A finite criterion for total sign skew symmetry in order three

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Abstract

We give a finite criterion for total sign skew symmetry of integer matrices of order three. The resulting decision procedure uses at most $H-2$ mutations on a cyclic input, where H is the sum of its three edge products. We construct a family showing that the linear order of this bound is optimal.

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1 Introduction

Matrix mutation governs the exchange matrices of cluster algebras [6, 7]. Skew symmetrizability is preserved under mutation, but sign skew symmetry alone need not be. Deciding whether sign skew symmetry persists after every finite mutation sequence is therefore a question about the entire mutation class. In order three, the acyclic case is covered by the unfolding theorem of Huang and Li. We give a terminating procedure for the remaining cyclic case, allowing the six off-diagonal absolute entries to be independent.

Throughout the paper, $B=(b_{ij})$ is a 3×3 integer matrix, and the vertex labels are $1,2,3$. It is *sign skew symmetric* if $b_{ii}=0$ and, for each $i\neq j$, either $b_{ij}=b_{ji}=0$ or $b_{ij}b_{ji}<0$. For an integer x, write $[x]_+=\max(x,0)$. Mutation in direction $k\in\{1,2,3\}$ is the integer matrix $\mu_k B$ defined by

$$(\mu_k B)_{ij}=\begin{cases}-b_{ij}, & i=k \text{ or } j=k,\\ b_{ij}+[b_{ik}]_+[b_{kj}]_+-[-b_{ik}]_+[-b_{kj}]_+, & i,j\neq k.\end{cases} \tag{1.1}$$

This formula is defined even when the resulting matrix fails sign skew symmetry. A *mutation word* is a finite list $w=(i_1,\dots,i_t)$ of vertices; its length is t, and its value on B is $\mu_{i_t}\cdots\mu_{i_1}B$. The empty word has length zero and value B. A sign skew symmetric matrix is *totally sign skew symmetric*, abbreviated TSSS, if every mutation word has a sign skew symmetric value. A word whose value fails this condition is called a *failure word*.

The directed graph of a sign skew symmetric matrix has an arrow $i\to j$ exactly when $b_{ij}>0$. We call the matrix *cyclic* if this graph is an oriented triangle, and *acyclic* if the graph has no directed cycle. These two cases exhaust sign skew symmetric matrices of order three. A matrix is *skew symmetrizable* if there is a diagonal matrix D with positive real entries such that DB is skew symmetric.

For a cyclic matrix, define the edge products and triangle products by

$$c_1=|b_{23}b_{32}|, \qquad c_2=|b_{13}b_{31}|, \qquad c_3=|b_{12}b_{21}|, \tag{1.2}$$

$$r=|b_{12}b_{23}b_{31}|, \qquad s=|b_{21}b_{32}b_{13}|, \tag{1.3}$$

$$p=\min(r,s), \qquad q=\max(r,s).$$

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Thus c_i belongs to the edge opposite vertex i, whereas p, q order the products around the two orientations of the triangle. All these numbers are positive integers, and

$$c_1c_2c_3 = pq. \tag{1.4}$$

We use the height and its directional increments

$$H = c_1 + c_2 + c_3, \qquad \delta_i = c_jc_k - p - q \quad \text{for } \{i, j, k\} = \{1, 2, 3\}. \tag{1.5}$$

These quantities are evaluated at the current matrix whenever mutations are performed. Lemma 2.2 will show that a cyclic mutation in direction i changes H by δ_i.

Consider the following procedure on a sign skew symmetric input. Retain the original vertex labels and start with the empty word w.

(1) If the current matrix is acyclic, return YES. If it is cyclic and $p = q$, return YES.

(2) Otherwise choose i with c_i maximal, breaking ties by the smallest vertex label.

   (a) If $c_i > q$, apply μ_i, append i to w, and return YES.
   (b) If $p \le c_i \le q$, apply μ_i, append i to w, and return NO.
   (c) If $c_i < p$ and $\delta_i \ge 0$, return YES.
   (d) If $c_i < p$ and $\delta_i < 0$, apply μ_i, append i to w, and return to step (1).

