

A CONTRACTIBLE OPEN FOUR-MANIFOLD WITH A COMPLETE METRIC OF UNIFORMLY POSITIVE SCALAR CURVATURE

ANONYMOUS CONTRIBUTOR TO MATHEMATICS

ABSTRACT. We construct a smooth contractible open four-manifold that is not homeomorphic to $\mathbb{R}^4$ and admits a complete Riemannian metric with scalar curvature at least one. This gives a negative answer to a question raised by Chang–Weinberger–Yu and Chodosh–Máximo–Mukherjee.

1. INTRODUCTION

A complete metric with uniformly positive scalar curvature imposes restrictions on the topology of an open manifold, including its topology at infinity. For contractible manifolds, these restrictions are particularly strong in dimension three: Chang–Weinberger–Yu proved that a contractible open three-manifold admitting such a metric is homeomorphic to $\mathbb{R}^3$ [3, Theorem 1]. The standard $\mathbb{R}^n$, for $n \geq 3$, admits a complete metric with uniformly positive scalar curvature, for instance the torpedo metric; for its construction, see [4, Section 2.3].

In dimension four, Chodosh–Máximo–Mukherjee proved that if the interior of a compact contractible smooth four-manifold admits a complete metric with uniformly positive scalar curvature, then the compact manifold is homeomorphic to the four-ball [2, Theorem 1.2]. For Mazur manifolds, they obtained a diffeomorphism conclusion [2, Theorem 1.1]. The question whether the standard $\mathbb{R}^4$ is the only contractible smooth four-manifold admitting a complete metric with uniformly positive scalar curvature was raised by Chang–Weinberger–Yu [3, Section 5] and reiterated by Chodosh–Máximo–Mukherjee [2, Question 1.5].

The following construction gives a negative answer.

Theorem 1.1. *There exists a smooth, connected, one-ended, contractible open four-manifold M carrying a complete Riemannian metric g with $\text{Scal}_g \geq 1$. The manifold M is not simply connected at infinity and hence is not homeomorphic to $\mathbb{R}^4$.*

We obtain a closed four-manifold Z by circle surgery on $\#^2(S^1 \times S^3)$, and then remove tubular neighborhoods of two circles to produce a compact cobordism W between two copies of $S^1 \times S^2$. We join copies $W_0, W_1, \dots$ in a sequence and cap the first boundary with $C = S^2 \times D^2$; see Figure 1.

Let K_n be the compact manifold formed by the cap C and the first n copies of W . Each K_n is simply connected, with $H_2(K_n; \mathbb{Z}) \cong \mathbb{Z}$ and all other reduced integral homology groups equal to zero. Attaching the next copy of W makes every class in $H_2(K_n; \mathbb{Z})$ null-homologous, while providing a new generator for $H_2(K_{n+1}; \mathbb{Z})$. Since every singular cycle in M lies in a finite stage, all reduced integral homology groups of M vanish. The manifold M is also simply connected, so the Hurewicz and Whitehead theorems imply that it is contractible.

On the other hand, both boundary inclusions into W induce injective homomorphisms on fundamental groups. Consequently, outside every compact subset of M there is a loop that cannot be contracted in $M \setminus C$: although it contracts in M , every contraction must meet the cap. Thus M is not simply connected at infinity and is not homeomorphic to $\mathbb{R}^4$.

