

FINITE VCG AND GROVES PAYMENTS IN LEAN

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ABSTRACT. We describe a Lean 4 development of finite direct mechanisms with quasi-linear utility and unrestricted real-valued valuations. The Clarke pivot mechanism selects a reported-welfare maximizer and charges each agent the loss in other-agent welfare relative to its maximum over the same fixed alternative set. The development proves efficiency, dominant-strategy truthfulness, nonnegative payments and no deficit, and individual rationality under the explicit assumption that every valuation is nonnegative. It also proves that every efficient, dominant-strategy incentive-compatible mechanism on the full valuation domain has Groves-form payments, with a term independent of the paying agent's report. We explain the finite-maximization interface, the report-update invariants, and the characterization argument using outcome-forcing reports and positive perturbations. The exposition is tied to a specific source revision and separates proved implementations from statement-only comparator placeholders. These are classical mechanism-design results; no new mathematical theorem or formalization-priority claim is made.

1. SCOPE AND CONTEXT

The Vickrey–Clarke–Groves tradition connects efficient allocation to truthful reporting through quasi-linear payments. Vickrey's auction analysis, Clarke's public-goods pivot rule, and Groves' more general payment scheme provide the classical context [9, 3, 5]. Characterization results associated with Green and Laffont and with Holmström explain when efficient, strategy-proof mechanisms must have Groves form [4, 6]. This paper explains one finite, unrestricted-domain Lean development of these ideas. It does not claim to reproduce every domain or participation assumption in those original papers.

All implementation descriptions refer to revision `aad0c41b99053cf4657d91f7446279557f87ae18` of *vcg-mechanism-lean* [7]. The repository's public interface contains four results about its Clarke pivot mechanism and a general Groves-payment necessity theorem. The latter is named `VCG.vcg_greenLaffont`. Our presentation states its exact finite-domain meaning before explaining the proof. The source fixes Lean `v4.35.0-rc2`; the Mathlib dependency uses the same release tag and resolves to revision `065356127b1dc0016f66b7283ce0ce2c4055aa55`. Mathlib supplies real arithmetic, finite sums, finite maximization, and supporting tactics [8].

The contribution of this note is an inspectable correspondence between the mathematics and the artifact: the sign convention for payments, the exact individual-rationality hypothesis, the full report domain used in the characterization, and the separation between a proof implementation and a statement-only challenge. The mathematical arguments are classical. We make no assertion that this is the first formalization of any result.

There is prior machine-checked VCG work. The Isabelle Archive of Formal Proofs entry by Caminati, Kerber, Lange, and Rowat formalizes combinatorial VCG auctions, including well-defined allocation and pricing and nonnegative prices [2]. Barthe and coauthors verify generic VCG truthfulness using relational refinement types [1]. These developments use different formal systems and models. They provide relevant context for the supporting VCG properties; we do not claim that they prove the particular Groves-form necessity theorem discussed here.

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2. THE FINITE DIRECT-MECHANISM MODEL

Let N be a finite nonempty set of agents and A a finite nonempty set of alternatives. We work under `Fintype`, `Nonempty`, and `DecidableEq` assumptions for both types, as explicitly stated in the comparator theorems of `Challenge.lean`. A profile is a function

$$v \in \mathcal{V} := \mathbb{R}^{N \times A}, \quad v_i(a) = v(i, a).$$

Every real-valued report is allowed by `VCG.Valuation`. There is no sign restriction in this profile type. Nonnegative valuations enter only as an additional hypothesis of the individual-rationality theorem.

For $a \in A$, define total and other-agent welfare by

$$(1) \quad W(v, a) = \sum_{j \in N} v_j(a), \quad S_i(v, a) = \sum_{j \in N \setminus \{i\}} v_j(a).$$

Thus $W(v, a) = v_i(a) + S_i(v, a)$. The definitions `VCG.welfare` and `VCG.othersWelfare` use sums over `Finset.univ` and `Finset.univ.erase i`, respectively. The finite sum identity is the bridge between an individual utility expression and an objective involving all agents.