Theorem 1.1. *The procedure terminates on every sign skew symmetric integer matrix B of order three and returns YES if and only if B is* TSSS*. For a cyclic input it applies at most $H(B) - 2$ mutations. A NO answer supplies a failure word. A YES answer supplies a word leading to an acyclic matrix, a skew symmetrizable matrix, or a cyclic matrix with $\delta_1, \delta_2, \delta_3 \ge 0$.*

An arbitrary integer input is first checked for sign skew symmetry. Failure of that initial check gives a failure word of length zero; otherwise Theorem 1.1 applies.

The cyclic terminal condition is the following result.

Theorem 1.2. *Let B be a cyclic sign skew symmetric integer matrix of order three. If $\delta_i \ge 0$ for all i, then every mutation word has a cyclic sign skew symmetric value.*

By (1.4), its hypothesis is equivalent to

$$\max(c_1, c_2, c_3)(p + q) \le pq. \tag{1.6}$$

The proof tracks the last direction of a reduced mutation word. The inequalities retained after that direction force every different next direction to remain cyclic. This supplies the terminal condition needed for descent: if the largest edge product does not satisfy it, the selected mutation either terminates the procedure or strictly decreases the positive integer H.

The number of mutations needed to detect failure cannot be bounded independently of the input.

Theorem 1.3. *For each integer $N \ge 2$, let*

$$B_N = \begin{pmatrix} 0 & N & -(N+1) \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}. \tag{1.7}$$

The shortest failure word for B_N has length $N - 1$.

Fomin and Zelevinsky proved that skew symmetrizable matrices are TSSS [6, Proposition 4.5]. They also constructed cyclic TSSS matrices which are not skew symmetrizable. Their argument uses edge products, one triangle product, and inequalities attached to a specified direction [6, Proposition 4.7 and Lemma 4.8]. Theorem 1.2 follows the same directional induction, using both triangle products to allow every first direction. Huang and Li proved that every acyclic integer sign skew symmetric matrix is TSSS [9, Theorem 2.17].

Descent and minimal representatives for skew symmetric matrices of order three were studied by Assem, Blais, Brüstle and Samson [2] and by Beineke, Brüstle and Hille [3]. Seven treated skew symmetrizable matrices [11]; see also Akagi [1, Remark 1.1]. When $p = q$, taking square roots of the edge products identifies (1.6) with the corresponding inequalities for minimal representatives. For independent entries, the edge products alone no longer determine total sign skew symmetry, as Example 6.1 demonstrates.

Berenstein, Fomin and Zelevinsky announced a complete classification of TSSS integer matrices of order three after [4, Problem 1.28]; the separate rank-three work is listed as unpublished drafts in [5]. The present contribution is an explicit decision procedure with a height bound and a family giving a linear lower bound. Section 6 identifies the cyclic recurrence with a Fricke cubic, and Appendix A compares the terminal inequalities with those of Fomin and Zelevinsky.

AI use. The main results of this paper were generated in a single conversation with GPT-6-Astra-Pro. Codex was subsequently used to reconstruct the LaTeX source from the supplied manuscript, revise its exposition, and check the displayed calculations and proof dependencies.

2 Mutation formulas

Changing B to $-B$ reverses every arrow and commutes with mutation. A cyclic matrix can therefore be written, after this change if needed, as

$$B = \begin{pmatrix} 0 & a & -c' \\ -a' & 0 & b \\ c & -b' & 0 \end{pmatrix}, \qquad a, a', b, b', c, c' \in \mathbb{Z}_{>0}. \tag{2.1}$$

Here the primes distinguish independent entries, not derivatives or mutated quantities. In these coordinates, $(c_1, c_2, c_3) = (bb', cc', aa')$ and $(r, s) = (abc, a'b'c')$. Substitution in (1.1) gives

$$\mu_1 B = \begin{pmatrix} 0 & -a & c' \\ a' & 0 & b - a'c' \\ -c & ac - b' & 0 \end{pmatrix}, \tag{2.2}$$

$$\mu_2 B = \begin{pmatrix} 0 & -a & ab - c' \\ a' & 0 & -b \\ c - a'b' & b' & 0 \end{pmatrix}, \tag{2.3}$$

$$\mu_3 B = \begin{pmatrix} 0 & a - b'c' & c' \\ bc - a' & 0 & -b \\ -c & b' & 0 \end{pmatrix}. \tag{2.4}$$