The metric is constructed on the same pieces. The Gromov–Lawson surgery construction and standardization near the two circles give W a positive-scalar-curvature metric with identical product metrics near

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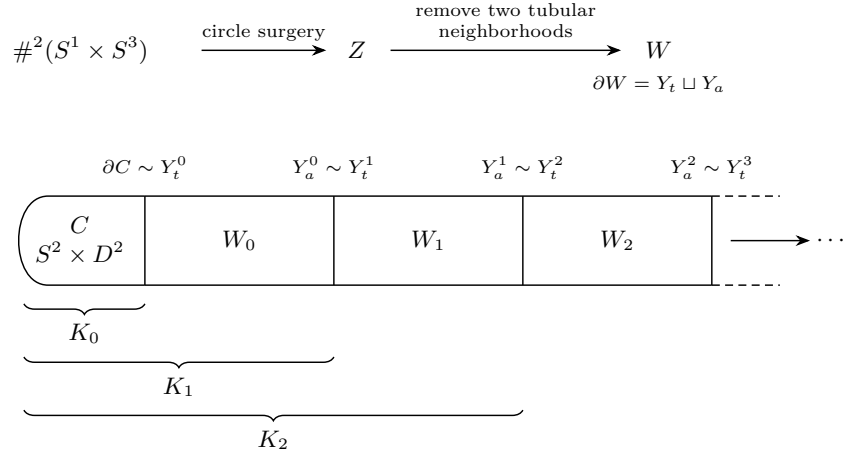


FIGURE 1. The construction of W and the infinite union. The incoming and outgoing boundaries of W_j are denoted by Y_t^j and Y_a^j . Each vertical seam represents an entire $S^1 \times S^2$ boundary, and $\sim$ denotes identification by the gluing map. The compact set K_n consists of the cap and the first n copies of W . The outlines show only how the pieces are attached; they do not assert that W is a product cobordism.

its boundary components. Copies of this metric fit together smoothly. Repeating a single compact metric gives a uniform positive lower bound for scalar curvature, and crossing each new copy requires a fixed positive amount of length. The resulting metric is complete.

Our example does not arise as the interior of a compact smooth four-manifold. Indeed, suppose that M were diffeomorphic to $\text{int } N$ for some compact smooth four-manifold N . Since the inclusion $\text{int } N \hookrightarrow N$ is a homotopy equivalence and M is contractible, N would also be contractible. The complete uniformly psc metric on M would then satisfy the hypotheses of Chodosh–Máximo–Mukherjee [2, Theorem 1.2], which imply that N is homeomorphic to D^4 . Hence M would be homeomorphic to $\mathbb{R}^4$, contradicting Theorem 1.1. Thus the hypothesis that the open manifold is the interior of a compact manifold is essential in their result.

Section 2 gives the topological construction, and Section 3 constructs the metric. All homology groups below have integral coefficients. A metric is called *psc* if its scalar curvature is positive, and *uniformly psc* if the scalar curvature has a positive constant lower bound.

2. THE TOPOLOGICAL CONSTRUCTION

2.1. The compact cobordism. The construction uses the Baumslag–Solitar group [1]

$$(2.1) \quad G = BS(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle.$$

Its abelianization is generated by $[t]$, whereas $[a] = 0$. On the other hand, setting $t = 1$ in the defining relation forces $a = 1$, so $G/\langle\langle t \rangle\rangle = 1$. Both a and t have infinite order: the affine transformations $a(x) = x + 1$ and $t(x) = 2x$ satisfy the relation and have infinite order. These elementary properties will have different consequences for finite unions and for neighborhoods of infinity.

Start with $X = \#^2(S^1 \times S^3)$ and identify $\pi_1(X)$ with the free group $F(a, t)$. Choose pairwise disjoint smoothly embedded circles c_a, c_t, c_r representing the conjugacy classes of a, t , and $r = tat^{-1}a^{-2}$, respectively. General position allows these representatives to be chosen embedded and pairwise disjoint. Their normal bundles are oriented rank-three bundles over S^1 and hence are trivial. Choose framings, and let $\nu(c_i) \cong S^1 \times D^3$ denote the corresponding tubular neighborhoods. Perform circle surgery along c_r by setting

$$(2.2) \quad Z = (X \setminus \text{int } \nu(c_r)) \cup_{S^1 \times S^2} (D^2 \times S^2).$$

Since c_a and c_t are disjoint from $\nu(c_r)$, after shrinking the tubular neighborhoods if necessary, they remain embedded circles in Z .

