Explicit non-inducible rooted cluster morphisms

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Abstract

The augmentation of the Markov cluster algebra is explicit but not inducible as a rooted cluster morphism. This answers a question of Chang and Zhu negatively.

1 Rooted morphisms

Rooted cluster morphisms were introduced by Assem, Dupont, and Schiffler [1]. Chang and Zhu [4, Definition 2.8] call such a morphism *explicit* if its values on the initial cluster determine it uniquely, and *inducible* if it extends to the ambient Laurent rings. They ask whether every explicit rooted cluster morphism is inducible [4, Problem 1]. We give a counterexample using the Markov cluster algebra.

The construction uses its positive grading, recorded by Muller [6, Section 11.2], and the propagation of zero values under mutation. The latter argument appears in Castronovo–Gorsky–Simental–Speyer [3, arXiv version, Section 3.4.1] and Beyer–Muller [2, Section 5.2] in their study of deep points. Here we use these facts to obtain uniqueness of the augmentation among homomorphisms to reduced rings, and hence the required rooted cluster morphism.

All rings are commutative, and all ring homomorphisms preserve the identity. We work with coefficient-free cluster algebras over $\mathbb{Z}$. A seed is a pair $\Sigma = (\mathbf{x}, B)$, where $\mathbf{x} = (x_1, \dots, x_n)$ consists of algebraically independent variables over $\mathbb{Q}$, $n \geq 1$, and $B = (b_{ij})_{1 \leq i,j \leq n}$ is a skew-symmetrizable integer matrix. Thus there are positive integers d_i such that $d_i b_{ij} = -d_j b_{ji}$ for all i, j. Every variable in $\mathbf{x}$ is exchangeable; there are no frozen variables. Mutation in direction $k \in \{1, \dots, n\}$ replaces x_k by

$$x_k' = \frac{\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{x_k},$$

where an empty product is 1, and replaces B by the matrix with entries

$$b_{ij}' = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & i, j \neq k. \end{cases}$$

The cluster algebra $\mathcal{A}(\Sigma)$ is the $\mathbb{Z}$-subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by all variables obtained by finite sequences of mutations. A rooted cluster algebra retains the choice of Σ. By the Laurent phenomenon [5, Theorem 3.1],

$$\mathcal{A}(\Sigma) \subseteq \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

Following [4], its ambient ring is $L_{\Sigma,\mathbb{Q}} = \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

For another seed $\Sigma' = (\mathbf{y}, B')$, a rooted cluster morphism $f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ is a ring homomorphism with the following properties. Each initial variable maps to an initial variable of Σ' or to an integer, and f commutes with every biadmissible mutation sequence. Write μ_u for mutation at the cluster variable u, acting on a seed and its labelled variable positions. A finite sequence $(u_1, \dots, u_\ell)$, with $\ell \geq 0$, is *admissible* if each u_j is exchangeable after the preceding mutations; it is *biadmissible* if its image sequence under f is also admissible in the target. More precisely, for every such sequence and every initial variable v,

$$f\big(\mu_{u_\ell} \cdots \mu_{u_1}(v)\big) = \mu_{f(u_\ell)} \cdots \mu_{f(u_1)}\big(f(v)\big).$$

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Each mutation replaces the variable at its chosen position and leaves the other positions unchanged; on the right, integers are left fixed. The empty sequence acts as the identity. These are the coefficient-free instances of the axioms in [4].

Precisely, f is explicit if every rooted cluster morphism $g : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ satisfying $g(x_i) = f(x_i)$ for $1 \le i \le n$ equals f. It is inducible if there is a ring homomorphism $L_{\Sigma,\mathbb{Q}} \to L_{\Sigma',\mathbb{Q}}$ whose restriction is f.

2 The Markov example

Let x, y, z be algebraically independent over $\mathbb{Q}$, and let A be the cluster algebra with initial seed

$$\Sigma = \left((x, y, z), B = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix} \right).$$

This is the coefficient-free Markov cluster algebra. Matrix mutation gives $\mu_k(B) = -B$ and $\mu_k(-B) = B$ for $k = 1, 2, 3$. Consequently, in every cluster (a, b, c), mutation at a has exchange relation

$$aa' = b^2 + c^2, \tag{1}$$

with the analogous relations in the other two directions.

Theorem 1. *There is a ring homomorphism $\epsilon : A \to \mathbb{Z}$ sending every cluster variable to zero. For every coefficient-free rooted cluster algebra C, the composite*

$$f : A \xrightarrow{\epsilon} \mathbb{Z} \hookrightarrow C$$

is an explicit rooted cluster morphism which is not inducible.

Proof. Give a Laurent monomial $x^i y^j z^k$, with $i, j, k \in \mathbb{Z}$, degree $i + j + k$. A Laurent polynomial is homogeneous of degree d if all its nonzero monomials have degree d. The initial variables have degree 1, and (1) shows inductively that every cluster variable is homogeneous of degree 1. The Laurent phenomenon ensures that these are homogeneous elements of the Laurent polynomial ring.