A deterministic direct mechanism is a pair

$$x : \mathcal{V} \longrightarrow A, \quad p : \mathcal{V} \longrightarrow \mathbb{R}^N.$$

The quantity $p_i(v)$ is a charge paid by agent i to the mechanism. Under true valuation t_i and submitted profile w , that agent's quasi-linear utility is

$$(2) \quad U_i(t_i; w) = t_i(x(w)) - p_i(w).$$

No income constraint, nonlinear money preference, implementation cost, or outside option is represented.

Write $v[i \leftarrow r]$ for the profile obtained by replacing only agent i 's report with $r : A \rightarrow \mathbb{R}$. This is `Function.update v i r` in the source. The general mechanism is *efficient* if

$$(3) \quad W(v, a) \leq W(v, x(v)) \quad (v \in \mathcal{V}, a \in A),$$

and *dominant-strategy incentive compatible* (DSIC) if

$$(4) \quad v_i(x(v[i \leftarrow r])) - p_i(v[i \leftarrow r]) \leq v_i(x(v)) - p_i(v)$$

for every v, i, r . In this inequality, v_i is the true valuation of the agent whose report changes. The components v_j for $j \neq i$ may be any fixed reports. Their numerical values enter the mechanism, but the agent's true utility uses only its own true valuation.

3. CLARKE PIVOT ALLOCATION AND PAYMENTS

Because A is finite and nonempty, a maximum of $W(v, \cdot)$ exists. The artifact chooses an alternative $x^*(v)$ with

$$W(v, a) \leq W(v, x^*(v)) \quad \text{for every } a \in A.$$

It also defines the attained maximum

$$(5) \quad H_i(v) = \max_{a \in A} S_i(v, a).$$

Both constructions use `Finset.exists_max_image`; selected maximizers are obtained through `Classical.choice`. Consequently `VCG.xstar` and `VCG.pivotTerm` are noncomputable definitions, rather than an executable optimization algorithm. Ties are allowed, and none of the proofs needs a specified tie-breaking order.

Definition 3.1 (Clarke pivot charge). For each profile v , allocate $x^*(v)$ and charge agent i

$$(6) \quad p_i^*(v) = H_i(v) - S_i(v, x^*(v)).$$

At truthful reporting its utility is

$$(7) \quad u_i(v) = v_i(x^*(v)) - p_i^*(v) = W(v, x^*(v)) - H_i(v).$$

The definition `VCG.clarkePayment` gives the charges in (6). Utilities are recorded by `VCG.utility`, with identity (7) proved by `VCG.utility_eq`. The baseline in (5) maximizes the other agents' welfare over the *same* A as the original mechanism. Removing agent i from the sum does not create a new feasible-alternative set. This detail is material when comparing the model to allocation problems in which feasible outcomes change with participation.

Theorem 3.2 (Pivot mechanism properties). *The allocation and charges in Definition 3.1 have the following properties.*

- (1) For all v, a , $W(v, a) \leq W(v, x^*(v))$.
- (2) For all v, i, r , $U_i(v_i; v[i \leftarrow r]) \leq U_i(v_i; v)$, where U_i uses x^* and p^* .
- (3) For all v, i , $0 \leq p_i^*(v)$, and hence $0 \leq \sum_{i \in N} p_i^*(v)$.
- (4) If $v_j(a) \geq 0$ for every $j \in N$ and $a \in A$, then $u_i(v) \geq 0$ for every $i \in N$.

The first two results are `VCG.vcg_efficient` and `VCG.vcg_truthful`. Nonnegative charges are established by `VCG.clarkePayment_nonneg`, then summed in `VCG.vcg_noDeficit`. The last theorem is `VCG.vcg_individualRational`; its hypothesis, named `hnn`, is exactly $\forall j a, 0 \leq v_j(a)$.

