

# FINITE HAUSDORFF HYPERSPACES AND GROMOV–HAUSDORFF GEOMETRY

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ABSTRACT. For a nonempty finite metric space  $X$ , let  $\text{Exp}(X)$  be the space of its nonempty subsets equipped with the Hausdorff metric. We prove that the isometry type of  $\text{Exp}(X)$  determines  $X$  when  $X$  is strongly rigid, meaning that distinct unordered pairs of distinct points have distinct distances. We also prove that the hyperspace operation preserves the ordinary Gromov–Hausdorff distance locally under explicit separation conditions on the distance spectra. For finite ultrametric spaces, the hyperspace operation is an isometric embedding with respect to the Gromov–Hausdorff ultrametric. In contrast, we construct finite ultrametric spaces for which the ordinary Gromov–Hausdorff distance strictly decreases under the hyperspace operation. Examples of equal cardinality exist in every cardinality at least six and realize every ratio in  $[1/2, 1)$ . Their distances remain constant after the first hyperspace iteration. An explicit perturbation also yields strict contraction for finite strongly rigid metric spaces with all triangle inequalities strict.

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*Date:* 30 September 2026.

*2020 Mathematics Subject Classification.* Primary 54B20; Secondary 51F30, 54E35.

*Key words and phrases.* Hausdorff hyperspace, Gromov–Hausdorff distance, Gromov–Hausdorff ultrametric, strongly rigid metric space, ultrametric space.

## 1. INTRODUCTION

Let  $\text{Exp}(X)$  denote the Hausdorff hyperspace of a compact metric space  $X$ , that is, the space of its nonempty compact subsets with the Hausdorff metric. Mikhailov proved that the assignment

$$X \longmapsto \text{Exp}(X)$$

is nonexpanding on the Gromov–Hausdorff space [4]. It is therefore natural to ask whether the assignment preserves every Gromov–Hausdorff distance.

We first study whether the hyperspace isometry type determines the base. A finite metric space is strongly rigid if distinct unordered pairs of distinct points have distinct distances. We prove that if  $X$  and  $Y$  are finite strongly rigid metric spaces, then

$$\text{Exp}(X) \cong \text{Exp}(Y) \implies X \cong Y.$$

One proof recognizes singleton subsets from the cardinalities of their closed balls in the hyperspace. For a strongly rigid base with  $n$  points, a second proof reconstructs the base from any family of  $n$  hyperspace points whose  $\binom{n}{2}$  distances are all different. These arguments also imply quantitative local results for the ordinary distance. In a symmetric form,  $\text{Exp}$  preserves  $d_{\text{GH}}(X, Y)$  when it is smaller than one quarter of the minimum gap in the two finite distance spectra. For bases with equal cardinalities, the gap condition can be imposed on only one of the two spaces. Thus the ordinary hyperspace operation has explicit conditions for local distance preservation even though it is not globally isometric.

The answer depends on which Gromov–Hausdorff geometry is used. On finite ultrametric spaces we consider the Gromov–Hausdorff ultrametric  $u_{\text{GH}}$ , introduced by Zarichnyi [5]. For all nonempty finite ultrametric spaces  $U, V$ , we prove that

$$u_{\text{GH}}(\text{Exp}(U), \text{Exp}(V)) = u_{\text{GH}}(U, V).$$

Thus  $\text{Exp}$  is an isometric embedding in the non-Archimedean setting. The proof has two ingredients. First, a finite ultrametric space can be reconstructed from the isometry type of its Hausdorff hyperspace. Second, taking a Hausdorff hyperspace commutes with every closed quotient of a finite ultrametric space. The structural formula for  $u_{\text{GH}}$  due to Mémoli, Smith, and Wan [3] then yields the distance equality.

