

Signed Mukai diagonals and derived equivalences of fivefolds

September 2026

Abstract

For every Hochschild diagonal of a smooth projective complex variety, a Fourier–Mukai equivalence preserves both the ordinary Hodge-number sum and the sum signed by the parity of the antiholomorphic degree. The latter is the signature of a Hermitian form obtained from the generalized Mukai pairing. In dimension five these invariants determine $h^{0,3}$ and $h^{1,4}$ and reduce the unrestricted Hodge-number problem to two numerical parameters, one of which is the possible failure of invariance of $h^{0,2}$. Albanese methods remove this structure-sheaf ambiguity when the Albanese image has dimension at least three. The general fivefold case remains open, while a theorem of Abuaf proves invariance of all Hodge numbers for fivefolds with trivial canonical bundle.

1 Introduction

The conjecture that derived-equivalent smooth projective complex varieties have the same Hodge numbers is known in dimensions at most three. For fourfolds, the topological signature is a derived invariant and, together with the Hochschild–Kostant–Rosenberg decomposition and the known edge invariants, determines the full Hodge diamond [1]. The next unresolved case is dimension five. The purpose of this paper is to isolate the extra information contained in the signed Hochschild diagonals and to identify exactly what they leave undetermined.

There is a complete result under a natural geometric hypothesis. Abuaf proved that derived-equivalent smooth projective complex fivefolds with $K_X \simeq \mathcal{O}_X$ have the same Hodge numbers [2]; triviality of K_Y follows from compatibility with the Serre functor. His proof refines the Mukai pairing on the individual odd Hochschild diagonals. After composition with complex conjugation, the resulting Hermitian forms are preserved by Fourier–Mukai equivalences, and their signatures are computed using Lefschetz decomposition and the Hodge–Riemann bilinear relations. A fivefold Riemann–Roch identity at $c_1 = 0$,

$$\chi(X, \Omega_X^2) = -11\chi(X, \Omega_X^1),$$

then separates the remaining terms [5]. Thus the two numerical ambiguities found below both vanish when the canonical bundle is trivial. For unrestricted fivefolds they remain open.

There is also a complementary reduction governed by the Albanese map. The results of Caucci–Pareschi, Mammola, and Popa–Schnell imply that all $h^{0,q}$ are invariant as soon as the Albanese image has dimension at least three. Section 7 gives the precise stable/non-stable argument and describes the residual higher-direct-image defects when the Albanese image has dimension two or one.

Figure 1 records the resulting division of labor among the available invariance statements.