Lemma 2.1. *Let B be cyclic and $i \in \{1, 2, 3\}$.*

(i) *If $c_i < p$, then $\mu_i B$ is cyclic.*

(ii) *If $c_i > q$, then $\mu_i B$ is acyclic and sign skew symmetric.*

(iii) *If $p < q$ and $p \le c_i \le q$, then $\mu_i B$ is not sign skew symmetric.*

(iv) *If $c_i = p = q$, then both entries on the edge opposite i vanish in $\mu_i B$, which is acyclic and sign skew symmetric.*

Proof. For $i = 2$, the entries on the changed edge are $ab - c'$ and $c - a'b'$, and

$$ab - c' = \frac{r - c_2}{c}, \qquad a'b' - c = \frac{s - c_2}{c'}.$$

If $c_2 < p$, both displayed quantities are positive, so all three arrows reverse. If $c_2 > q$, both are negative, and the mutated graph is acyclic. Between p and q the two matrix entries have the same sign; at an endpoint with $p < q$, exactly one vanishes. If $c_2 = p = q$, both vanish. Cyclically permuting the labels in (2.1) proves the same assertions in the other two directions. □

For the next lemma and all later recurrence formulas, a prime on c_i, p, q, H or δ_i denotes its value at the mutated matrix.

Lemma 2.2. *Suppose that B and $\mu_i B$ are cyclic, and let $\{i, j, k\} = \{1, 2, 3\}$. Then*

$$c_i' = c_i + \delta_i, \qquad c_j' = c_j, \qquad c_k' = c_k, \tag{2.5}$$

$$p' = p + \delta_i = c_j c_k - q, \qquad q' = q + \delta_i = c_j c_k - p, \tag{2.6}$$

$$H' = H + \delta_i, \tag{2.7}$$

and

$$\delta_i' = -\delta_i, \qquad \delta_j' = \delta_j + (c_k - 2)\delta_i, \qquad \delta_k' = \delta_k + (c_j - 2)\delta_i. \tag{2.8}$$

Proof. For $i = 2$, Lemma 2.1 implies $c_2 < p$, so both $ab - c'$ and $a'b' - c$ are positive. Hence

$$c_2' = (ab - c')(a'b' - c) = aa'bb' - abc - a'b'c' + cc' = c_1 c_3 - p - q + c_2.$$

The other edge products do not change. The new triangle products are the unordered pair $\{c_1 c_3 - r, c_1 c_3 - s\}$. They are positive because $c_1 c_3 = pq/c_2 > q$; putting them in increasing order proves (2.6). Cyclic permutation proves these formulas for every i, and summation gives (2.7). Finally $p' + q' = p + q + 2\delta_i$, so

$$\delta_i' = c_j c_k - (p + q + 2\delta_i) = -\delta_i, \qquad \delta_j' = (c_i + \delta_i)c_k - (p + q + 2\delta_i).$$

The second expression is $\delta_j + (c_k - 2)\delta_i$; exchanging j and k gives the last identity. □

Remark 2.3. The labeled data $(c_1, c_2, c_3; p, q)$ determine the outcome of the next mutation in Lemma 2.1. If the outcome is cyclic, they also determine the next five products. Thus they determine the first exit from cyclic matrices along every word, including whether that exit is acyclic or fails sign skew symmetry. This permits comparison of failure words for matrices with identical product data.

3 Cyclicity under mutation

Lemma 3.1. *If B is cyclic and $\delta_i \geq 0$ for all i, then*

$$c_i \geq \frac{(p+q)^2}{pq} = 4 + \frac{(p-q)^2}{pq} \geq 4 \qquad (i = 1, 2, 3). \tag{3.1}$$

If $p < q$, then every $c_i \geq 5$.

