

NASH BARGAINING CHARACTERIZATION IN LEAN

ARTHUR FREITAS RAMOS, DAVID BARROS HULAK, AND RUY J. G. B. DE QUEIROZ

ABSTRACT. We describe a Lean 4 formalization of the classical two-person Nash bargaining characterization. A bargaining problem consists of a compact convex subset of the real plane, a feasible disagreement point, and a feasible outcome strictly improving both utilities. The Nash product is maximized over the individually rational part of that set. The development proves existence and uniqueness of the maximizer, verifies Pareto optimality, symmetry, positive affine invariance, and independence of irrelevant alternatives, and proves that these four axioms characterize the maximizer among feasible selection rules. We explain the algebraic midpoint proof of uniqueness and the normalization argument, including an explicit small-step tangent bound and a compact symmetric enlargement. The exposition is tied to an immutable repository snapshot and four declarations registered in Palomar. Historical registry verification is distinguished from the source inspection used to prepare this manuscript. The contribution is a documented formalization of established mathematics; no new bargaining theorem or formalization-priority claim is made.

1. THE RESULT AND ITS SCOPE

Nash's bargaining solution selects a cooperative outcome by maximizing the product of the players' utility gains above disagreement. His 1950 paper gives an axiomatic justification for this choice [2]. The mathematical question is distinct from the existence of a Nash equilibrium in a noncooperative game: here the input is a feasible utility set and a disagreement point, and the output is one utility pair.

This article explains the development `Arthur742Ramos/nash-bargaining-lean` at commit `0ddaa4bb858fa6fdf78074459eafada5e7938727` [4]. The selected declarations are registered as PALOMAR-2026-09-25-000013, version 1 [3]. The formalization uses Lean 4 and Mathlib [1, 5]. References to source files below always mean this pinned snapshot, rather than a moving branch.

The scope is precisely two players with real-valued utilities. Feasible sets are compact and convex, disagreement is feasible, and a strictly mutually beneficial outcome exists. No comprehensiveness or downward closure of the feasible set is assumed. The feasible set may contain outcomes below disagreement. The theorem concerns a utility-space selection rule; it does not construct a bargaining protocol, prove strategic implementation, or supply an executable optimization algorithm.

The formalization's useful content lies in making this domain explicit and carrying it through both directions of the characterization. Its proof separates compactness-based attainment from algebraic uniqueness, and gives a direct real-arithmetic version of the normalization argument. These are implementation and exposition contributions for a classical result, not claims of a new solution concept.

Date: October 1, 2026.

2020 Mathematics Subject Classification. 91A12, 68V20.

Key words and phrases. Nash bargaining, two-person bargaining, compact convex feasible sets, individual rationality, axiomatic characterization, Lean 4, Mathlib.

Author ORCID identifiers: Arthur Freitas Ramos, 0009-0003-3568-0325; David Barros Hulak, 0009-0002-8056-1774; Ruy J. G. B. de Queiroz, 0000-0003-1482-0977.

Copyright 2026 the authors. Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0).

2. BARGAINING PROBLEMS AND INDIVIDUAL RATIONALITY

Definition 2.1. A problem is a pair $P = (S, d)$ such that $S \subseteq \mathbb{R}^2$ is nonempty, compact, and convex, $d \in S$, and

$$(1) \quad \exists z \in S, \quad d_1 < z_1 \quad \text{and} \quad d_2 < z_2.$$

Write $\mathcal{B}$ for this class of problems. Its individually rational part is

$$\text{IR}(P) = \{u \in S : d_1 \leq u_1, d_2 \leq u_2\}.$$

The Nash product and maximizer predicate are

$$\begin{aligned} \text{NP}_d(u) &= (u_1 - d_1)(u_2 - d_2), \\ \text{Max}(P, x) &\iff x \in \text{IR}(P) \text{ and } \forall y \in \text{IR}(P), \text{NP}_d(y) \leq \text{NP}_d(x). \end{aligned}$$

These definitions translate `NashBargaining.Problem`, `nashProduct`, and `IsNashMaximizer` in `Basic.lean`. The Lean type bundles the set and point with proofs of all five conditions in Definition 2.1. The nonempty condition is retained explicitly although it follows from $d \in S$.