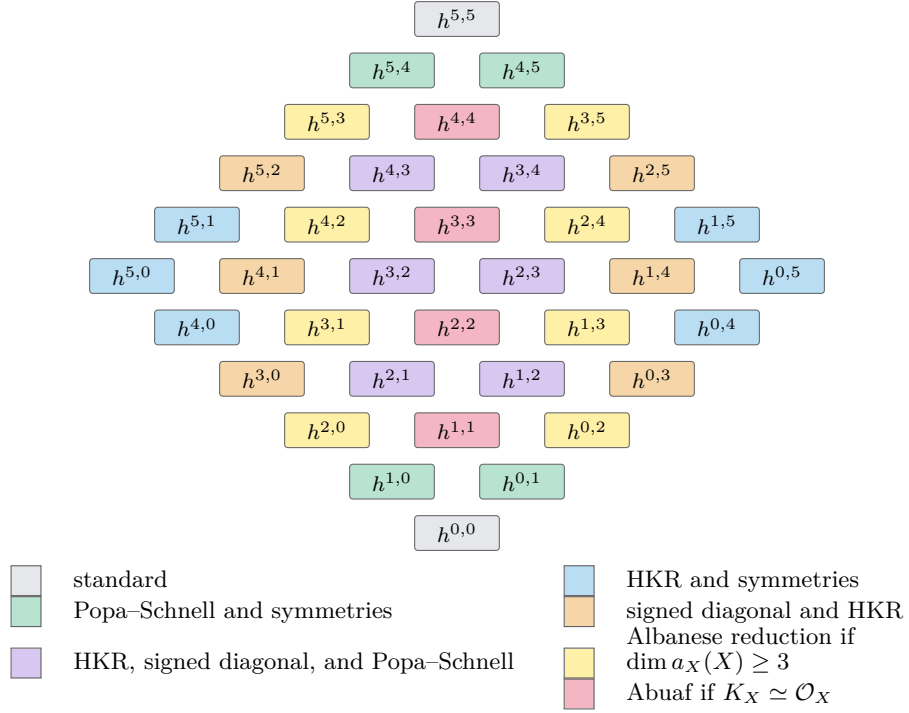


Figure 1: Derived-invariance status of the Hodge numbers of a smooth projective complex fivefold. Hodge symmetry and Serre duality are included in each label. The yellow cells form the structure-sheaf family and are settled under the stated Albanese hypothesis (or by Abuaf when K_X is trivial); the pink interior family is known under Abuaf's hypothesis. Both families remain open without additional hypotheses.

2 Ordinary and signed Hochschild diagonals

Let X be a smooth projective complex n -fold and put

$$V_k(X) = \bigoplus_{p-q=k} H^q(X, \Omega_X^p).$$

HKR identifies $V_k(X)$ with $\mathrm{HH}_k(X)$. Thus a Fourier–Mukai equivalence preserves

$$b_k(X) = \dim V_k(X) = \sum_{p-q=k} h^{p,q}(X). \quad (2.1)$$

The generalized Mukai pairing supplies a parity refinement.

Theorem 2.1 (Signed-diagonal theorem). *If X and Y are Fourier–Mukai equivalent smooth projective complex varieties, then for every integer k ,*

$$s_k(X) := \sum_{p-q=k} (-1)^q h^{p,q}(X) = s_k(Y). \quad (2.2)$$

Consequently the even- q and odd- q totals on each diagonal are separately derived invariants.

This is numerical information: it does not give canonical isomorphisms of the individual Hodge spaces.

3 The Hermitian form

For $\alpha = \sum_j \alpha_j \in H^*(X, \mathbf{C})$ set $\alpha^\vee = \sum_j i^j \alpha_j$. Under the modified HKR map, the cohomological Mukai pairing is

$$P_X(\alpha, \beta) = \int_X e^{c_1(X)/2} \alpha^\vee \beta. \quad (3.1)$$

Cohomological Fourier–Mukai transforms preserve every V_k , are defined over $\mathbf{Q}$, and are isometries for this pairing [4, 6].

Symmetrize with the dimension-dependent sign:

$$S_X(\alpha, \beta) = P_X(\alpha, \beta) + (-1)^n P_X(\beta, \alpha).$$

A graded-commutativity calculation gives

$$S_X(\alpha, \beta) = 2 \int_X \cosh\left(\frac{c_1(X)}{2}\right) \alpha^\vee \beta. \quad (3.2)$$

Multiplication by $\sqrt{\cosh(c_1(X)/2)}$ is a rational congruence, preserving every V_k , from this form to

$$Q_X(\alpha, \beta) = \int_X \alpha^\vee \beta. \quad (3.3)$$

Complex conjugation exchanges V_k and V_{-k} . On V_k define

$$\mathcal{H}_{X,k}(\alpha, \beta) = \begin{cases} Q_X(\alpha, \bar{\beta}), & n+k \text{ even,} \\ iQ_X(\alpha, \bar{\beta}), & n+k \text{ odd.} \end{cases} \quad (3.4)$$

This is Hermitian. Rationality makes a Fourier–Mukai transform commute with conjugation, so the isometry class, and in particular the signature, of $\mathcal{H}_{X,k}$ is derived invariant.

4 Lefschetz-string calculation

Choose an ample class L . A primitive class u of type (a, b) and total degree $r = a + b$ generates the string

$$u, Lu, \dots, L^{n-r}u.$$

Put $\ell = n - r$. Its contribution to the signed diagonal is

$$\sum_{j=0}^{\ell} (-1)^{b+j} = \begin{cases} 0, & \ell \text{ odd}, \\ (-1)^b, & \ell \text{ even}. \end{cases} \quad (4.1)$$

The Hermitian form pairs the j th and $(\ell - j)$ th members. If ℓ is odd, the entire string is hyperbolic. If $\ell = 2t$, all noncentral terms are hyperbolic and only $L^t u$ contributes. The Hodge–Riemann bilinear relations compute its sign. Summing the primitive strings gives

$$\boxed{\text{sig}(\mathcal{H}_{X,k}) = (-1)^{n(n-1)/2} s_k(X)}. \quad (4.2)$$

This proves Theorem 2.1. When $n + k$ is odd, both sides vanish: there is no central member in any relevant Lefschetz string.