Proof. For $\{i, j, k\} = \{1, 2, 3\}$, the hypotheses give $c_i c_j \geq p + q$ and $c_i c_k \geq p + q$. Their product is $c_i pq \geq (p + q)^2$. Division by pq proves (3.1). If $p < q$, its right-hand side is strictly larger than four, and c_i is an integer. □

Lemma 3.2. *In a cyclic matrix, $\delta_i \geq 0$ implies that $\mu_i B$ is cyclic and that every edge product is nondecreasing under this mutation.*

Proof. Using (1.4), we obtain

$$c_i = \frac{pq}{c_j c_k} \leq \frac{pq}{p+q} < p.$$

Lemma 2.1 gives cyclicity, and (2.5) gives the edge-product inequalities. □

To iterate this observation, we keep track of the last direction. For a vertex i, define the condition D_i on a cyclic matrix by

$$c_1, c_2, c_3 \geq 4, \qquad \delta_i \leq 0, \qquad \delta_j, \delta_k \geq -\delta_i \quad (\{i, j, k\} = \{1, 2, 3\}). \tag{3.2}$$

Lemma 3.3. *If D_i holds and $j \neq i$, then $\mu_j B$ is cyclic and satisfies D_j.*

Proof. We have $\delta_j \geq -\delta_i \geq 0$. Lemma 3.2 shows that the mutation is cyclic and all edge products remain at least four. If k is the third vertex, (2.8) gives

$$\delta'_j = -\delta_j, \qquad \delta'_i = \delta_i + (c_k - 2)\delta_j \geq \delta_i + 2\delta_j \geq \delta_j,$$

and

$$\delta'_k = \delta_k + (c_i - 2)\delta_j \geq \delta_k + 2\delta_j \geq \delta_j.$$

These are precisely the inequalities in D_j. □

Proof of Theorem 1.2. Lemma 3.1 gives $c_1, c_2, c_3 \geq 4$ initially. Choose any first direction i. Lemma 3.2 shows that the mutation is cyclic. By (2.8), $\delta'_i = -\delta_i \leq 0$, and each of the other two increments is at least $2\delta_i$. Thus D_i holds at the new matrix.

A word is *reduced* if consecutive letters are distinct. Lemma 3.3 now proves by induction on its length that every nonempty reduced word has a cyclic value. Mutation is an involution:

$$\mu_i^2 B = B. \tag{3.3}$$

Indeed, the second mutation reverses row i and column i again, and its increment at every other entry is the negative of the first increment in (1.1). Deleting consecutive equal letters therefore preserves the value of a word. Repeated deletion produces a reduced word, or the empty word, whose value is cyclic. This proves the assertion for every word. □

Example 3.4. For

$$E = \begin{pmatrix} 0 & 4 & -2 \\ -3 & 0 & 1 \\ 5 & -5 & 0 \end{pmatrix},$$

we have $(c_1, c_2, c_3; p, q) = (5, 10, 12; 20, 30)$ and $(\delta_1, \delta_2, \delta_3) = (70, 10, 0)$. Theorem 1.2 applies. Mutation in direction 3 changes the matrix but leaves all five products unchanged. The weak inequalities in (3.2) include this zero-increment case.

4 A finite criterion

Lemma 4.1. *Every acyclic integer sign skew symmetric matrix is* TSSS*. A cyclic matrix of order three is skew symmetrizable if and only if $p = q$, and in that case it is* TSSS*.*

Proof. The acyclic assertion is [9, Theorem 2.17]. For a cyclic matrix in the form (2.1), the equality $p = q$ is $abc = a'b'c'$. It implies that

$$D = \operatorname{diag}(a'b', ab', ab)$$

is positive and DB is skew symmetric. Conversely, multiplying the three pairwise equations for a positive diagonal symmetrizer yields $abc = a'b'c'$. Skew symmetrizability is preserved under mutation, with the same diagonal symmetrizer, by [6, Proposition 4.5]; hence it implies total sign skew symmetry. □

Lemma 4.2. *If a mutation word takes a sign skew symmetric matrix B to a* TSSS *matrix, then B is* TSSS*.*

Proof. By (3.3), the reversed word takes the terminal matrix back to B. Concatenate that reversed word with any word starting at B. Total sign skew symmetry at the terminal matrix implies that the concatenated word has a sign skew symmetric value. □

Proof of Theorem 1.1. At a cyclic matrix, (1.4) gives

$$\delta_i = \frac{pq}{c_i} - (p + q). \tag{4.1}$$

A maximal c_i therefore has a minimal δ_i. If the selected increment is nonnegative, all three increments are nonnegative and Theorem 1.2 applies.

Lemma 2.1 justifies the final mutations in steps (2a) and (2b), and shows that every continuing mutation in step (2d) remains cyclic. By (2.7), a continuing mutation satisfies $H' \le H - 1$. At every cyclic matrix, H is an integer at least three. There are at most $H(B) - 3$ continuing mutations and at most one final mutation, so the procedure terminates within the stated bound.

