}_w$. Finally, (6.32) puts the exact full-word density (6.54) between $e^{-2\eta_w^*}$ and $e^{2\eta_w^*}$. Multiplying the nonnegative exponential by this global density and taking logarithms changes the answer by at most $2\eta_w^*$, exactly the last error in (6.35). This proves (6.35) under P_w .

Step 6. Optimize the terminal upper tail. The first choice of tilt will cancel the endpoint Gaussian cost. A further adjustment supplies decay in the distance from the preferred endpoint, which is needed when summing all endpoint bands. The inverse of (6.34) is

$$\theta(r) = \log \frac{1 + G(x, x)r}{1 + (G(x, x) - 1)r}. \quad (6.63)$$

Take $r_0 = (gL)^{-1}$. Since $G(x, x) = gh + O(1)$ and $h^2/J = o(1)$,

$$\theta(r_0) = \frac{1}{gJ} + O(J^{-2} + h^2 J^{-3}), \quad \theta(r_0)U = 2J - c \log J + O(1), \quad bgr_0 = b/L. \quad (6.64)$$

Since $G_0 = 1$, $b = O(J^2)$, $Q \leq J^3$, and $M = e^{(C_h \log J)^4}$, every error term in (6.35) is $O(1)$ at r_0 , uniformly in the word. Write $\theta_x(r_0)$ for (6.63) at site x . Since $r_0 = O(J^{-1}) \leq c/h$, (6.35) applies; also $|V_y| \leq Ce^{2h}$ and (6.64) is uniform over $x \in V_y$. Thus Chernoff's inequality gives

$$\begin{aligned} P \left(\max_{x \in V_y} \sum_{i=1}^b X_i(x) \geq U \mid w \right) &\leq \sum_{x \in V_y} e^{-\theta_x(r_0)U} E \left[e^{\theta_x(r_0) \sum_{i=1}^b X_i(x)} \mid w \right] \\ &\leq Ce^{2h} e^{-2J + c \log J + b/L} = Ce^{-2L} J^c e^{b/L}. \end{aligned}$$

This is the first line of (6.36).

The first tilt's factor $e^{b/L}$ will cancel the traversal endpoint cost. We need additional decay in the endpoint gap to sum the ballot factor $1 + p$ over all endpoint bands. Choose C_0 to dominate the quadratic error $O(br^2)$ in (6.35), and set

$$F_b(r) = \theta(r)U - bgr - C_0 br^2.$$

The remaining moment errors are $O(1)$ uniformly on each fixed interval $0 \leq r \leq a/h$ allowed by (6.35). Hence Chernoff's inequality and the union over sites give a bound of $C \exp\{2h - F_b(r)\}$. An increase in F_b beyond $F_b(r_0)$ therefore improves the preceding tail estimate by the corresponding exponential factor. We now quantify that increase. Uniformly on every fixed interval $0 \leq r \leq a/h$,

$$\begin{aligned}\theta'(r) &= \frac{1}{(1 + G(x, x)r)(1 + (G(x, x) - 1)r)}, \\ -\theta''(r) &= \theta'(r) \left\{ \frac{G(x, x)}{1 + G(x, x)r} + \frac{G(x, x) - 1}{1 + (G(x, x) - 1)r} \right\}.\end{aligned}$$

Thus $c_a \leq \theta'(r) \leq C_a$, $c_a h \leq -\theta''(r) \leq C_a h$, and $|\theta'''(r)| \leq C_a h^2$. Since $b = O(J^2)$, after increasing J ,

$$-C_1 J^2 h \leq F_b''(r) \leq -c_1 J^2 h. \quad (6.65)$$

Also

$$\theta'(r_0) = 1 - \frac{2h}{L} + O(L^{-1} + h^2 L^{-2}). \quad (6.66)$$

Define b_* by $F_{b_*}'(r_0) = 0$. Since

$$b_* = \frac{U\theta'(r_0)}{g + 2C_0 r_0},$$

the relations $J = L + h + O(1)$ and $h^2 = o(J)$ give

$$b_* = 2L^2 - cL \log J + O(L + h^2 + h \log J) = 2L^2 - cL \log J + O(L).$$

Consequently

$$Y_* = \sqrt{2b_*} = 2L - \frac{c}{2} \log J + O(1), \quad (6.67)$$

The preferred gap is therefore $B_L + 1 - Y_* = O(\log J)$. More exactly,

$$F_b'(r_0) = (g + 2C_0 r_0)(b_* - b) = (g + o(1)) \frac{(Y_* - Y)(Y_* + Y)}{2}. \quad (6.68)$$

Fix a sufficiently small constant $\delta > 0$. If $\epsilon L \leq Y \leq B_L + 1$ and $|Y - Y_*| \leq \delta J$, Taylor's formula, (6.65), and the displacement

$$s = \frac{F_b'(r_0)}{C_1 J^2 h}$$

(with C_1 enlarged once) give

$$\sup_r F_b(r) - F_b(r_0) \geq \kappa \frac{(Y - Y_*)^2}{h}. \quad (6.69)$$

This displacement is feasible. If it is negative, the barrier and $h^{1/4} = O(\log J)$ imply $Y - Y_* = O(\log J)$, whence $|s| = O(\log J / (Jh)) = o(r_0)$. If it is positive, choosing δ small ensures $s < a/(2h)$. When $\epsilon L \leq Y \leq Y_* - \delta J$, choose instead a fixed $s = a_0/h > 0$, with a_0 small compared with δ/C_1 . Equations (6.65)–(6.68) then give

$$F_b(r_0 + s) - F_b(r_0) \geq \kappa J^2/h. \quad (6.70)$$

There is no range $Y \geq Y_* + \delta J$ because the upper barrier permits only an $O(\log J)$ overshoot above Y_* .

Finally, if $Y < \epsilon L$, take $r = a_0/h$ directly. The integral formula $\theta(r) = \int_0^r \theta'(t) dt$ gives $\theta(a_0/h) \geq c_a a_0/h$. First choosing ϵ and then a_0 small yields

$$F_b(a_0/h) \geq \kappa_0 J^2/h, \quad F_b(r_0) = O(J) = o(J^2/h). \quad (6.71)$$

Equations (6.69)–(6.71) supply the gain beyond the first tilt. Since $p - (B_L + 1 - Y_*) = Y_* - Y$, this is the decay claimed in (6.36). The expansion (6.67) is uniform in x , so a common choice

$$p_* = B_L + 1 - 2L + \frac{c}{2} \log J$$

differs from each site's preferred gap by $O(1)$. The inequality $(u - v)^2 \geq u^2/2 - v^2$ absorbs this bounded shift by changing C, κ . Thus the union over sites retains the same decay factor, proving both endpoint alternatives in (6.36).