3.1. The truthfulness invariant. Other-agent welfare does not change when only agent i changes its report:

$$(8) \quad S_i(v[i \leftarrow r], a) = S_i(v, a), \quad H_i(v[i \leftarrow r]) = H_i(v).$$

The first identity follows term by term in the sum excluding i . For the second, the artifact uses attainment on each side and the maximum bound on the other side, then applies antisymmetry. This avoids requiring the chosen maximizing *alternatives* to agree: only the maximum *values* must agree. The lemmas are `VCG.othersWelfare_update` and `VCG.pivotTerm_update`.

Put $w = v[i \leftarrow r]$. Expanding the charge and using (8) gives

$$U_i(v_i; w) = W(v, x^*(w)) - H_i(v).$$

Since $x^*(v)$ maximizes welfare computed using v ,

$$W(v, x^*(w)) - H_i(v) \leq W(v, x^*(v)) - H_i(v) = U_i(v_i; v).$$

This proves item 2. In Lean, `VCG.trueUtility_eq_of_agree` first proves the welfare expression under agreement outside i , and `VCG.vcg_truthful` then applies it to an update.

The formal theorem compares a truthful profile with a one-agent deviation. Its universal quantification gives dominant-strategy truthfulness against arbitrary reports by the other agents: given their reports w_{-i} and agent i 's true valuation t_i , apply the theorem to the hybrid profile (t_i, w_{-i}) . There is no assumption that the other agents' submitted reports match their true valuations.

3.2. Payments and utility. Equation (5) bounds $S_i(v, x^*(v))$, so (6) is nonnegative for arbitrary real valuations. Summing this inequality proves item 3. No deficit here means nonnegative aggregate charges; the model contains no separate cost term.

For item 4, choose a_i attaining $H_i(v)$. Then

$$H_i(v) = S_i(v, a_i) \leq v_i(a_i) + S_i(v, a_i) = W(v, a_i) \leq W(v, x^*(v)).$$

Subtracting $H_i(v)$ and using (7) proves the claim. Although this proof uses only $v_i(a_i) \geq 0$ for the chosen agent, the exported theorem assumes nonnegativity of *all* entries. A weaker hypothesis should not be attributed to that theorem merely because its proof would support a refinement.

4. THE GROVES PAYMENT CHARACTERIZATION

The pivot rule is one member of a larger family. A Groves-form payment has an agent-specific term depending only on other agents' reports and subtracts their welfare at the selected outcome.

Theorem 4.1 (Finite unrestricted-domain necessity). *Let $x : \mathcal{V} \rightarrow A$ and $p : \mathcal{V} \rightarrow \mathbb{R}^N$ satisfy efficiency (3) and DSIC (4) on the full domain $\mathcal{V} = \mathbb{R}^{N \times A}$. There is a function $h : N \times \mathcal{V} \rightarrow \mathbb{R}$ such that*

$$(9) \quad v_{-i} = w_{-i} \implies h_i(v) = h_i(w),$$

$$(10) \quad p_i(v) = h_i(v) - S_i(v, x(v)) \quad (v \in \mathcal{V}, i \in N).$$

Here $v_{-i} = w_{-i}$ means $v_j = w_j$ as functions on A for every $j \neq i$.

This is the conclusion of `VCG.vcg_greenLaffont`. The source represents h_i as a function of a full profile and separately proves (9). It does not define a quotient type of other-agent profiles. The theorem identifies the payment *form*; it neither fixes h_i to the pivot maximum nor imposes individual rationality or budget conditions on arbitrary h_i .