Lemma 2.1. *The closed oriented four-manifold Z admits a psc metric, has fundamental group G , and has the integral homology of $S^1 \times S^3$. The class $[c_t]$ generates $H_1(Z)$, and $[c_a] = 0$.*

Proof. The product $S^1 \times S^3$ admits a psc metric. The Gromov–Lawson connected-sum and surgery constructions therefore give psc metrics on X and Z , since the surgery in (2.2) has codimension three [5, Theorem A].

Removing an embedded circle from a four-manifold does not change the fundamental group: loops and homotopies of loops can be made disjoint from the circle by relative transversality, since their dimensions are at most two. Removing a tubular neighborhood has the same effect, by radial retraction. Van Kampen’s theorem applied to (2.2) thus gives $\pi_1(Z) = F(a, t) / \langle\langle r \rangle\rangle = G$. The assertions about $H_1(Z)$ follow by abelianization.

We have $\chi(X) = -2$, and circle surgery increases the Euler characteristic by two. Thus $\chi(Z) = 0$. Poincaré duality gives $b_3(Z) = b_1(Z) = 1$, and the Euler characteristic then gives $b_2(Z) = 0$. Since $H_1(Z) \cong \mathbb{Z}$, the universal coefficient theorem identifies $H^2(Z)$ with $\text{Hom}(H_2(Z), \mathbb{Z})$. The latter vanishes because $H_2(Z)$ is finitely generated of rank zero. Duality now gives $H_2(Z) = 0$, including its torsion. The remaining groups follow from connectedness and duality. $\square$

Choose disjoint framed tubular neighborhoods $N_t, N_a \cong S^1 \times D^3$ of c_t, c_a in Z , and set

$$(2.3) \quad W = Z \setminus \text{int}(N_t \sqcup N_a), \quad \partial W = Y_t \sqcup Y_a.$$

The framings identify both boundary components with $S^1 \times S^2$. We regard Y_t as the incoming boundary and Y_a as the outgoing boundary. In Y_i , write $\lambda_i = S^1 \times *$ for the longitude and $\mu_i = * \times S^2$ for the meridian sphere. If either tubular neighborhood is replaced by a smaller concentric one with respect to the chosen framing, the complement changes only by the addition of a product collar $(S^1 \times S^2) \times [0, 1]$. In particular, the resulting framed complement is diffeomorphic to W , and the boundary identifications, including the longitude and meridian classes, are unchanged up to isotopy.

Lemma 2.2. *The cobordism W is connected, with $\pi_1(W) \cong G$. With suitable base paths, the boundary inclusions induce injective homomorphisms on fundamental groups, sending λ_t, λ_a to t, a , respectively.*

The inclusion $Y_t \hookrightarrow W$ induces isomorphisms on H_1 and H_3 and the zero map on H_2 , while $Y_a \hookrightarrow W$ induces an isomorphism on H_2 .

Proof. General position shows that W is connected and that $W \hookrightarrow Z$ induces an isomorphism on fundamental groups. Under the identification $\pi_1(Z) = G$ from Lemma 2.1, the longitudes represent t, a . Their infinite orders imply injectivity of the boundary homomorphisms. Abelianization gives the assertion about H_1 .

Excision and the Thom isomorphism give $H_k(Z, W) \cong H_{k-3}(c_t \sqcup c_a)$, so the only nonzero relative groups are $\mathbb{Z}^2$ in degrees four and three. Order their generators by t, a . After choosing orientations, the map $H_4(Z) \rightarrow H_4(Z, W)$ is $1 \mapsto (1, 1)$. The map $H_3(Z) \rightarrow H_3(Z, W)$ records intersection numbers with the two circles and is $1 \mapsto (1, 0)$ by Lemma 2.1. Since $H_2(Z) = 0$ and both maps are injective, the long exact sequence gives $H_4(W) = 0$ and

$$\begin{aligned} H_3(W) &\cong \mathbb{Z}^2 / \langle (1, 1) \rangle, \\ H_2(W) &\cong \mathbb{Z}^2 / \langle (1, 0) \rangle. \end{aligned}$$