Step 7. Account for the initial and final portions. In the applications, all core occupation belongs to completed pieces. A lower source block begins and ends on interfaces outside D_y . An upper block with $k \leq C_0 \log J$ starts at the origin, which is outside D_y , and ends at σ_y , also outside D_y . Before the first inward crossing of radius M the path cannot visit V_y . After any visit to V_y , reaching the block's outer endpoint forces a later crossing of radius eM , so that inward crossing and all intervening core occupation form a completed piece. Thus the initial and final incomplete portions contribute exactly zero to every $X(x)$ used here. Upper cells with $k > C_0 \log J$, including the cells near the walk's starting point, are treated separately by the one-site Green estimate in (6.81). ■

6.3. A LOCAL UPPER BOUND FOR THE MAXIMUM.

PROPOSITION 6.3. *For every fixed $\beta > 0, c > 2$ and $A < \infty$,*

$$P \left(\max_{\|x\| \leq e^J J^{-\beta}} L_{\tau_{e^J}}(x) \geq u_{J,c} \right) \leq J^{-2\beta+c-2+o(1)} + O(J^{-A}). \quad (6.72)$$

The $o(1)$ is uniform when β, c range over compact subsets of $(0, \infty)$ and $(2, \infty)$, respectively.

Proof of Proposition 6.3. We work with one terminal cell at a time and extend the walk to a disk centered at that cell. This puts the source law, traversal barrier, and terminal word in the same centered geometry. The traversal estimate and the terminal exponential moment then have opposite Gaussian factors; after their cancellation, the ballot prefactor supplies the required power J^{-2} .

Step 1. Choose cells and exclude barrier crossings. Use the upper buffer

$$h = (C_h \log J)^4, \quad M = e^h, \quad L = J - h,$$

with C_h fixed large. Cover the target disk by original M -cells obtained from the dilated image of Rosen's predetermined net F_L , rounding each image center to its nearest lattice point. The chart has uniformly bounded distortion on the relevant compact patch, so these centers are cM -separated and form a CM -cover. After altering the fixed core and terminal radius factors, every lattice point lies in a core, the overlap multiplicity is bounded, and Lemmas 5.3 and 6.2 apply to the same cells. We continue to denote their radii by $M/2$ and eM . For a cell center with $\|y\| > M$, use the exact index from (5.62),

$$u_y = \log \frac{3e^J}{\|y\|}, \quad k = k(y) = \lceil u_y \rceil, \quad m = k + 1. \quad (6.73)$$

Cells with $\|y\| \leq M$ are assigned to the final sitewise range $k > C_0 \log J$. Uniform distortion of the fixed chart makes (6.73) equal to the spherical source index (5.2) up to a bounded additive error, which is absorbed by the barrier slack. Every target cell has

$$k \geq s := \beta \log J + O(1). \quad (6.74)$$

For this fixed cell continue the walk from τ_{e^J} to $\sigma_y = \tau_{D(y, 3e^J)}$. By (5.42), a high local time in the cell before τ_{e^J} is also high before σ_y . All counts, words, terminal bridges, and the source index below refer to this single y -centered extended path. This domination is applied separately to each cell before the final union bound.

First exclude a crossing of the fixed positive-intercept coefficient-two barrier (6.37) at or before level L . Since $h > (4 \log J)^4$ after choosing C_h large, all these levels lie in the range of (5.69)–(5.71). Restricting the source shells to $k \geq s$ and using (5.44) gives

$$\begin{aligned} P(\text{some bad prefix in the target}) &\leq C \sum_{k \geq s} \sum_{l=k+1}^L e^{2(l-k) + \frac{1}{2}(l \wedge (J-l))^{1/4}} e^{-2l - (l \wedge (J-l))^{1/4}} \\ &\leq J^{-2\beta + o(1)}. \end{aligned} \tag{6.75}$$

The direct lattice estimates (5.64)–(5.71), for both the representative count and the probability of a crossing at one center, apply through the source-shell range $k \leq C_0 \log J$.

Step 2. Use the source-inclusive endpoint estimate. Fix an original cell with $s \leq k \leq C_0 \log J$, where its source index is $k = k(y)$ in (6.73). The extension to σ_y makes all its radial interfaces centered at y , as required by Lemma 5.3; our choice of h satisfies $h/\log J \rightarrow \infty$ and $h = o(J)$. On the complement of the crossing event in Step 1, its prefix satisfies the barrier (6.37) and hence the no-crossing event in (5.45). Put $Y = \sqrt{2T_{y,L}^S}$, and use the same small cutoff $\epsilon_0 = \epsilon$ as in Lemma 6.2. Take $\epsilon_0 L \leq q \leq B_L + 1$. With $p = B_L + 1 - q$, the bound (5.45) gives

$$P(\text{prefix good}, Y \in I_q) \leq C(k+1)^C (1+p)L^{-2} \exp\{-q^2/(2L)\}. \tag{6.76}$$

Here $I_q = [q, q+1]$, and $B_L = B_L^J$ in that lemma because $L = J - h$. The estimate already includes the source population and its last death decision; no conditioning on a source height is needed.

Step 3. Combine the traversal and terminal costs. The complementary endpoint range is controlled directly by the terminal tilt. If $Y < \epsilon_0 L$, then $b \leq \epsilon_0^2 L^2/2 + O(L)$, and the saturated line of (6.36), uniformly in the radial word, gives

$$P(\text{terminal mark} \mid w) \leq C e^{-2L} J^c \exp\{\epsilon_0^2 L/2 - \kappa J^2/h + O(1)\}. \tag{6.77}$$

After summing this over at most $C e^{2L}$ original cells, the result is $o(J^{-A})$ for every fixed A , because $h = o(J)$, after fixing ϵ_0 sufficiently small. This is why neither (5.7) nor (5.45) was claimed in the extinction range.