4.1. Equal outcomes imply equal payments. Fix i and other reports, and abbreviate their welfare by $S(a)$. Consider two reports r, s by agent i that select the same outcome c . Apply DSIC with r as the true report and s as the deviation:

$$r(c) - p_i(s) \leq r(c) - p_i(r).$$

Here $p_i(r)$ abbreviates the payment at the full profile with these fixed other reports. Reversing r and s proves the opposite payment inequality, so $p_i(r) = p_i(s)$. The source uses this fact as a local lemma inside `VCG.groves_key`. It requires no continuity of the payment rule.

4.2. Forcing an alternative. For any $c \in A$ and $\varepsilon > 0$, the unrestricted report domain permits

$$(11) \quad r^{c,\varepsilon}(d) = -S(d) + \begin{cases} \varepsilon, & d = c, \\ 0, & d \neq c. \end{cases}$$

Total welfare at d becomes ε when $d = c$ and 0 otherwise. Efficiency therefore forces $x = c$, even if the mechanism has arbitrary tie-breaking at other profiles. By the preceding argument, every positive ε forcing c has the same payment as the report $r^{c,1}$. Any original report that selects c also has that payment. This step explains why restricting all reports to nonnegative valuations would change the characterization argument: (11) need not belong to such a restricted domain.

4.3. Squeezing the payment difference. Let two original profiles v, w agree outside i , and put $a = x(v)$, $b = x(w)$. If $a = b$, equal payments and equality of the relevant other-agent welfare already give

$$(12) \quad p_i(w) + S_i(w, x(w)) = p_i(v) + S_i(v, x(v)).$$

Suppose $a \neq b$. Apply DSIC to the forced report $r^{a,\varepsilon}$ with deviation $r^{b,\varepsilon}$. Since payments at these reports equal the original payments at v and w , respectively,

$$S(a) - S(b) \leq p_i(w) - p_i(v) + \varepsilon.$$

Applying DSIC in the reverse direction gives

$$p_i(w) - p_i(v) \leq S(a) - S(b) + \varepsilon.$$

Both inequalities hold for every positive ε . The real order lemma `le_of_forall_pos_le_add` removes the perturbation on each side, yielding

$$p_i(w) - p_i(v) = S(a) - S(b).$$

Because the other reports agree, this is (12). Thus `VCG.groves_key` proves that $p_i(v) + S_i(v, x(v))$ is invariant under changes to i 's report. The proof uses finite welfare sums, efficiency, and real order; no differentiability, continuity, integration, or limiting sequence is introduced.

4.4. Choosing an independent term. Let $v^0 = v[i \leftarrow 0]$, where 0 is the identically zero report. The artifact defines

$$h_i(v) = p_i(v^0) + S_i(v^0, x(v^0)).$$

If v, w agree outside i , their zeroed profiles are equal, so $h_i(v) = h_i(w)$. Applying (12) to v and v^0 and rearranging establishes (10). This completes the necessity theorem.

For context, the usual sufficiency argument is the same welfare comparison as in the pivot proof: for an efficient x and any term satisfying (9), a unilateral deviation changes utility to $W(v, x(v[i \leftarrow r])) - h_i(v)$, which cannot exceed $W(v, x(v)) - h_i(v)$. The exported general theorem in this artifact is the *necessity* direction. We do not attribute a separately exported general sufficiency theorem to the source.

5. PROOF ORGANIZATION AND TRUST BOUNDARY

The artifact divides its implementation into four substantive modules. `VCG.Basic` defines valuations, finite welfare maximizers, pivot terms, charges, and truthful utility. `VCG.Truthful` defines true utility at a reported profile and proves update invariance, truthfulness, and efficiency. `VCG.Properties` proves nonnegative charges, no deficit, and individual rationality. `VCG.GreenLaffont` proves the invariant (12) and the general payment characterization. `VCG.Main` publicly imports all four; `Solution.lean` publicly imports `VCG.Main`.