Under the connecting homomorphisms, the standard basis vectors map, up to signs, to the images of $[Y_t], [Y_a]$ in degree three and of $[\mu_t], [\mu_a]$ in degree two. Thus the image of $[Y_t]$ generates $H_3(W)$, while $[\mu_t]$ maps to zero and $[\mu_a]$ maps to a generator of $H_2(W)$. This proves the remaining assertions. $\square$

2.2. The infinite union. Let $W_0, W_1, \dots$ be copies of W , with boundary components Y_t^j, Y_a^j . Glue Y_a^j to Y_t^{j+1} by the product diffeomorphism

$$(2.4) \quad \phi(e^{i\theta}, x) = (e^{-i\theta}, x).$$

Choose the boundary product coordinates consistently so that ϕ reverses the induced boundary orientations. Cap Y_t^0 by $C = S^2 \times D^2$, using the product map $\psi(x, e^{i\theta}) = (e^{-i\theta}, x)$ from ∂C to Y_t^0 . Orient C so that ψ reverses the boundary orientations as well.

Set $K_0 = C$ and define

$$(2.5) \quad K_n = C \cup \bigcup_{j=0}^{n-1} W_j \quad (n \geq 1), \quad M = \bigcup_{n \geq 0} K_n.$$

All gluings use collars. Each seam consequently has a neighborhood diffeomorphic to $(S^1 \times S^2) \times (-\varepsilon, \varepsilon)$. The pieces form a locally finite chain, and there is no point added at its infinite end. Thus M is a smooth open four-manifold, $K_n \subset \text{int } K_{n+1}$, and every compact subset of M lies in the interior of some K_n . For $n \geq 1$, the only boundary component of K_n is Y_a^{n-1} .

Lemma 2.3. *Each K_n is simply connected, has the homology of $S^2 \times D^2$, and its boundary inclusion induces an isomorphism on H_2 . The inclusion $K_n \hookrightarrow K_{n+1}$ induces the zero map on H_2 .*

Proof. The first three assertions hold for $K_0 = C$. Assume they hold for K_n and consider $K_{n+1} = K_n \cup_{Y_t^n} W_n$. The incoming longitude represents t_n in $\pi_1(W_n)$ and is null-homotopic in K_n . Van Kampen's theorem gives

$$\pi_1(K_{n+1}) \cong G / \langle\langle t \rangle\rangle = 1.$$

For homology, use the Mayer–Vietoris sequence with collar neighborhoods of the two pieces. The maps from the common boundary $Y \cong S^1 \times S^2$ to their direct-sum homology groups are, up to signs,

$$\begin{aligned} H_3(Y) &\longrightarrow H_3(K_n) \oplus H_3(W_n), & x &\longmapsto (0, x), \\ H_2(Y) &\longrightarrow H_2(K_n) \oplus H_2(W_n), & x &\longmapsto (x, 0), \\ H_1(Y) &\longrightarrow H_1(K_n) \oplus H_1(W_n), & x &\longmapsto (0, x). \end{aligned}$$

The degree-three map is an isomorphism and the degree-two map is injective. Exactness therefore gives $H_4(K_{n+1}) = H_3(K_{n+1}) = 0$. The degree-one map is also an isomorphism. Hence exactness gives

$$H_2(K_{n+1}) \cong \frac{H_2(K_n) \oplus H_2(W_n)}{H_2(K_n) \oplus \{0\}} \cong H_2(W_n).$$

Under this identification, the inclusion $W_n \hookrightarrow K_{n+1}$ induces an isomorphism on H_2 , while the inclusion $K_n \hookrightarrow K_{n+1}$ induces the zero map. Connectedness and simple connectivity give the assertions in degrees zero and one, and the remaining homology groups vanish.