For the number $b = T_L + O(1)$ of completed outward pieces in the band I_q ,

$$b = q^2/2 + O(L).$$

Since $q \geq \epsilon_0 L$, uniformly in this band the actual terminal gap satisfies $B_L + 1 - \sqrt{2b} = p + O(1)$. The bounded shift is absorbed by the same quadratic inequality used to choose a common p_* above. Conditional on the radial word, Lemma 6.2 applies; the start/end observation at the end of that lemma shows that all visits to V_y belong to its completed pieces. Multiply (6.76) by (6.36). Since $b = q^2/2 + O(L)$, the combined endpoint cost is

$$\exp\left\{-\frac{q^2}{2L} + \frac{b}{L}\right\} = e^{O(1)}.$$

Thus the cancellation is uniform throughout the central range. The remaining ballot factor $1 + p$ is summable because of the additional terminal decay:

$$\sum_{0 \leq p \leq B_L + 1} (1 + p) \exp \left\{ -\kappa \min \left(\frac{(p - p_*)^2}{h}, \frac{J^2}{h} \right) \right\} = J^{o(1)}. \quad (6.78)$$

Indeed the nonsaturated Gaussian sum is

$$O((1 + |p_*|)\sqrt{h} + h) = J^{o(1)},$$

while its saturated complement contributes $O(J^2)e^{-\kappa J^2/h} = o(1)$. The small- Y part has the saturated factor in (6.36). Thus, uniformly over all endpoint bands and all radial words,

$$P(\text{this original cell is good and marked}) \leq (k + 1)^C e^{-2L} J^{c-2+o(1)}. \quad (6.79)$$

The terminal maximum has already included all sites of the original cell, so the remaining union bound is over the original cells.

There are $O(e^{2(L-k)})$ original cells in shell k . Summing (6.79) and using (6.74) yields

$$\sum_{k=s}^{C_0 \log J} e^{2(L-k)} (k + 1)^C e^{-2L} J^{c-2+o(1)} \leq J^{-2\beta+c-2+o(1)}. \quad (6.80)$$

Step 4. Treat cells near the starting point. It remains to treat $k > C_0 \log J$. For a lattice site in shell k , Green-function identities and Lemma 3.1 give

$$P_0(H_x < \sigma_y) \leq C(k + 1)/J$$

and, conditionally on the hit,

$$P_x(L_{\sigma_y}(x) \geq u_{J,c}) \leq C e^{-2J} J^c.$$

There are $O(e^{2J-2k})$ sites in that shell. Their total contribution is

$$J^{c-1-2C_0+o(1)}. \quad (6.81)$$

Choose C_0 after A, c to make (6.81) $O(J^{-A})$. The barrier-crossing bound (6.75) is absorbed by (6.80) because $c > 2$. These two estimates and the sitewise remainder therefore give (6.72). The word-cap and digital row errors are superpolynomial by Lemma 5.3 and are included in $O(J^{-A})$. ■

7. The terminal maximum and the lower bound

The coarse construction stops at depth $L = J - h$ and produces a cell with $b = 2L^2 - 2L \log L + O(L)$ completed terminal pieces. Their mean occupation, about gb , misses a contribution of order Jh from the last h logarithmic scales. Maximizing over the terminal disk recovers this contribution: after conditioning on the entrance and exit pairs, the pieces are independent, and their normalized centered occupation field has covariance close to $2g$ times the killed Green function. A Gaussian approximation and the discrete Gaussian free field maximum then give a gain of $2\sqrt{2}g\sqrt{b}h$, which restores the missing depth.

We prove this estimate for one fixed cell before combining it with coarse-root existence. Uniform conditional failure bounds allow us to sum over the candidates and obtain a disk maximum with positive probability. Independent walk segments then yield the almost-sure lower bound at deterministic times.

7.1. **CONDITIONAL OCCUPATION IN ONE TERMINAL DISK.** Recall that $D_y = D(y, eM)$ is the killing disk, $V_y = D(y, M/2)$ is its core, and $X_i(x)$ is the occupation of x by the i th completed bridge. Write $G = G_{D_y}$ and retain the conditional moments from Section 6:

$$m_x(\omega) = E^\omega X(x), \quad q_{x,x'}(\omega) = E^\omega[X(x)X(x')], \quad \omega = (a, e).$$

Here a is the entrance vertex, e is the directed exit edge, and $\Lambda_M(a, e) = \lambda_A(a)H(a, e)$ is the reference endpoint law. As before, $\Lambda_M f = \sum_{a,e} \Lambda_M(a, e)f(a, e)$ denotes its average. We need the empirical means and second moments to be close to these reference averages for every site and pair of sites. The terminal scale below is chosen so that this simultaneous control and the final error both fit within the $J \log J$ margin.

LEMMA 7.1 (The terminal maximum). *Fix $1 < \xi < 2$ and put*

$$h = (\log J)^\xi, \quad M = e^h, \quad L = J - h.$$

Suppose an admissible centered block word has $G_0 \leq m_0$ blocks and b completed terminal pieces, where

$$b = 2L^2 - 2L \log L + O(L) \tag{7.1}$$

with a uniform remainder constant. Put $t_J = \log J / (J \log \log J)$. Call the endpoint realization moment regular if, for every $x, x' \in V_y$,

$$\begin{aligned} \left| \frac{1}{b} \sum_{i \leq b} m_x(\omega_i) - \Lambda_M m_x \right| &\leq 2t_J, \\ \left| \frac{1}{b} \sum_{i \leq b} q_{x,x'}(\omega_i) - \Lambda_M q_{x,x'} \right| &\leq 2t_J(1 + G(x, x')). \end{aligned} \tag{7.2}$$

Uniformly over these words,

$$P(\text{moment regularity fails} \mid w) \leq CM^4 e^{-cbt_J^2} = o(J^{-A}) \tag{7.3}$$

for every fixed $A > 0$.

Condition on the intrinsic sigma-field $\mathcal{R}_y$ defined in Section 5. It records the exterior pieces, the word, and every entrance and directed-exit pair, while erasing the completed bridges and their durations. For every moment-regular realization, under the conditional law given $\mathcal{R}_y$,

$$\max_{x \in V_y} \sum_{i \leq b} X_i(x) = gb + 2\sqrt{2} g\sqrt{b} h + O_P \left(J\sqrt{h} + \frac{J \log J}{\log \log J} \right). \tag{7.4}$$

The remainder is uniform over such realizations. In particular, for every fixed $\eta > 0$, the conditional probability that this maximum is at least $u_{J,2+\eta}$ tends to one uniformly.