The restriction to $\text{IR}(P)$ is essential. For example, take $d = (0, 0)$ and $S = \{(t, t) : -2 \leq t \leq 1\}$. The raw product is t^2 , whose maximum on S is attained at $(-2, -2)$, whereas its maximum on $\text{IR}(P)$ is attained at $(1, 1)$. Thus a statement about maximization over all of S would misdescribe the formal theorem, even for a compact convex problem satisfying (1).

Strict simultaneous improvement ensures that the maximum product is positive. It excludes the flat zero-product case: on $S = [0, 1] \times \{0\}$ with $d = (0, 0)$, every individually rational point has product zero, so uniqueness fails. A strictly improving point need not be an interior point of S in the plane. The admissible diagonal segment in the preceding example already has empty planar interior.

Definition 2.2. A solution is a function $F : \mathcal{B} \rightarrow \mathbb{R}^2$ satisfying $F(S, d) \in S$ for every problem. No individual-rationality condition is included in the definition of a solution.

The corresponding Lean type `NashBargaining.Solution` is a subtype of functions with a feasibility proof. This matters for the converse: the characterization starts with any feasible rule satisfying the four axioms, rather than assuming in advance that it selects an individually rational point.

3. THE FOUR AXIOMS

The predicates in `Basic.lean` have the following mathematical content. Every quantified problem below belongs to $\mathcal{B}$.

- (1) *Pareto optimality.* If $y \in S$ and $F(P)_1 \leq y_1, F(P)_2 \leq y_2$, then $y = F(P)$. This rules out weak domination with at least one strict utility gain.
- (2) *Symmetry.* Define $\text{swap}(u_1, u_2) = (u_2, u_1)$. If $\text{swap}(S) = S$ and $d_1 = d_2$, then $F(P)_1 = F(P)_2$. This is a condition on symmetric problems, not a separately assumed equivariance law for every permutation of an arbitrary problem.
- (3) *Positive affine invariance.* For

$$A(u) = (a_1 u_1 + b_1, a_2 u_2 + b_2), \quad a_1, a_2 > 0,$$

if $Q = (A(S), A(d))$, then $F(Q) = A(F(P))$. Both the feasible set and disagreement point transform together; the two scales and translations may be different.

- (4) *Independence of irrelevant alternatives (IIA).* If $P = (S, d)$, $Q = (T, d)$, $S \subseteq T$, and $F(Q) \in S$, then $F(P) = F(Q)$. The smaller problem must remain in $\mathcal{B}$.

Their conjunction is `NashBargaining.NashAxioms`. In particular, IIA compares sets at the same disagreement point. It does not assert invariance under arbitrary changes to disagreement or contraction to sets outside the specified domain.

4. THE FORMAL THEOREM INTERFACE

Theorem 4.1. *For every $P \in \mathcal{B}$ there exists exactly one $x \in \mathbb{R}^2$ such that $\text{Max}(P, x)$. If a function N selects a maximizer for every problem, its induced feasible solution satisfies the four axioms. Conversely, for every feasible solution F satisfying the axioms, every $P \in \mathcal{B}$, and every x with $\text{Max}(P, x)$, one has $F(P) = x$.*

The registered namespace is `NashBargaining.Palomar`. Its four declarations are:

- `nashMaximizerExists`: $\forall P, \exists x, \text{Max}(P, x)$;
- `nashMaximizerUnique`: any two maximizers for one problem agree;
- `nashSatisfiesAxioms`: any pointwise maximizer selector satisfies the axioms;
- `axiomsCharacterizeNash`: an axiomatic feasible rule agrees with any given maximizer.

The registered declarations in `Solution.lean` are wrappers around implementation theorems in the main namespace. The existence and uniqueness assertions do not themselves define a particular selector. Classically, existence permits choosing one for each problem, and uniqueness makes all such selections extensionally equal. The axiom-verification theorem is deliberately expressed for an arbitrary function together with its pointwise maximizer proofs.