Each YES terminal matrix is TSSS by Lemma 4.1 or Theorem 1.2, and Lemma 4.2 transfers this property to the input. For a NO answer, Lemma 2.1 shows that the recorded word fails sign skew symmetry at its last step. These alternatives prove the equivalence and the assertions about the returned words. □

Corollary 4.3. *A cyclic sign skew symmetric integer matrix B which is not* TSSS *has a failure word of length at most $H(B) - 2$.*

Proof. Theorem 1.1 returns NO on this input, together with a word of the required length. □

The returned word can be checked by applying (1.1) to the input and testing the terminal condition. All comparisons use integers. During descent, every absolute matrix entry is a positive integer factor of an edge product and is at most $H(B)$. Products formed in a mutation and entries at the final step consequently have $O(\log H(B))$ bits. The procedure uses $O(H(B))$ integer arithmetic operations. This is a bound in the numerical height, which can be exponentially larger than its bit length.

5 Shortest failure words

For a matrix which is not TSSS, let $L(B)$ denote the least length of a failure word. A shortest failure word is reduced by (3.3).

Proof of Theorem 1.3. For the matrix B_N in (1.7), direct calculation gives

$$(c_1, c_2, c_3; p, q) = (4, N+1, N; 2N, 2N+2), \tag{5.1}$$

$$(\delta_1, \delta_2, \delta_3) = (N^2 - 3N - 2, -2, 2). \tag{5.2}$$

When $N = 2$, we have $c_1 = p = 4 < q$, so mutation in direction 1 fails and $L(B_2) = 1$. When $N = 3$, every $c_i < p = 6$, so all one-letter words remain cyclic. Mutating direction 2 gives $(c_1, c_2, c_3; p, q) = (4, 2, 3; 4, 6)$; mutation in direction 1 then fails. Thus $L(B_3) = 2$.

For $N \geq 4$, all edge products are at least four and $N^2 - 3N - 2 \geq 2$, so D_2 holds. By Lemma 3.3, every reduced word starting in a direction different from 2 remains cyclic. Any shortest failure word must therefore start with 2.

After this mutation the product data are

$$(4, N-1, N; 2N-2, 2N).$$

Exchanging vertex labels 2 and 3 gives exactly the data for B_{N-1}. Remark 2.3 matches the first cyclic exits for corresponding words. An acyclic exit cannot lead to a later failure, by Lemma 4.1. Inductively, a shortest failure word for B_{N-1} therefore gives a failure word of the same length at $\mu_2 B_N$, and every failure word at $\mu_2 B_N$ has length at least $L(B_{N-1})$. It follows that

$$L(B_N) = 1 + L(B_{N-1}) = N - 1. \qquad \square$$

Since $H(B_N) = 2N + 5$, this family satisfies

$$L(B_N) = \frac{H(B_N) - 7}{2}. \tag{5.3}$$

In particular, no uniform upper bound $f(H)$ for shortest failure length can satisfy $f(H)/H \to 0$ as $H \to \infty$.

6 Fricke cubics

The equality $p = q$ is the compatibility relation for skew symmetrizability. Without that relation, fixing the three edge products leaves triangle-product information which can change the answer to the decision problem.

Example 6.1. Consider

$$P_- = \begin{pmatrix} 0 & 12 & -10 \\ -10 & 0 & 1 \\ 12 & -6 & 0 \end{pmatrix}, \qquad P_+ = \begin{pmatrix} 0 & 8 & -12 \\ -15 & 0 & 2 \\ 10 & -3 & 0 \end{pmatrix}.$$

Both have $(c_1, c_2, c_3) = (6, 120, 120)$, but their other products and increments are

	(p, q)	$(\delta_1, \delta_2, \delta_3)$
P_-	$(144, 600)$	$(13656, -24, -24)$
P_+	$(160, 540)$	$(13700, 20, 20)$.