Proof. All probabilities, moments, and covariances of bridge interiors in this proof are conditional on $\mathcal{R}_y$. We will show that the normalized centered field has maximum $2\sqrt{2} gh + O_P(\sqrt{h})$. Multiplication by $\sqrt{b}$ and restoration of the means then give (7.4); the final calculation checks that its deterministic part reaches the required level.

Step 1. Conditional moments and their common center. Fix the entrance and directed-exit pairs $(a_i, e_i)_{i \leq b}$. The erased occupation vectors are independent killed bridges under the conditional law. With $H(a, e)$ denoting the exit-edge probability from a , the one-visit and two-visit decompositions give

$$E^{a,e} X(x) = \frac{G(a, x)H(x, e)}{H(a, e)} \tag{7.5}$$

and

$$\begin{aligned} E_a[X(x)X(x'); E = e] &= G(a, x)G(x, x')H(x', e) \\ &\quad + G(a, x')G(x', x)H(x, e) - \mathbf{1}_{\{x=x'\}}G(a, x)H(x, e). \end{aligned} \quad (7.6)$$

The subtraction accounts for the visit counted in both orders when $x = x'$. Dividing (7.6) by $H(a, e)$ gives $q_{x,x'}(a, e)$.

To apply the endpoint concentration estimate, first normalize the two moment functions. The fixed-ratio Harnack estimates yield

$$\|m_x/C\|_\infty \leq 1, \quad \left\| \frac{q_{x,x'}}{C(1 + G(x, x'))} \right\|_\infty \leq 1 \quad (7.7)$$

uniformly in $x, x' \in V_y$. These are exactly the functions in the finite Green class (6.28), whose exit-edge averages are (6.40)–(6.41). Thus the word concentration estimate (6.33) applies to at most CM^4 functions. Its deterministic bias and likelihood correction are small enough: admissibility gives $G_0 \leq m_0$ and $Q \leq J^3$, while the choice $h = (\log J)^\xi$ with $\xi > 1$ makes $M = e^h$ grow faster than every power of J . Hence

$$G_0/b + \varepsilon_M + Q/M = o(t_J), \quad \eta_w^* = O(1). \quad (7.8)$$

Apply (6.33) with $t = t_J/C_1$, where C_1 is a sufficiently large fixed multiple of the normalization constant in (6.28). After restoring the normalizations in (7.7), the tolerances are those in (7.2). Thus

$$\begin{aligned} P((7.2) \text{ fails} \mid w) &\leq CM^4 \exp \left\{ -c \frac{\log^2 J}{(\log \log J)^2} \right\} \\ &= o(J^{-A}) \end{aligned} \quad (7.9)$$

for every fixed $A > 0$. Indeed, $\log M^4 = 4h$, whereas $bt_J^2 \asymp \log^2 J / (\log \log J)^2$; the latter dominates both h and $\log J$ because $\xi < 2$. The estimate is uniform over the normalized reset kernels, which are integrated out in the conditional endpoint law. This proves (7.3). The smaller terminal scale is needed here because all pairs of sites are screened simultaneously; the upper bound used the one-point exponential estimate (6.35) before its spatial union bound.

We next identify the reference averages, which determine the mean and covariance of the occupation field. Multiplying each conditional moment by $\Lambda_M(a, e) = \lambda_A(a)H(a, e)$ cancels its denominator $H(a, e)$. Summing over e uses $\sum_e H(x, e) = 1$; averaging the remaining entrance factor gives $d_x = \sum_a \lambda_A(a)G(a, x)$ from (6.42). The Green comparison (6.43)–(6.44) therefore yields

$$\begin{aligned} \Lambda_M m_x &= d_x, \\ \Lambda_M q_{x,x'} &= d_x G(x, x') + d_{x'} G(x', x) - \mathbf{1}_{\{x=x'\}} d_x, \\ \sup_{x \in V_y} |d_x - g| &\leq C\varepsilon_M. \end{aligned} \quad (7.10)$$

In particular, a moment-regular realization satisfies

$$\max_{x \in V_y} \left| \frac{1}{b} \sum_{i \leq b} E^{\omega_i} X_i(x) - g \right| \leq Ct_J + O(\varepsilon_M). \quad (7.11)$$

Center each bridge and normalize the sum:

$$\tilde{X}_i(x) = X_i(x) - m_x(\omega_i), \quad U_x = b^{-1/2} \sum_{i \leq b} \tilde{X}_i(x).$$

Conditional independence gives the covariance explicitly as

$$\text{Cov}(U_x, U_{x'}) = \frac{1}{b} \sum_{i \leq b} (q_{x,x'}(\omega_i) - m_x(\omega_i)m_{x'}(\omega_i)).$$

The moment screen replaces the average of $q_{x,x'}$ by its reference value with error $O(t_J(1+h)) = o(1)$, since $\sup_{x,x'} G(x, x') \leq Ch$. In (7.10), replacing $d_x, d_{x'}$ by g costs $O(\varepsilon_M h) = o(1)$, and symmetry gives $gG(x, x') + gG(x', x) = 2gG(x, x')$. The diagonal correction and the average of $m_x(\omega_i)m_{x'}(\omega_i)$ are bounded by (7.7). Consequently

$$\max_{x, x' \in V_y} |\text{Cov}(U_x, U_{x'}) - 2gG_{D_y}(x, x')| \leq C. \quad (7.12)$$

Thus the reference averages give a nearly constant mean g and a covariance equal to $2gG$ up to a bounded error. These are the two inputs needed for the Gaussian approximation and comparison.

Step 2. Approximation of the maximum by a Gaussian maximum. The number of coordinates is $P = |V_y| = e^{2h+O(1)}$, whereas there are only $b \asymp J^2$ independent vectors. We therefore need an approximation whose dimension dependence is logarithmic.