The separate `Challenge.lean` repeats public definitions and theorem statements with twelve intentional `sorry` placeholders: seven nontrivial definition bodies and five theorem proofs. Its valuation abbreviation is explicit. This statement-only comparator input is not a proof of the results. It imports `Mathlib` rather than the implementation modules. The implementation route is `Solution`, whose source and imported `VCG` modules contain no `sorry`, `admit`, custom `axiom`, or `unsafe` declaration tokens at the pinned revision.

The comparator configuration lists eight definition names and five theorem names and permits `propext`, `Classical.choice`, and `Quot.sound`. The repository’s verification script requests a build, standalone challenge compilation, declaration-kind checks, placeholder scanning, an axiom audit, and comparator checks. These are different checks: matching interfaces does not make a placeholder proof valid, and source-token scanning does not replace kernel checking. The script and metadata describe the intended reproducibility procedure; this paper makes no claim to have executed a fresh Lean build in the manuscript preparation environment.

The version pins determine the Lean and dependency environment. They do not establish that the prose matches every source theorem or that an historical attribution is correct. Those are source-review obligations. Likewise, a kernel-checked mathematical result does not supply a runtime auction system, a verified optimizer, or economic guarantees outside the formal hypotheses.

6. EXAMPLES AND LIMITATIONS

The pivot model does not imply exact budget balance. For two agents and two alternatives a, b , take

$$(v_1(a), v_1(b)) = (10, 0), \quad (v_2(a), v_2(b)) = (0, 6).$$

The welfare-maximizing alternative is a , with total welfare 10. The pivot terms are $H_1 = 6$ and $H_2 = 10$, so the charges are $p_1^* = 6$ and $p_2^* = 0$. Aggregate revenue is 6, whereas budget balance in the zero-cost model would require revenue 0. Utilities are 4 and 0. This example is explanatory arithmetic, not an additional named theorem in the artifact.

Individual rationality cannot be read as unconditional on the full real valuation domain. With one agent and one alternative valued at -1 , the other-agent welfare and pivot payment are 0, and utility is -1 . The exported nonnegativity assumption rules out this example. There is also no participation decision in the formal model, so the theorem’s “individual rationality” means utility at least zero under its stated hypothesis rather than comparison to a separately modeled reservation utility.

The characterization does not single out Clarke pivot charges. Replacing $H_i(v)$ by an arbitrary other-report-dependent $h_i(v)$ retains Groves form and truthful incentives for efficient allocation,

but may produce negative charges or negative utility. Conversely, the necessity proof uses the full unrestricted real report domain throughout; the result should not be transferred unchanged to a restricted type space, a randomized allocation rule, or an infinite alternative set.

No welfare-maximization complexity bound is formalized. Classical choice provides selected maximizers for finite nonempty A and yields a mathematical mechanism, while executable optimization and deterministic tie-breaking algorithms remain separate implementation questions. Coalition resistance, Bayesian incentive constraints, approximate efficiency, externalities in valuations, and feasibility changes upon agent removal are outside this development.

7. REPRODUCIBILITY AND MANUSCRIPT PROVENANCE

The pinned source repository [7] provides the theorem implementations, dependency manifest, comparator configuration, and verification script. Its software license is BSD-3-Clause. This manuscript and its two-file L^AT_EX project are licensed CC BY 4.0; the manuscript license does not change the repository’s software license. The submitted project consists of `main.tex` and `references.bib`. A standard pdfL^AT_EX, BibT_EX, pdfL^AT_EX, pdfL^AT_EX sequence produces the reading copy without bundling Lean source or prebuilt bibliography files.

The manuscript text was generated primarily with OpenAI GPT-6.1, with human direction of the scope and some human understanding of the work. No comprehensive personal, line-by-line human review is claimed. AI-assisted source comparison and an independent AI mathematical and bibliographic review were used in manuscript preparation; these are not represented as human peer review. The repository’s own formalization metadata records a distinct implementation-assistance history involving GPT-6-Luna. Lean’s proof checking applies to the formal implementation, not to the surrounding exposition, citations, or authorship assertions.

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