Finally, $\partial K_{n+1} = Y_a^n$, and the boundary inclusion factors as $Y_a^n \hookrightarrow W_n \hookrightarrow K_{n+1}$. The first inclusion induces an isomorphism on H_2 by Lemma 2.2, and the second does so by the preceding calculation. Thus the boundary inclusion also induces an isomorphism on H_2 . This completes the induction and proves the last assertion for every $n \geq 0$. $\square$

Proposition 2.4. *The manifold M is contractible and has exactly one end.*

Proof. Every loop in M has compact image and lies in a finite stage, where it contracts by Lemma 2.3. Thus $\pi_1(M) = 1$. Singular chains also have compact support, so homology commutes with this increasing union. In degree two, Lemma 2.3 gives

$$H_2(M) \cong \varinjlim (\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \cdots) = 0.$$

All other reduced homology groups vanish as well. A simply connected acyclic space of CW homotopy type is contractible, by the Hurewicz and Whitehead theorems. These apply because M is a smooth manifold.

For every n , the complement of $\text{int } K_n$ is the connected chain $W_n \cup W_{n+1} \cup \cdots$. Its interior in M is connected as well. Moreover, every neighborhood of infinity contains one of these complements, so M has one end. $\square$

2.3. The topology at infinity. A manifold is *simply connected at infinity* if for every compact subset A there is a compact subset $B \supset A$ such that every loop outside B contracts outside A . We show that the manifold M fails this condition.

Set $U_n = M \setminus \text{int } K_n$ and $U_{n,m} = W_n \cup \cdots \cup W_m$ for $m \geq n$. Choose compatible base paths along a proper ray through the pieces, and write $G_j = \pi_1(W_j)$. The gluing map (2.4) identifies a_j with t_{j+1}^{-1} . Thus van Kampen's theorem gives

$$(2.6) \quad \pi_1(U_{n,m}) \cong G_n *_{\langle a_n \rangle = \langle t_{n+1}^{-1} \rangle} G_{n+1} * \cdots *_{\langle a_{m-1} \rangle = \langle t_m^{-1} \rangle} G_m$$

when $m > n$, and $\pi_1(U_{n,n}) = G_n$.

Proposition 2.5. *The manifold M is not simply connected at infinity and is not homeomorphic to $\mathbb{R}^4$.*

Proof. Fix $n \geq 0$. By Lemma 2.2, both boundary inclusions into W induce injections on fundamental groups. The normal form theorem for free products with amalgamation therefore shows that each factor G_j , $n \leq j \leq m$, in (2.6) embeds in $\pi_1(U_{n,m})$, and that each inclusion $U_{n,m} \hookrightarrow U_{n,m+1}$ induces an injection on fundamental groups [6, I, §1.2, Theorem 1]. By local finiteness, the compact image of any loop or null-homotopy of a loop in U_n is contained in some $U_{n,m}$. Hence the inclusion $W_j \hookrightarrow U_n$ induces an injection $G_j \hookrightarrow \pi_1(U_n)$ for every $j \geq n$.

Let $B \supset K_n$ be any compact subset of M , and choose $m > n$ such that $B \subset \text{int } K_m$. Push a longitude of Y_a^m representing a_m into $\text{int } W_m$, and denote the resulting loop by γ_m . Since $\text{int } W_m \cap K_m = \emptyset$, this loop lies outside B . The element a_m is nontrivial in G_m , so γ_m is not null-homotopic in U_n . Since $M \setminus K_n \subset U_n$, it is not null-homotopic in $M \setminus K_n$ either. As B was arbitrary, this disproves simple connectivity at infinity with the fixed compact set $A = K_n$.