Let $Y_i = (Y_i(x) : x \in V_y)$ be independent centered Gaussian vectors with the respective conditional covariances of $\tilde{X}_i$, and set

$$\Gamma_x = b^{-1/2} \sum_{i \leq b} Y_i(x).$$

Then Γ has the conditional covariance of U . Chernozhukov–Chetverikov–Kato [4, Theorem 3.1], applied under the conditional bridge law, gives, for every Borel set $A \subset \mathbb{R}$ and $\delta > 0$,

$$\begin{aligned} P(\max_{x \in V_y} U_x \in A) &\leq P(\max_{x \in V_y} \Gamma_x \in A^{C_7 \delta}) \\ &\quad + \frac{C_8 \log^2 P}{\delta^3 \sqrt{b}} \{L_b + M_{b,X}(\delta) + M_{b,Y}(\delta)\}. \end{aligned} \quad (7.13)$$

Here $A^r = \{s : \text{dist}(s, A) \leq r\}$, the constants are universal, and the error depends on the following third moments:

$$\begin{aligned} L_b &= \max_{x \in V_y} \frac{1}{b} \sum_{i \leq b} E^{\omega_i} |\tilde{X}_i(x)|^3, \\ M_{b,X}(\delta) &= \frac{1}{b} \sum_{i \leq b} E^{\omega_i} \left[\max_{x \in V_y} |\tilde{X}_i(x)|^3; \max_{x \in V_y} |\tilde{X}_i(x)| > \frac{\delta \sqrt{b}}{\log P} \right]. \end{aligned}$$

The definition of $M_{b,Y}$ is the same with Y_i in place of $\tilde{X}_i$ and expectation under its Gaussian law. We will use $\delta = 1$, so it remains to bound these moments and the maximum of Γ .

A bridge reaches a fixed core site with probability of order at most h^{-1} ; conditional on reaching it, its number of visits has an exponential tail on scale h . Retaining both factors makes the third-moment error small enough. The rank-one identity (6.38) gives this tail estimate directly. Indeed, at $\theta = c_0/h$, with $c_0 > 0$ fixed and sufficiently small, (6.34) gives $r_\theta \leq C/h$, and hence

$$E^\omega(e^{\theta X(x)} - 1) = r_\theta m_x(\omega) \leq C/h.$$

For $s \geq h$, exponential Markov's inequality gives $P^\omega(X(x) > s) \leq Ch^{-1}e^{-cs/h}$. To see the same scale for smaller s , the generating function in (6.38) identifies $X(x)$ as zero with

probability $1 - m_x(\omega)/G(x, x)$ and, otherwise, as a geometric random variable of mean $G(x, x)$. Since $G(x, x) \asymp h$ on V_y , we obtain, uniformly in the endpoint pair,

$$P^\omega(X(x) > s) \leq Ch^{-1}e^{-cs/h}, \quad E^\omega|X(x) - E^\omega X(x)|^3 \leq Ch^2. \quad (7.14)$$

Here and below $s \geq 0$. Integrating the tail proves the moment bound, and therefore $L_b = O(h^2)$.

For $\delta = 1$, the truncation threshold is

$$T = \frac{\sqrt{b}}{\log P} \asymp \frac{J}{h}.$$

A union bound over P coordinates, followed by integration of (7.14), gives the truncated third moment. To make the truncation explicit, put $V_i = \max_{x \in V_y} |\tilde{X}_i(x)|$. Since $P \leq Ce^{2h}$ and $T \asymp J/h \rightarrow \infty$, (7.14) implies $P(V_i > t) \leq CP h^{-1} e^{-ct/h}$ for $t \geq T$. Therefore

$$\begin{aligned} E[V_i^3; V_i > T] &= T^3 P(V_i > T) + 3 \int_T^\infty t^2 P(V_i > t) dt \\ &\leq CP h^{-1} (T^3 + hT^2 + h^2T + h^3) e^{-cT/h}. \end{aligned}$$

Here $T/h \asymp J/h^2$ and $\log P = O(h)$, which gives

$$\frac{1}{b} \sum_{i \leq b} E^{\omega_i} \left[\max_{x \in V_y} |\tilde{X}_i(x)|^3; \max_{x \in V_y} |\tilde{X}_i(x)| > T \right] \leq \exp\{-cJ/h^2 + Ch\}. \quad (7.15)$$

The polynomial factors are absorbed by the negative exponent, since J/h^2 dominates h and $\log J$. The matching Gaussian vectors have coordinate variances at most Ch , by the second derivative of (6.38). Their truncated third-moment contribution is bounded by $\exp\{-cJ^2/h^3 + Ch\}$ after the same union bound. Finally, $\log^2 P/\sqrt{b} = O(h^2/J)$ and $L_b = O(h^2)$, so the error in (7.13) is at most

$$O(h^4/J) + \exp\{-cJ/h^2 + Ch\} = o(1). \quad (7.16)$$

All estimates depend on the endpoint data only through the uniform bounds already established.

Step 3. The Gaussian maximum. We now compare the Gaussian field Γ from Step 2 with $\sqrt{2g}\phi$, where ϕ is the zero-boundary discrete Gaussian free field in D_y :

$$E[\phi_x \phi_{x'}] = G_{D_y}(x, x').$$

The extremal-process result of Biskup–Louidor [3, (1.2), (1.6), and Theorem 2.1], together with its stated consequence for maxima on open subsets following (1.8), gives tightness of the maximum on a fixed inner subdisk after centering by

$$m_M = 2\sqrt{g} \log M - \frac{3}{4} \sqrt{g} \log \log M.$$

The digital disks here approximate a disk, and V_y is a fixed nonempty subdisk strictly inside it. In particular,

$$\max_{x \in V_y} \phi_x = 2\sqrt{g} h + O_P(\log h). \quad (7.17)$$

Only this localization, rather than the full limiting point process, is used below.

To pass to expectations, convergence in distribution alone is insufficient. The tightness just stated places every median of the maximum at $m_M + O(1)$. Its maximal coordinate

variance is $O(h)$, so Gaussian concentration places the mean within $O(\sqrt{h})$ of a median. Therefore

$$E \max_{x \in V_y} \phi_x = 2\sqrt{g}h + O(\sqrt{h}). \quad (7.18)$$

The covariance error in (7.12) has no sign, so we compare expected maxima by Gaussian interpolation. The following smooth approximation to the maximum makes a uniform entrywise covariance error sufficient. For $\lambda > 0$, define

$$F_\lambda(z) = \lambda^{-1} \log \sum_{x \in V_y} e^{\lambda z_x}.$$

If $p_x = e^{\lambda z_x} / \sum_{x'} e^{\lambda z_{x'}}$, then

$$\partial_{xx'} F_\lambda = \lambda(p_x \mathbf{1}_{\{x=x'\}} - p_x p_{x'}), \quad \sum_{x, x'} |\partial_{xx'} F_\lambda| \leq 2\lambda.$$