Theorem 1.2 applies to P_+. Every one-letter word on P_- remains cyclic, whereas

$$\mu_2 P_- = \begin{pmatrix} 0 & -12 & 2 \\ 10 & 0 & -1 \\ -48 & 6 & 0 \end{pmatrix}, \qquad \mu_3\mu_2 P_- = \begin{pmatrix} 0 & 0 & -2 \\ -38 & 0 & 1 \\ 48 & -6 & 0 \end{pmatrix}.$$

The last matrix fails sign skew symmetry. The initial matrix P_- also has two distinct directions decreasing H, even though every edge product is at least six.

The cyclic recurrence preserves two integers which serve as parameters of a cubic surface. Define

$$U = H - p, \qquad V = H - q. \tag{6.1}$$

The coordinates $x_i = c_i - 2$ below translate the edge products by two so that the cubic has the usual Fricke form.

Proposition 6.2. *Along any mutation sequence all of whose matrices are cyclic, U and V are constant. The translated edge products $x_i = c_i - 2$, for $i = 1, 2, 3$, satisfy*

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 - (U + V - 8)(x_1 + x_2 + x_3) \\ + (6 - U)(6 - V) - 8 = 0. \end{aligned} \tag{6.2}$$

A cyclic mutation in direction i acts by

$$x_i' = x_jx_k - x_i + U + V - 8, \qquad x_j' = x_j, \qquad x_k' = x_k, \quad \{i, j, k\} = \{1, 2, 3\}. \tag{6.3}$$

Proof. Equations (2.6) and (2.7) show that $H-p$ and $H-q$ are unchanged. Substituting $c_i = x_i+2$ into (1.4) gives

$$(x_1 + 2)(x_2 + 2)(x_3 + 2) = (x_1 + x_2 + x_3 + 6 - U)(x_1 + x_2 + x_3 + 6 - V).$$

Expanding and cancelling the mixed quadratic terms yields (6.2). In (2.5), substitute $p = H - U$, $q = H - V$ and $H = x_1 + x_2 + x_3 + 6$ to obtain (6.3). □

For fixed parameters A_1, A_2, A_3, A_4, the standard Fricke cubic is the affine surface

$$X_1^2 + X_2^2 + X_3^2 + X_1X_2X_3 = A_1X_1 + A_2X_2 + A_3X_3 + A_4.$$

Its coordinate involution in direction i exchanges the two roots of this quadratic equation in X_i, fixing the other coordinates. Explicitly it is $X_i' = A_i - X_jX_k - X_i$; see [10, p. 741]. Taking

$$X_i = -x_i \ (i = 1, 2, 3), \qquad A_1 = A_2 = A_3 = 8 - U - V, \qquad A_4 = 8 - (6 - U)(6 - V)$$

identifies this equation and its involutions with (6.2)–(6.3).

For B_N, the invariant values are $U = 5$ and $V = 3$. The translated triple $(2, N-1, N-2)$ satisfies

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 5.$$

These triples lie in the classical families on Markoff cubics; see [3, Lemma 4.1] and [8, Section 5.1]. Theorem 1.3 determines the first failure of sign skew symmetry in the associated matrix mutation problem.

A Biased inequalities

For a cyclic matrix, choose one triangle product $R \in \{p, q\}$. Consider the inequalities

$$R > c_i \geq R/2 \geq c_j, c_k \geq 6, \qquad \{i, j, k\} = \{1, 2, 3\}. \tag{A.1}$$

The choice $R = r$ gives the condition called i-biased in [6]; an odd permutation of the vertex labels exchanges r and s. Lemma 4.8 of that paper states that mutation in a direction $j \neq i$ preserves cyclicity and gives the corresponding condition at j. The next example describes what can happen in direction i.

Example A.1. For

$$Q = \begin{pmatrix} 0 & 2 & -2 \\ -3 & 0 & 2 \\ 3 & -5 & 0 \end{pmatrix},$$

we have $(c_1, c_2, c_3) = (10, 6, 6)$ and $r = 12$, so (A.1) holds at vertex 1. However,

$$\mu_1 Q = \begin{pmatrix} 0 & -2 & 2 \\ 3 & 0 & -4 \\ -3 & 1 & 0 \end{pmatrix}, \qquad \mu_2\mu_1 Q = \begin{pmatrix} 0 & 2 & -6 \\ -3 & 0 & 4 \\ 0 & -1 & 0 \end{pmatrix}.$$

Thus Q is not TSSS.

For E in Example 3.4, every word using only vertices 2 and 3 retains $c_1 = 5$. These matrices remain cyclic by Theorem 1.2 but never satisfy (A.1), which requires every edge product to be at least six.