The complement of every closed Euclidean ball in $\mathbb{R}^4$ is simply connected, so $\mathbb{R}^4$ is simply connected at infinity. Since this property is preserved by homeomorphisms, M is not homeomorphic to $\mathbb{R}^4$. $\square$

3. THE METRIC CONSTRUCTION

It remains to put a complete uniformly psc metric on M . We first construct a psc metric on W with the same product metric near both boundary components. We can then put an isometric copy of this metric on every W_j and glue the copies smoothly along their product collars. In particular, the scalar curvature lower bound and the distance between the two boundary components of each W_j are independent of j .

Lemma 3.1. *The cobordism W admits a psc metric g_W with the same product metric near both boundary components. More precisely, there are constants $\varepsilon, \ell, \rho > 0$ such that each Y_i , $i = t, a$, has a product neighborhood $[0, \varepsilon) \times Y_i$ in W on which*

$$(3.1) \quad g_W = ds^2 + g_Y, \quad g_Y = \ell^2 d\theta^2 + \rho^2 g_{S^2}.$$

Here $s = 0$ is the boundary and s increases into W . The formula for g_Y uses the fixed identification of each Y_i with $S^1 \times S^2$ from the topological construction. The variable $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is the circle coordinate, and g_{S^2} is the unit round metric on S^2 .

Proof. By Lemma 2.1, the closed manifold Z admits a psc metric. We will arrange that this metric is a product on the outside of each deleted tubular neighborhood.

The fixed tubular coordinates identify N_t and N_a with copies of $S^1 \times D^3$, where D^3 is the closed unit ball. Extend these coordinates slightly beyond the boundaries of N_t and N_a , keeping the two enlarged neighborhoods disjoint. A radial reparametrization outside D^3 then gives an open embedding

$$\Phi : (S^1 \sqcup S^1) \times \mathbb{R}^3 \longrightarrow Z$$

that agrees with the fixed parametrizations on $(S^1 \sqcup S^1) \times D^3$. Thus $r = 1$ corresponds to $Y_t \sqcup Y_a$, where r is the radial coordinate in $\mathbb{R}^3$.

Choose $\ell > 0$ and a standard torpedo metric on $\mathbb{R}^3$ of the form

$$g_{\text{tor}} = dr^2 + \eta(r)^2 g_{S^2},$$

with positive scalar curvature and $\eta(r) = \rho$ for $r \geq r_0$, where $0 < r_0 < 1$. Such a choice is obtained by scaling a standard torpedo metric. Use the same metric $\ell^2 d\theta^2$ on both circles and the same g_{tor} on both normal factors. Since the circle metric is flat, the product $\ell^2 d\theta^2 + g_{\text{tor}}$ has positive scalar curvature.

Fix $R > 1$. Applying the standardization theorem to the two circles together, which have codimension three in Z , gives a psc metric g_Z such that

$$(3.2) \quad \Phi^* g_Z = \ell^2 d\theta^2 + g_{\text{tor}} \quad \text{on } (S^1 \sqcup S^1) \times B_R^3;$$

see [4, Theorem 1.2 and Remark 1.3]. Here B_R^3 is the open ball of radius R .

Set $g_W = g_Z|_W$. In these coordinates, the regions $1 \leq r < R$ lie in W . Since $r_0 < 1$, their metric is exactly

$$dr^2 + \ell^2 d\theta^2 + \rho^2 g_{S^2}.$$

Taking $s = r - 1$ and $\varepsilon = R - 1$ gives the required product neighborhoods. At $s = 0$, the circle and sphere coordinates are the fixed boundary coordinates. Both neighborhoods use the same constants ℓ and ρ . $\square$

We next equip the cap $C = S^2 \times D^2$ with a psc metric whose product collar matches that of W . Choose a smooth nonincreasing function $q : [0, 1] \rightarrow [0, 1]$ which equals 1 near zero and 0 near one, and set

$$L = \frac{\ell}{\int_0^1 q(u) du}, \quad \zeta(r) = \int_0^r q(t/L) dt, \quad 0 \leq r \leq L.$$

Then $\zeta(r) = r$ near zero, $\zeta(r) = \ell$ near L , and $\zeta'' \leq 0$.