Gaussian interpolation between Γ and $\sqrt{2g}\phi$ therefore gives

$$\begin{aligned} & \left| EF_\lambda(\Gamma) - EF_\lambda(\sqrt{2g}\phi) \right| \\ & \leq \frac{1}{2} \max_{x, x'} |\text{Cov}(\Gamma_x, \Gamma_{x'}) - 2gG(x, x')| \sup_z \sum_{x, x'} |\partial_{xx'} F_\lambda(z)| \\ & \leq C\lambda. \end{aligned}$$

Since

$$0 \leq F_\lambda(z) - \max_x z_x \leq \frac{\log P}{\lambda},$$

the difference of the two expected maxima is bounded by $C\lambda + 2 \log P / \lambda$. Balancing these errors with $\lambda \asymp \sqrt{\log P}$ yields

$$\left| E \max_x \Gamma_x - E \max_x \sqrt{2g}\phi_x \right| \leq C \sqrt{\log P}. \quad (7.19)$$

Both fields have maximal coordinate variance $O(h)$. Combining Gaussian concentration with (7.18)–(7.19) gives

$$\max_x \Gamma_x = 2\sqrt{2}gh + O_P(\sqrt{h} + \log h) = 2\sqrt{2}gh + O_P(\sqrt{h}). \quad (7.20)$$

Apply (7.13) to each of the two half-lines. For any $K > 0$, the probability that $\max_x U_x$ is below $2\sqrt{2}gh - K\sqrt{h}$ is bounded by the corresponding Gaussian lower-tail probability with its threshold increased by C_7 , plus the error in (7.16). The upper tail is bounded by the Gaussian upper tail with its threshold decreased by C_7 , plus the same error. Equation (7.20) thus transfers to $\max_x U_x$, uniformly in the regular endpoint data. This argument compares distributions; it requires no pathwise coupling of the maxima.

Step 4. Restoring the means and the terminal depth. The conditional means are not exactly constant in x , but (7.11) gives

$$\sup_{x \in V_y} \left| \sum_{i \leq b} m_x(\omega_i) - gb \right| \leq Cb(t_J + \varepsilon_M) = O\left(\frac{J \log J}{\log \log J}\right).$$

Using $\sum_i X_i(x) = \sum_i m_x(\omega_i) + \sqrt{b}U_x$, we have

$$\left| \max_{x \in V_y} \sum_{i \leq b} X_i(x) - gb - \sqrt{b} \max_{x \in V_y} U_x \right| \leq \sup_{x \in V_y} \left| \sum_{i \leq b} m_x(\omega_i) - gb \right|.$$

The centered maximum estimate from Step 3 and $\sqrt{b} = O(J)$ now prove (7.4).

Finally, (7.1) gives

$$gb + 2\sqrt{2}g\sqrt{b}h = 2gJ^2 - 2gJ \log J + O(h^2 + h \log J + J). \quad (7.21)$$

Indeed, (7.1) implies $\sqrt{b} = \sqrt{2}L - (\log L)/\sqrt{2} + O(1 + (\log L)^2/L)$. Thus the terminal gain is $4gLh + O(h \log L + h)$, while $gb = 2gL^2 - 2gL \log L + O(L)$. Substituting $L = J - h$ uses

$$2gL^2 + 4gLh = 2gJ^2 - 2gh^2, \quad L \log L = J \log J + O(h \log J),$$

which gives (7.21). This is how the terminal maximum restores the contribution lost by stopping the coarse construction at L . Since $\xi < 2$,

$$J\sqrt{h} + \frac{J \log J}{\log \log J} + h^2 + h \log J + J = o(J \log J).$$

The level in (7.21) exceeds $u_{J,2+\eta}$ by $\eta g J \log J + o(J \log J)$, which proves the final assertion. $\blacksquare$

7.2. A POSITIVE PROBABILITY OF A LARGE DISK MAXIMUM. The terminal estimate applies to one fixed cell under its own coarse conditioning. We cannot simply select a coarse root and then apply it: the selection may reveal bridge interiors of other cells. Instead, we sum the failure probabilities over all candidate roots. The bounded expected number of roots makes this sum tend to zero, even though their terminal disks may overlap.

PROPOSITION 7.2 (The disk maximum). *For every fixed $\eta > 0$, there is $p_\eta > 0$ such that, for all sufficiently large J ,*

$$P(M_{\tau_{e^J}} \geq u_{J,2+\eta}) \geq p_\eta. \quad (7.22)$$

Proof. Take $h = (\log J)^\xi$, $1 < \xi < 2$, and $L = J - h$. Apply the lower-root construction (5.16)–(5.28) at depth L , with terminal physical radius $M = e^h$. Restrict the centers to a fixed compact subdisk so that their enlarged terminal domains are contained in $D(0, e^J)$. This keeps a fixed positive fraction of the available area, and the estimates are uniform on the restricted set. The endpoint band in (5.21), with $\rho_L = 2 - (\log L)/L$, is

$$\sqrt{2T_L} \in [2L - \log L + z_0, 2L - \log L + z_0 + C_*].$$

Squaring gives the completed-piece count $b = 2L^2 - 2L \log L + O(L)$ in (7.1).

Lemma 5.3 transfers the probability of existence of a coarse root directly from Brownian motion; the pair estimate is used on the Brownian side. At each root, the terminal pieces are ordered separately within the m_0 source intervals. The inter-source connectors are exposed and omitted from the occupation sums. Their occupation is nonnegative, so omitting it preserves a lower bound for the full-walk maximum. For the lattice coarse-root count Z_L^S , the count assertion of that lemma gives

$$P(Z_L^S > 0) \geq p_0, \quad EZ_L^S \leq C. \quad (7.23)$$

We first check that imposing moment regularity loses only $o(1)$ in the probability of root existence. Recall that the lattice event $I_y^{S,\circ}$ retains only the radial barrier and terminal band; the Brownian angular screens were dropped in its definition. Thus it is determined by the

complete radial words. The scalar radial-word comparison (5.33) and Rosen's Appendix Theorem 9.1(a), as used in (5.82), give

$$P(I_y^{S,\circ}) \leq Ce^{-2L}, \quad \sum_y P(I_y^{S,\circ}) = EZ_L^S \leq C, \quad (7.24)$$

where the sum is over the deterministic net of $O(e^{2L})$ centers. For every complete radial word realizing $I_y^{S,\circ}$, (7.3) bounds failure of moment regularity by the same $\epsilon_J = o(J^{-A})$. Consequently,

$$\begin{aligned} & E \sum_y \mathbf{1}_{I_y^{S,\circ}} \mathbf{1}_{\{\text{moment regularity fails at } y\}} \\ &= \sum_y \sum_{w: w \text{ realizes } I_y^{S,\circ}} P(w) P(\text{failure at } y \mid w) \\ &\leq \epsilon_J \sum_y P(I_y^{S,\circ}) = \epsilon_J EZ_L^S = o(1). \end{aligned} \quad (7.25)$$