5. ATTAINMENT AND ALGEBRAIC UNIQUENESS

5.1. Compactness gives a maximum. For $P = (S, d)$, let

$$K = S \cap \{u : d_1 \leq u_1\} \cap \{u : d_2 \leq u_2\}.$$

The coordinate half-spaces are closed, so K is compact. It is nonempty by (1). The map $u \mapsto \text{NP}_d(u)$ is continuous as a product of two continuous coordinate differences. The extreme-value theorem therefore supplies a point of maximum on K .

In `Existence.lean`, attainment follows from Mathlib’s `IsCompact.exists_isMaxOn` theorem. The proof then unpacks membership in K into feasibility and the two individual-rationality inequalities. Convexity is not used in this attainment step, although it remains part of the input type and is needed for uniqueness and characterization.

5.2. Positive product and a midpoint identity. Suppose x, y both maximize. Put

$$a = x_1 - d_1, \quad b = x_2 - d_2, \quad c = y_1 - d_1, \quad e = y_2 - d_2.$$

The gains are initially nonnegative. Comparing each maximizer with the other gives $ab = ce = M$. Comparing with the strictly improving witness gives $M > 0$, hence $a, b, c, e > 0$.

If $a = c$, cancellation in $ab = ce$ gives $b = e$, so $x = y$. Otherwise, convexity places $z = (x + y)/2$ in S . It is individually rational, with gains $u = (a + c)/2$, $v = (b + e)/2$. The crucial equality is

$$(2) \quad 4c(uv - M) = b(a - c)^2.$$

It follows by expanding and substituting $ce = ab$. Its right side is strictly positive when $a \neq c$, while $4c > 0$. Therefore $uv > M$, contradicting maximality of x .

This is the proof in `Uniqueness.lean`. It expresses strictness through a polynomial identity and positivity, without introducing logarithms, derivatives, or a strict-concavity library. The product function is not concave on the entire nonnegative quadrant; the equal-product relation is a substantive ingredient in (2).

6. WHY MAXIMIZER SELECTION SATISFIES THE AXIOMS

The first theorem of `Characterization.lean` proves each conjunct for an arbitrary maximizer selector N . Uniqueness is the common final step.

For Pareto optimality, if y weakly dominates $N(P)$, it is individually rational and its product is at least that of $N(P)$ by monotonicity of multiplication on nonnegative gains.

Thus y is also a maximizer; uniqueness gives $y = N(P)$. This argument checks domination by all feasible points, not merely by an externally restricted Pareto frontier.

On a symmetric problem, swapping $N(P)$ preserves feasibility, individual rationality, and the product. Swapping every comparison point shows that the swapped point is again a maximizer. Uniqueness then forces the two coordinates of $N(P)$ to coincide.

For a positive affine map A , direct expansion gives

$$(3) \quad \text{NP}_{A(d)}(A(u)) = a_1 a_2 \text{NP}_d(u).$$

The positive factor preserves the ordering of products, and positive scales preserve the individual-rationality inequalities. Every point of $A(S)$ has a preimage in S , so $A(N(P))$ is a maximizer for the transformed problem. Uniqueness identifies it with $N(Q)$.

Finally, in the IIA situation, $N(Q) \in S$ and disagreement is unchanged. Its larger-set maximality restricts to every individually rational point in S , making it a maximizer for P . Uniqueness again proves the required equality. Neither this argument nor the axiom grants a contraction that loses the selected point.

7. THE CONVERSE VIA NORMALIZATION AND A BOUNDED ENLARGEMENT

We now explain the second theorem of `Characterization.lean`. Let F satisfy the four axioms, and fix a maximizer x for $P = (S, d)$. The positivity argument above yields $x_i - d_i > 0$. Define

$$(4) \quad \phi(u) = \left(\frac{u_1 - d_1}{x_1 - d_1}, \frac{u_2 - d_2}{x_2 - d_2} \right), \quad U = \phi(S).$$

This positive affine map sends d to $(0, 0)$ and x to $(1, 1)$. It preserves compactness and convexity, and the transformed problem $Q = (U, (0, 0))$ remains admissible. By (3), $(1, 1)$ is a Nash maximizer of Q .