In polar coordinates on the disk, with boundary $r = L$ and angular coordinate θ , define

$$h_D = dr^2 + \zeta(r)^2 d\theta^2.$$

Since $\zeta(r) = r$ near zero, the metric h_D is Euclidean near the center and extends smoothly across it. For $r > 0$, its Gaussian curvature is

$$K_{h_D} = -\frac{\zeta''}{\zeta} \geq 0.$$

Hence the product metric $g_C = \rho^2 g_{S^2} + h_D$ has scalar curvature

$$\text{Scal}_{g_C} = \frac{2}{\rho^2} - \frac{2\zeta''}{\zeta} \geq \frac{2}{\rho^2} > 0$$

for $r > 0$. Near the disk center, h_D is flat, so $\text{Scal}_{g_C} = 2/\rho^2$ there as well. Near ∂C , setting $u = L - r$ gives

$$g_C = du^2 + \rho^2 g_{S^2} + \ell^2 d\theta^2.$$

The maps $\phi : Y_a^j \rightarrow Y_t^{j+1}$ used to glue consecutive copies of W , and $\psi : \partial C \rightarrow Y_t^0$ used to attach the cap, are isometries for these boundary metrics. Put a copy of g_W on every W_j and put g_C on C , and glue the pieces along these maps using the product collars. Near each gluing hypersurface, use the negative of the inward collar coordinate on one side and the positive inward collar coordinate on the other. After identifying the boundary coordinates by the gluing map, the metric on both sides is

$$ds^2 + g_Y.$$

It is therefore smooth across each gluing hypersurface and has positive scalar curvature there. Since the pieces form a locally finite chain, this defines a smooth psc metric g on M .

Proof of Theorem 1.1. Propositions 2.4 and 2.5 give the topological assertions. Since g_C and g_W have positive scalar curvature on the compact manifolds C and W , respectively, the constant

$$\sigma = \min \left\{ \min_C \text{Scal}_{g_C}, \min_W \text{Scal}_{g_W} \right\}$$

is positive. Every W_j carries an isometric copy of g_W , so the glued metric satisfies $\text{Scal}_g \geq \sigma$ on M .

To prove completeness, fix $p \in C$ and let $\delta = \text{dist}_{g_W}(Y_t, Y_a) > 0$, where the distance is intrinsic to W . Fix $N \geq 1$, let $q \in M \setminus K_N$, and let γ be any piecewise smooth curve from p to q . For each $j = 0, \dots, N-1$, consider the first time that γ reaches Y_a^j , and take the segment from its last preceding visit to Y_t^j up to that time. Since γ can leave W_j only through Y_t^j or Y_a^j , this segment lies in W_j and has length at least δ .

These segments occur in order and have disjoint parameter interiors, so $L_g(\gamma) \geq N\delta$. Taking the infimum over all such curves gives $d_g(p, q) \geq N\delta$.

For every $R > 0$, choose N so that $N\delta > R$. Then $\overline{B}_g(p, R) \subset K_N$. Since K_N is compact and the ball is closed in K_N , the ball is compact. Thus (M, g) is proper, and the Hopf–Rinow theorem gives completeness.

Finally, set $\tilde{g} = \sigma g$. Constant rescaling preserves completeness and gives $\text{Scal}_{\tilde{g}} = \sigma^{-1} \text{Scal}_g \geq 1$. $\square$

On the use of AI. In a series of conversations concerning the above question, GPT-6 constructed a highly intricate counterexample and developed the associated proofs. Its contributions include the topological construction and the metric construction. The human researchers’ work consists of posing the research problem, simplifying the counterexample construction, completing the writing of this manuscript, and interpreting and verifying GPT-6’s arguments. The human authors take full responsibility for the content, accuracy, and integrity of this manuscript. The authors will not submit this manuscript for publication.

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