The radial nature of $I_y^{S,\circ}$ is needed for this averaging: the endpoint failure estimate is conditional on w , so its bound must be weighted by $P(w)$. Equations (7.23) and (7.25) therefore show that a moment-regular lattice root exists with probability at least $p_0 - o(1)$.

For a fixed root y , use its intrinsic erasure sigma-field $\mathcal{R}_y$. The event $\mathcal{G}_y$ that y is a moment-regular coarse root is $\mathcal{R}_y$ -measurable. Given this sigma-field, the erased pieces have the independent killed bridge laws used in Lemma 7.1. Let

$$\mathcal{A}_y = \left\{ \max_{x \in V_y} \sum_{i \leq b} X_i(x) \geq u_{J,2+\eta} \right\}.$$

The uniform conclusion of that lemma gives a deterministic $\tilde{\epsilon}_J \downarrow 0$ such that

$$P(\mathcal{G}_y \cap \mathcal{A}_y^c) = E[\mathbf{1}_{\mathcal{G}_y} P(\mathcal{A}_y^c \mid \mathcal{R}_y)] \leq \tilde{\epsilon}_J P(\mathcal{G}_y). \quad (7.26)$$

Consequently

$$\begin{aligned} P(\exists y : \mathcal{G}_y \cap \mathcal{A}_y) &\geq P(\exists y : \mathcal{G}_y) - \sum_y P(\mathcal{G}_y \cap \mathcal{A}_y^c) \\ &\geq p_0 - o(1) - \tilde{\epsilon}_J EZ_L^S \geq p_0/2 \end{aligned} \quad (7.27)$$

for all sufficiently large J . Each summand uses the intrinsic bridge disintegration for its own cell; the union bound does not require disjoint terminal disks. On the event in (7.27), the full local-time maximum is at least the terminal maximum of a successful cell. This proves (7.22). ■

7.3. AN ALMOST-SURE LOWER BOUND AT DETERMINISTIC TIMES. The disk estimate gives one attempt with probability bounded away from zero. We make $(\log N)^a$ independent attempts within the first N increments. Each attempt has a smaller time budget, so its logarithmic radius decreases by about $(a/2) \log \log N$. Choosing a small keeps the resulting loss below the available $J \log J$ margin, while the probability that all attempts fail is summable.

PROPOSITION 7.3 (The deterministic-time maximum). *Let $N_j = \lfloor e^{2j} \rfloor$. For every $\eta > 0$, almost surely,*

$$M_{N_j} \geq u_{j,2+\eta}$$

for all sufficiently large j .

Proof. Reserve part of the target margin for shortening the independent blocks: first fix $0 < \delta < \eta$, and then choose $a > 0$ with $\delta + 2a < \eta$. Proposition 7.2 gives a disk-success probability $p_\delta > 0$. By Lemma 3.1, choose a fixed C large enough that

$$\sup_R P_0(\tau_R > CR^2) \leq p_\delta/2.$$

For the walk started at the center of a radius- R disk,

$$P(M_{\tau_R} \geq u_{\log R, 2+\delta}, \tau_R \leq CR^2) \geq p_\delta - P(\tau_R > CR^2) \geq p_\delta/2.$$

Translation invariance therefore gives the same success probability $p = p_\delta/2$ for a segment of length CR^2 from any starting site, uniformly for sufficiently large R .

Partition the first N_j increments into

$$q_j = \lfloor (\log N_j)^a \rfloor$$

disjoint blocks of length $m_j = \lfloor N_j/q_j \rfloor$, ignoring the remainder. Use the sites immediately after the increments in each block to define its within-block local times. The translated block paths depend on disjoint increment sets and are independent. Their local times count disjoint sets of observation times, so each block maximum is bounded by M_{N_j} . Changing the endpoint convention changes a local time by at most a bounded amount, which the strict margin $\delta + 2a < \eta$ absorbs. Choose R_j by

$$\log R_j = \frac{1}{2} \log(m_j/C) = j - \frac{a}{2} \log j + O(1).$$

The loss in logarithmic radius is therefore $(a/2) \log j + O(1)$. Substituting this into the quadratic leading term gives

$$2g(\log R_j)^2 = 2gj^2 - 2agj \log j + O(j + (\log j)^2),$$

while $g(\log R_j) \log \log R_j = gj \log j + O(j + (\log j)^2)$. Hence

$$u_{\log R_j, 2+\delta} = u_{j, 2+\delta+2a+o(1)} \geq u_{j, 2+\eta} \quad (7.28)$$

for all sufficiently large j . The coefficient $2a$ is the cost of splitting the time budget into q_j blocks. Each block therefore succeeds at the target level with probability at least p , and independence implies

$$P(M_{N_j} < u_{j, 2+\eta}) \leq (1-p)^{q_j} \leq e^{-cj^a}. \quad (7.29)$$

The right-hand side is summable, so Borel–Cantelli proves the assertion. $\blacksquare$

8. Escape below the critical scale

The localized disk estimate must exclude a favorite at every time in a block, not just at its endpoints. Monotonicity of local times makes this possible: use the global maximum at the left endpoint and the local time in the target disk at the right endpoint.