For each integer $d \ge 0$, let A_d be the subgroup of A consisting of its homogeneous elements of degree d, together with zero. Since cluster variables generate A over $\mathbb{Z}$, the Laurent grading restricts to

$$A = \bigoplus_{d \ge 0} A_d, \qquad A_0 = \mathbb{Z}, \qquad A_d A_e \subseteq A_{d+e} \quad (d, e \ge 0).$$

Projection onto A_0 therefore defines a ring homomorphism

$$\epsilon \left(\sum_{d \ge 0} a_d \right) = a_0 \qquad (a_d \in A_d),$$

where each sum has finitely many nonzero terms. In particular, $\epsilon(1) = 1$, and ϵ kills every cluster variable.

We next prove uniqueness with a more general target. A ring is *reduced* if its only nilpotent element is zero. Let R be a reduced ring, and let $g : A \to R$ be a ring homomorphism satisfying $g(x) = g(y) = g(z) = 0$. Suppose that g kills all variables in a cluster (a, b, c). Mutate at a to obtain (a', b, c), and then at b to obtain a variable $b^\star$. The second exchange relation is

$$bb^\star = (a')^2 + c^2.$$

Applying g gives $g(a')^2 = 0$, hence $g(a') = 0$. The same argument works in each direction. Induction on the length of a mutation sequence shows that g kills every cluster variable. Since these variables generate A, the map g is ϵ followed by the canonical map $\mathbb{Z} \to R$.

For the composite f in the statement, all initial variables map to the integer zero. There is no nonempty biadmissible sequence, since its first image would have to be an exchangeable variable of C. Thus f is a

rooted cluster morphism. The algebra C is a subring of a rational function field, so it is reduced. The preceding uniqueness statement applies to every rooted cluster morphism with the same initial values as f, proving that f is explicit.

Finally, an extension $\widetilde{f}$ to the ambient rings would send the invertible element x to zero. This would give

$$1 = \widetilde{f}(xx^{-1}) = \widetilde{f}(x)\widetilde{f}(x^{-1}) = 0,$$

which is impossible in the ambient ring of C. □

Taking $C = A$ gives an endomorphism answering Chang–Zhu Problem 1. Its image is $\mathbb{Z}$, and it satisfies $f^2 = f$.

The role of reduced targets can also be seen in a quotient of A. Let $I = (x, y, z)A$ be the ideal generated by the initial variables, and let $A_{>0} = \bigoplus_{d>0} A_d = \ker \epsilon$. For an ideal J, its radical is $\sqrt{J} = \{a : a^m \in J \text{ for some integer } m \geq 1\}$; the reduction of a ring S is $S_{\mathrm{red}} = S/\sqrt{(0)}$.

Proposition 2. *One has $\sqrt{I} = A_{>0}$ and $(A/I)_{\mathrm{red}} \cong \mathbb{Z}$, but A/I is not reduced.*

Proof. Apply the uniqueness argument in the proof of Theorem 1 to the quotient map $A \to A/\sqrt{I}$. Every cluster variable lies in $\sqrt{I}$, so $A_{>0} \subseteq \sqrt{I}$. Conversely, $I \subseteq \ker \epsilon$ and $A/\ker \epsilon \cong \mathbb{Z}$ is reduced, which gives $\sqrt{I} \subseteq \ker \epsilon = A_{>0}$. The stated isomorphism follows.

Let $u = (y^2 + z^2)/x$ be the mutation of x, and let v be the mutation of y in the cluster (u, y, z). Then $u^2 = yv - z^2 \in I$. If $u \in I$, taking the degree 1 part of an expression $u = xh_1 + yh_2 + zh_3$, with $h_i \in A$, would give $u \in \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z$. This contradicts its Laurent expansion $u = x^{-1}y^2 + x^{-1}z^2$. Thus the class of u in A/I is nonzero and has square zero. □

AI use. ChatGPT was used to develop the counterexample and its proof. Codex was used to check the arguments, search the literature, and prepare the manuscript.

References

[1] I. Assem, G. Dupont, and R. Schiffler, *On a category of cluster algebras*, J. Pure Appl. Algebra **218** (2014), no. 3, 553–582.

[2] J. Beyer and G. Muller, *Deep points of cluster algebras*, Int. Math. Res. Not. IMRN **2025** (2025), no. 4, rnaf027, 39 pp.

[3] M. Castronovo, M. Gorsky, J. Simental, and D. E. Speyer, *Cluster deep loci and mirror symmetry*, Commun. Amer. Math. Soc. **6** (2026), 1021–1086; arXiv:2402.16970v2.

[4] W. Chang and B. Zhu, *On rooted cluster morphisms and cluster structures in 2-Calabi–Yau triangulated categories*, J. Algebra **458** (2016), 387–421; arXiv:1410.5702v2.

[5] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[6] G. Muller, *Locally acyclic cluster algebras*, Adv. Math. **233** (2013), 207–247.

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