5 What parity separation determines

On a diagonal at depth r from the edge, write

$$k = -n + r, \quad x_j = h^{j, n-r+j} \quad (0 \leq j \leq r).$$

Hodge symmetry and Serre duality give $x_j = x_{r-j}$. The two invariants are

$$b_{-n+r} = \sum_{j=0}^r x_j, \quad s_{-n+r} = (-1)^{n-r} \sum_{j=0}^r (-1)^j x_j. \quad (5.1)$$

For $r = 2$ they separate the two symmetry orbits. In all dimensions this proves derived invariance of

$$h^{0, n-2} \quad \text{and} \quad h^{1, n-1}. \quad (5.2)$$

At depth three, however, only $h^{0, n-3} + h^{1, n-2}$ is forced. This is the first general edge ambiguity.

6 The exact fivefold remainder

For $n = 5$, the diagonal $k = -3$ is

$$V_{-3} = H^{0,3} \oplus H^{1,4} \oplus H^{2,5}.$$

Hodge and Serre symmetry give

$$b_{-3} = 2h^{0,3} + h^{1,4}, \quad s_{-3} = -2h^{0,3} + h^{1,4}.$$

Therefore

$$\boxed{h^{0,3} = \frac{b_{-3} - s_{-3}}{4}}, \quad \boxed{h^{1,4} = \frac{b_{-3} + s_{-3}}{2}}. \quad (6.1)$$

Together with derived invariance of $h^{0,1}$ [8], this determines $h^{0,q}$ for $q = 0, 1, 3, 4, 5$. The structure-sheaf problem is reduced to

$$h^{0,2}(X) \stackrel{?}{=} h^{0,2}(Y), \quad (6.2)$$

equivalently to invariance of $\chi(\mathcal{O}_X)$.

For the full Hodge diamond, the ordinary and signed diagonals together with the two symmetries leave exactly two parameters. If $\delta^{p,q} = h^{p,q}(X) - h^{p,q}(Y)$, they may be chosen as

$$\delta^{0,2} = a, \quad \delta^{1,3} = -a, \quad \delta^{1,1} = c, \quad \delta^{2,2} = -c, \quad (6.3)$$

with the remaining nonzero entries forced by Hodge and Serre symmetry. The parameter a is the structure-sheaf obstruction; c is an independent interior obstruction.

7 Albanese reductions

Let $a_X : X \rightarrow \text{Alb}(X)$ be the Albanese map of a smooth projective complex fivefold and set

$$c(X) = 5 - \dim a_X(X). \quad (7.1)$$

This notation is unrelated to the interior parameter c in (6.3). Recall that an equivalence is *Rouquier-stable* if its Rouquier isomorphism carries $\{\text{id}_X\} \times \text{Pic}^0(X)$ onto $\{\text{id}_Y\} \times \text{Pic}^0(Y)$.

The cases of small Albanese fiber dimension combine as follows. If $c(X) = 0$, Caucci–Pareschi prove invariance of every $h^{0,q}$ [3]. If the equivalence is Rouquier-stable and $c(X) = 1$, Mammola proves that X and Y are birational, so their $h^{0,q}$ agree. If it is Rouquier-stable and $c(X) = 2$, Mammola proves

$$\chi(K_X) = \chi(K_Y). \quad (7.2)$$

Since $\chi(K_X) = -\chi(\mathcal{O}_X)$ for a fivefold, this kills the sole remaining structure-sheaf ambiguity $h^{0,2}$ [7, Theorems 3.15–3.16].