PROPOSITION 8.1 (Escape below the critical scale). *For every $\gamma > \frac{1}{2}$, almost surely,*

$$\liminf_{n \rightarrow \infty} \frac{R_n^-}{\sqrt{n}/(\log n)^\gamma} = \infty. \quad (8.1)$$

Proof. Fix $\gamma > 1/2$ and a positive integer spatial multiplier A_0 . Choose

$$0 < \delta < \frac{2\gamma - 1}{4}.$$

Set $N_j = \lfloor e^{2j} \rfloor$. Proposition 7.3 gives, for all sufficiently large j ,

$$M_{N_j} \geq u_{j,2+\delta}. \quad (8.2)$$

Consider any $n \in [N_j, N_{j+1}] \cap \mathbb{N}$. If $x \in \mathcal{F}_n$ lies in

$$B\left(0, A_0 \frac{\sqrt{n}}{(\log n)^\gamma}\right),$$

then monotonicity of local times and (8.2) give

$$L_{N_{j+1}}(x) \geq L_n(x) = M_n \geq M_{N_j} \geq u_{j,2+\delta}.$$

Since $n \leq N_{j+1}$ and $\log n \geq \log N_j$, the same site lies in

$$B(0, C_{A_0} e^{j+1} j^{-\gamma}) \quad (8.3)$$

throughout the block.

Take

$$K_j = C_* e^{j+1} \sqrt{\log j}, \quad J'_j = \log K_j.$$

By the maximal inequality in Lemma 3.1, C_* can be chosen so that

$$\sum_j P(\tau_{K_j} \leq N_{j+1}) < \infty. \quad (8.4)$$

For large j , the ball (8.3) is contained in

$$B(0, K_j (J'_j)^{-\gamma}).$$

Moreover $J'_j = j + O(\log \log j)$, and the fixed extra coefficient δ gives

$$u_{j,2+\delta} \geq u_{J'_j,2+2\delta} \quad (8.5)$$

for all large j , because

$$u_{j,2+\delta} - u_{J'_j,2+2\delta} = \delta g j \log j + O(j \log \log j) > 0.$$

Thus the extra coefficient δ absorbs the enlargement of the stopping disk.

Choose independently an upper-error exponent $A_{\text{err}} > 2$. If (8.2) holds and the early exit in (8.4) does not occur, any such favorite forces

$$\max_{x \in B(0, K_j (J'_j)^{-\gamma})} L_{\tau_{K_j}}(x) \geq u_{J'_j,2+2\delta}.$$

Proposition 6.3, with $\beta = \gamma$ and $c = 2 + 2\delta$, bounds the probability of this last event by

$$(J'_j)^{-2\gamma+2\delta+o(1)} + O(j^{-A_{\text{err}}}). \quad (8.6)$$

Our choice of δ makes the power in (8.6) strictly smaller than -1 . Hence (8.6) is summable. The lower bound (8.2) holds eventually almost surely, and Borel–Cantelli excludes both upper events and early exits eventually. The block implication therefore proves that eventually no favorite in the j -th block lies in the ball with multiplier A_0 . Intersecting over $A_0 \in \mathbb{N}$ proves (8.1). ■

Proof of Theorem 1.1. Propositions 4.5 and 8.1 give the two assertions for each fixed exponent. Intersect the probability-one events for rational $\gamma > 1/2$ and for the single critical exponent $1/2$. If $\gamma > 1/2$ is real, choose a rational $q \in (1/2, \gamma)$; then

$$\frac{R_n^-}{a_\gamma(n)} = \frac{R_n^-}{a_q(n)} (\log n)^{\gamma-q} \longrightarrow \infty.$$

For $\gamma \leq 1/2$, the inequality $R_n^+/a_\gamma(n) \leq R_n^+/a_{1/2}(n)$ gives the other assertion on the same event. This proves the simultaneous formulation. ■

Appendix A. Notation guide

We use n for physical time and $L = J - h$ for the retained traversal depth; J and h are the outer and terminal logarithmic radii. The table collects notation used across subsections.

Symbol	Meaning	Where used
$L_n(x), M_n$	Site local time and its spatial maximum.	Section 1
$\mathcal{F}_n, R_n^-, R_n^+$	Favorite set and its nearest and farthest distances from the origin.	Section 1
$a_\gamma(n)$	$\sqrt{n}/(\log n)^\gamma$.	Theorem 1.1
$g, u_{J,c}$	$g = 2/\pi$ and $u_{J,c} = 2gJ^2 - cgJ \log J$.	Both disk estimates
$D(x, r), \tau_r$	Lattice disk and exit time from $D(0, r)$.	Section 3
G_D	Green function for the walk killed on leaving D .	Sections 3 and 6
z, N, K	Geometric weight, $N = (1 - z)^{-1}$ and clock with $P(K = n) = (1 - z)z^n$.	Section 4
X_K^*, U_x	Priority-selected favorite and independent site priorities.	Lemma 4.2
σ, ℓ	First and last visits to the selected favorite.	Lemma 4.2
$\mathcal{B}, \tau_c$	Exposed excursion/suffix tuple and terminal-arm killing time.	Lemma 4.2
J, h, M, L	Outer logarithmic radius, terminal logarithmic radius, $M = e^h$, and retained traversal depth $L = J - h$.	Sections 5–7
r_j, ϑ_j	Conformal radius $r_0 e^{-j}$ and geodesic radius $2 \arctan(r_j/2)$.	Section 5
$\rho_L, \alpha_{z,\pm}$	Barrier slope $2 - (\log L)/L$ and its upper/lower curved barriers.	Section 5
F_L, k_y	Deterministic net of centers and a center's source shell.	Section 5
$T_{y,j}^{1,m_0}$	Traversal count within the first m_0 complete source excursions.	Section 5
I_y°, S_y	Truncated Brownian root event and connector safety event.	Section 5
χ_y, Ψ_J	Centered conformal coordinate and dilated stereographic chart.	Section 5
w, Q, G_0	Complete radial word, number of shell decisions, and number of nonempty source blocks.	Section 6
D_y, A_y, V_y, E_y	Terminal killing disk, entrance interface, core, and directed exit edges.	Section 6
$\omega_i, X_i(x)$	Entrance/exit pair and occupation at x of its killed bridge.	Section 6
$m_x, q_{x,x'}$	First and second occupation moments under the bridge with prescribed endpoints.	Sections 6–7
$P_w, \tilde{P}_w$	True complete-word endpoint law and its sequentially normalized comparison law.	Section 6
Λ_M	Reference entrance/exit measure obtained from harmonic measure at the center.	Section 6
$\mathcal{R}_y$	Intrinsic information exposing exterior paths and bridge endpoints while erasing bridge interiors.	Sections 5 and 7
Z_L^S	Number of lattice centers satisfying the radial lower barrier and terminal endpoint band.	Sections 5 and 7

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