The following proposition identifies a short path from the terminal inequalities of Theorem 1.2 to a biased matrix.

Proposition A.2. *Suppose that B is cyclic, $p < q$, and every $\delta_i \geq 0$. Relabel the vertices so that $c_1 \leq c_2 \leq c_3$. Unless*

$$(c_1, c_2, c_3; p, q) = (5, 5, 6; 10, 15) \quad \textit{or} \quad (6, 6, 6; 8, 27), \tag{A.2}$$

mutation in direction 1 *gives a matrix satisfying* (A.1) *at vertex* 1 *with $R = p'$. In either exceptional case, successive mutations in directions* 1 *and* 2 *give a matrix satisfying* (A.1) *at vertex* 2.

Proof. Lemma 3.1 gives $c_1 \geq 5$, and $p + q \leq c_1c_2$. In particular $q < c_1c_2$, so

$$p = \frac{c_1c_2c_3}{q} > c_3. \tag{A.3}$$

If $c_2 = 5$, then $c_1 = 5$ and

$$25c_3 = pq \le \frac{(p+q)^2}{4} \le \frac{625}{4}.$$

Thus c_3 is 5 or 6. The factor pairs of 125 with distinct ordered factors have sums at least 30, while the only pair for 150 with sum at most 25 is $(10, 15)$. This gives the first tuple in (A.2).

Assume $c_2 \ge 6$. Mutation in direction 1 is cyclic, and

$$c_1' = c_2c_3 - p - q + c_1, \qquad p' = c_2c_3 - q.$$

We have $p' - c_1' = p - c_1 > 0$. Also $q \ge \sqrt{c_1c_2c_3} \ge c_1^{3/2} > 2c_1$, and (1.4) gives

$$2c_1' - p' = \frac{(p - c_1)(q - 2c_1)}{c_1} > 0. \tag{A.4}$$

Since $c_2' = c_2 \ge 6$ and $c_3' = c_3 \ge c_2$, the required inequalities hold whenever $p' \ge 2c_3$.

Suppose $p' < 2c_3$. Then $q > (c_2 - 2)c_3$, whence $p < c_1c_2/(c_2 - 2)$. The inequality $p + q \le c_1c_2$, together with (1.4), gives

$$c_3 \le p - \frac{p^2}{c_1c_2}.$$

The function $t \mapsto t - t^2/(c_1c_2)$ is strictly increasing for $0 < t < c_1c_2/(c_2 - 2)$, because $c_2 \ge 6$. Consequently

$$c_2 \le c_3 < \frac{c_1c_2(c_2 - 3)}{(c_2 - 2)^2}.$$

It follows that

$$c_1 > \frac{(c_2 - 2)^2}{c_2 - 3} = c_2 - 1 + \frac{1}{c_2 - 3}.$$

Since $c_1 \le c_2$ are integers, $c_1 = c_2$. Substitution into the preceding bound yields

$$c_2 \le c_3 < c_2 + 1 - \frac{4}{(c_2 - 2)^2} < c_2 + 1,$$

so $c_1 = c_2 = c_3$. Now (A.3) gives

$$c_1 < p < \frac{c_1^2}{c_1 - 2} = c_1 + 2 + \frac{4}{c_1 - 2}, \qquad p \mid c_1^3.$$

If $c_1 \ge 7$, then p is $c_1 + 1$ or $c_1 + 2$. The first possibility implies $c_1 + 1 \mid 1$, and the second implies $c_1 + 2 \mid 8$, both impossible. For $c_1 = 6$, the possibilities are 7 and 8, and only 8 divides 216. Thus the second exceptional tuple is $(6, 6, 6; 8, 27)$.

For the two exceptional tuples, mutations in directions 1 and 2 give

$$(5, 5, 6; 10, 15) \longmapsto (10, 5, 6; 15, 20) \longmapsto (10, 30, 6; 40, 45),$$
$$(6, 6, 6; 8, 27) \longmapsto (7, 6, 6; 9, 28) \longmapsto (7, 11, 6; 14, 33).$$

Every transition is cyclic by Lemma 2.1. The final tuples satisfy (A.1) at vertex 2, with $R = 40$ and $R = 14$, respectively. □

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