It remains to remove the Rouquier-stability qualification. In the non-stable case, the off-diagonal projections of the Rouquier isomorphism give positive-dimensional abelian varieties in both connected automorphism groups. Popa–Schnell’s structure theorem exhibits both varieties as étale-locally trivial fibrations over positive-dimensional abelian varieties, and hence

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 0 \quad (7.3)$$

[8, Theorem A(2)]. The stable and non-stable cases therefore give the unconditional conclusion

Theorem 7.1 (Albanese-image reduction). *If X and Y are derived-equivalent smooth projective complex fivefolds and $\dim a_X(X) \geq 3$, then*

$$\boxed{h^{0,q}(X) = h^{0,q}(Y) \quad \text{for every } q.} \quad (7.4)$$

Equivalently, the parameter a in (6.3) vanishes; the interior parameter c need not be determined by this argument.

The first residual case is $\dim a_X(X) = 2$, or $c(X) = 3$. The Leray decomposition gives

$$\chi(K_X) = \sum_{p=0}^3 (-1)^p \chi(R^p a_{X*} K_X). \quad (7.5)$$

Under Rouquier stability, existing relative-canonical results control the $p = 0, 1, 3$ terms; the sole defect is $\chi(R^2 a_{X*} K_X)$ [7]. Factoring the Albanese map through a smooth model of its image, $X \xrightarrow{f} Z \rightarrow \text{Alb}(X)$, relative duality over the smooth locus identifies

$$R^2 f_* K_X \simeq (R^1 f_* \mathcal{O}_X)^\vee \otimes K_Z. \quad (7.6)$$

Thus the defect is the weight-one relative Picard/Hodge bundle. If the general Albanese threefold fiber F has $h^1(F, \mathcal{O}_F) = 0$, this sheaf vanishes; the corresponding conclusion on the Fourier–Mukai partner and the non-stable case then give invariance of $h^{0,2}$. Consequently, any counterexample over an Albanese surface must have irregular threefold fibers.

When $\dim a_X(X) = 1$, the residual defect involves both $R^1 f_* \mathcal{O}_X$ and $R^2 f_* \mathcal{O}_X$; vanishing of both on the general Albanese fourfold fiber again forces invariance of $h^{0,2}$. When the Albanese image is a point, this method supplies no additional constraint.

Contributors, affiliations, and support

No individual authorship is designated for this working paper. The AI system produced the mathematical synthesis and original draft. Benjamin Antieau (Northwestern University) selected the problem, supplied the source material, and directed revision. Mathematical claims should be independently checked before journal submission. The comparison with Roland Abuaf’s fivefold theorem is included explicitly; the signed-diagonal argument is presented independently.

References

- [1] Derived invariance of the signature and hodge numbers of fourfolds. Hexagon preprint hexagon:2609.00104v1, 2026. URL <https://hexagonmath.org/2609.00104v1>. Contributors: Benjamin Antieau; Andrei Căldăraru; Ruoxi Li; Akhil Mathew; Joshua Mundinger; Noah Olander; Martin Olsson.
- [2] Roland Abuaf. Hirzebruch signature theorem on Hochschild homology, with applications to derived invariance of Hodge numbers, September 2026. URL <https://arxiv.org/abs/2609.08917v3>. arXiv:2609.08917v3 [math.AG].
- [3] Federico Caucci and Giuseppe Pareschi. Derived invariants arising from the Albanese map. *Algebraic Geometry*, 6(6):730–746, 2019. doi: 10.14231/AG-2019-031.
- [4] Andrei Căldăraru. The Mukai pairing, II: the Hochschild–Kostant–Rosenberg isomorphism. *Advances in Mathematics*, 194(1):34–66, 2005. doi: 10.1016/j.aim.2004.05.012.
- [5] Alexander S. Haupt, Andre Lukas, and K. S. Stelle. M-theory on Calabi–Yau five-folds. *Journal of High Energy Physics*, 2009(5):069, 2009.
- [6] Daniel Huybrechts. *Fourier–Mukai Transforms in Algebraic Geometry*. Oxford Mathematical Monographs. Oxford University Press, 2006.
- [7] Stefano Mammola. *Derived Invariance of Higher Direct Image Sheaves of the Canonical Bundle*. PhD thesis, Università degli Studi di Milano-Bicocca, 2023. URL <https://hdl.handle.net/10281/455418>. Ph.D. thesis, academic year 2021/2022.

- [8] Mihnea Popa and Christian Schnell. Derived invariance of the number of holomorphic 1-forms and vector fields. *Annales scientifiques de l'École Normale Supérieure*, 44(3):527–536, 2011.