7.1. A tangent bound for every feasible point.

Lemma 7.1. *Every $z \in U$ satisfies $z_1 + z_2 \leq 2$.*

Proof. Assume $c = z_1 + z_2 - 2 > 0$. Set

$$K = |(z_1 - 1)(z_2 - 1)|, \quad K_i = |z_i - 1|,$$

and choose the explicit positive number

$$(5) \quad t = \min \left\{ \frac{1}{2}, \frac{c}{2(K+1)}, \frac{1}{2(K_1+1)}, \frac{1}{2(K_2+1)} \right\}.$$

The point $w = (1 - t)(1, 1) + tz$ belongs to U by convexity. The bounds $tK_i \leq 1/2$ show $w_i = 1 + t(z_i - 1) \geq 1/2$, so w is individually rational even if z itself is not. Expanding the product,

$$w_1 w_2 - 1 = tc + t^2(z_1 - 1)(z_2 - 1).$$

Since $tK \leq c/2$, the right side is at least $tc/2 > 0$. This contradicts maximality of $(1, 1)$ on the individually rational part of U . $\square$

The source uses nested binary minima to realize (5). This quantitative argument proves a supporting-line inequality for *all* of U . It is important not to infer the inequality directly from $z_1 z_2 \leq 1$: that comparison is available only for individually rational z , and even there requires convexity near $(1, 1)$ to obtain the linear bound. The proof avoids a differentiability premise.

7.2. Constructing an admissible symmetric comparison problem. The half-space $H = \{u : u_1 + u_2 \leq 2\}$ contains U , but is not compact and is therefore not itself an admissible bargaining set. The source repairs this by a bounded enlargement. The compact set

$$V = U \cup \text{swap}(U)$$

has upper and lower bounds for both coordinate projections. From any such bounds M_1, m_1, M_2, m_2 , set

$$B = |M_1| + |m_1| + |M_2| + |m_2| + 1 \geq 1, \quad W = [-B, B]^2 \cap H.$$

The bounds imply $V \subseteq [-B, B]^2$, while Lemma 7.1 and its swapped version give $V \subseteq H$. Consequently $U \subseteq W$.

The set W is compact, convex, and invariant under swapping. It contains both $(0, 0)$ and $(1, 1)$, so $R = (W, (0, 0))$ is an admissible problem. Symmetry gives $F(R) = (r, r)$. Feasibility gives $2r \leq 2$. If $r < 1$, the point $(1, 1) \in W$ would dominate $F(R)$ and Pareto optimality would force equality, a contradiction. Thus $r = 1$ and

$$F(R) = (1, 1).$$

Because $U \subseteq W$ and $(1, 1) \in U$, IIA gives $F(Q) = F(R) = (1, 1)$. Positive affine invariance then yields $\phi(F(P)) = F(Q) = \phi(x)$. The positive scales make ϕ injective, so $F(P) = x$, completing the converse.

This use of a symmetric box intersected with a half-space is the actual formal construction. The proof does not require a compact convex hull of $U \cup \text{swap}(U)$, nor an application of the axioms to an unbounded half-space. It also explains why individual rationality need not be postulated for F : the axioms determine its output through the comparison problem.

8. SOURCE ORGANIZATION AND VERIFICATION EVIDENCE

The implementation is small enough to inspect by module. Definitions and axiom predicates are in `NashBargaining/Basic.lean`; attainment is in `Existence.lean`; midpoint uniqueness is in `Uniqueness.lean`; both axiom directions are in `Characterization.lean`; and the four registered wrappers are in `Solution.lean`. The proof bodies use Mathlib’s topology and convexity interfaces together with real-arithmetic tactics, including `ring`, `linarith`, `nlinarith`, `positivity`, and `field_simp`.

`Challenge.lean` repeats the public definitions and the four target theorem statements, with four deliberate `sorry` holes. It is the statement-side challenge, not the implementation proof. The comparator configuration maps those declarations to `Solution.lean`. The inspected implementation files contain no `sorry` or `admit` proof placeholders; this textual observation is distinct from a kernel check of their complete dependency closure.

The immutable Palomar record reports the following historical evidence [3]:

- Lean toolchain `leanprover/lean4:v4.35.0-rc2` and Mathlib revision `065356127b1dc0016f66b7283ce0ce2c4055aa55`;
- verification on September 24, 2026, at 20:19:45 UTC, with workflow run `36053394619`;
- permitted axioms `propext`, `Quot.sound`, and `Classical.choice`, with external checker entries for `nanoda` and `con-ron`;
- registration on September 25, 2026, and preserved source at the commit identified in Section 1.

The record also reports a neutral editorial outcome with no warnings. That status is not a claim of journal peer review or an endorsement by Nash’s original publisher.

Preparation of this manuscript involved retrieving and inspecting the pinned definitions, proof files, comparator configuration, and dependency metadata, and checking the statement-side and solution-side file hashes against the registry record. We did not perform a fresh Lean build or rerun the registry’s external checkers for this manuscript. The verification claim therefore rests on the dated registry evidence, while the present explanation rests on

source inspection. Mathematical validity is relative to the encoded hypotheses and Lean’s foundational assumptions; neither compilation nor registration certifies a broader economic interpretation.

9. AUTHORSHIP DISCLOSURE AND LIMITATIONS

The manuscript was primarily drafted with GPT 6.1 under the authors’ direction. The submitting author reports understanding some parts of the work; this disclosure does not claim a complete independent human reconstruction or line-by-line human validation of every formal proof. The pinned `formalization.yaml` separately records AI-assisted proof engineering using `gpt-6-luna` (Codex CLI). These are distinct disclosures: the manuscript-drafting model is not inferred to be the historical proof-engineering model, and model identity is not verification evidence.

The manuscript and its source are licensed CC BY 4.0. The referenced Lean repository declares BSD-3-Clause; the manuscript license does not relicense that code. The registry lists Arthur Freitas Ramos as the formalization’s author and responsible maintainer. The present manuscript’s three-author byline should not be read as an altered registry attribution or as a statement of undocumented individual contributions.

The development covers neither degenerate problems lacking strictly positive mutual gains nor bargaining among more than two players. It establishes no convergence rates, numerical method, equilibrium implementation, or empirical fairness claim. Its mathematical value is the explicit, source-traceable verification of the classical characterization within the stated compact convex two-person model.

REFERENCES

- [1] Leonardo de Moura and Sebastian Ullrich. The Lean 4 theorem prover and programming language. In *Automated Deduction—CADE 28*, volume 12699 of *Lecture Notes in Computer Science*, pages 625–635. Springer, 2021. https://doi.org/10.1007/978-3-030-79876-5_37.
- [2] John F. Nash, Jr. The bargaining problem. *Econometrica*, 18(2):155–162, 1950. <https://doi.org/10.2307/1907266>.
- [3] Palomar Registry. Nash bargaining solution axiomatic characterization. PALOMAR-2026-09-25-000013, version 1, registered September 25, 2026, 2026. <https://palomar-registry.org/entry?id=PALOMAR-2026-09-25-000013&version=1>.
- [4] Arthur Freitas Ramos. Nash bargaining solution axiomatic characterization. Lean 4 source repository, immutable commit `0ddaa4bb858fa6fdf78074459eafada5e7938727`, 2026. <https://github.com/Arthur742Ramos/nash-bargaining-lean/tree/0ddaa4bb858fa6fdf78074459eafada5e7938727>.
- [5] The mathlib Community. The Lean mathematical library. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs*, pages 367–381. ACM, 2020. <https://doi.org/10.1145/3372885.3373824>.