Here “non-Archimedean” refers to the distance  $u_{\text{GH}}$ , not merely to restricting the ordinary Gromov–Hausdorff distance  $d_{\text{GH}}$  to ultrametric spaces. This distinction is essential. We construct finite ultrametric spaces  $X_0, Y_0$  such that

$$2d_{\text{GH}}(X_0, Y_0) = 3, \quad 2d_{\text{GH}}(\text{Exp}(X_0), \text{Exp}(Y_0)) = 2.$$

The contraction follows from the sizes of the components at the largest distance. The three top components of  $X_0$  all have two points, whereas those of  $Y_0$  have sizes 1, 1, 3. At the base level a two-point component cannot cover

a three-point component below distortion three. After passing to hyperspaces, the seven support blocks have sizes

$$(3, 3, 3, 9, 9, 9, 27) \quad \text{and} \quad (1, 1, 1, 7, 7, 7, 7),$$

and the former blocks cover the latter at distortion two.

The same phenomenon persists at equal cardinality. For every  $0 < a < b \leq 2a$  and sufficiently separated top scale  $H$ , there are six-point ultrametric spaces  $X, Y$  such that

$$2d_{\text{GH}}(X, Y) = b, \quad 2d_{\text{GH}}(\text{Exp}(X), \text{Exp}(Y)) = a.$$

Appending common singleton top components yields such examples in every cardinality at least six, and varying  $a/b$  realizes every contraction ratio in  $[1/2, 1)$ . The relevant invariant is not the total cardinality of a target support block. At distortion  $a$ , only the number of its closed  $a$ -classes consumes source capacity. We formulate this observation as a necessary and sufficient criterion in terms of quotient cardinalities.

For these examples, the ordinary distance remains constant after the first hyperspace iteration. The bound  $1/2$  describes the ratios attained by this construction, not a lower bound for arbitrary finite ultrametric spaces. We explain the restriction on this construction in theorem 5.7.

Repeated distance levels are not responsible for the global failure. We also construct an explicit perturbation consisting of a six-point and a five-point strongly rigid metric space. Every positive distance in each space is distinct and every triangle inequality is strict, while

$$d_{\text{GH}}(X, Y) = 1.504, \quad d_{\text{GH}}(\text{Exp}(X), \text{Exp}(Y)) \leq 1.0075.$$

Consequently, the Hausdorff hyperspace operation is not an isometric embedding for the ordinary Gromov–Hausdorff distance, even on finite metric spaces in general position. This does not contradict strongly rigid injectivity. Strict shortening of a positive distance is different from identifying two isometry classes.

**Organization.** After recalling the two Gromov–Hausdorff distances in section 2, we prove strongly rigid injectivity and the two local theorems in section 3. Finite-ultrametric reconstruction and the  $u_{\text{GH}}$ -isometry theorem follow in section 4. The ordinary-distance counterexamples are established in section 5.

**Conventions and notation.** All metric spaces are nonempty. We write  $X \cong Y$  when  $X$  and  $Y$  are isometric and  $|X|$  for the cardinality of a finite set  $X$ . The notation  $\text{Exp}(X)$  denotes the Hausdorff hyperspace. We distinguish the ordinary Gromov–Hausdorff distance  $d_{\text{GH}}$  from the Gromov–Hausdorff ultrametric  $u_{\text{GH}}$ .

**Use of AI.** This paper is a revised presentation of the author’s AI-assisted research report [2], published on Zenodo as version 1.0 on 13 August 2026.

The author proposed using the closed-ball profile  $\beta_A^X(r)$  to recognize singleton subsets in the Hausdorff hyperspace. OpenAI Codex assisted with mathematical exploration, including the search for proof constructions and counterexamples, proof auditing, finite computation, language editing, and preparation of the L<sup>A</sup>T<sub>E</sub>X manuscript. The author selected the mathematical statements included in this paper and takes sole responsibility for its mathematical content and final manuscript.

## 2. PRELIMINARIES

We recall the metrics on hyperspaces and on isometry classes that are used below. For a metric space  $X$ , a nonempty subset  $A \subseteq X$ , and  $x \in X$ , write

$$d_X(x, A) = \inf_{a \in A} d_X(x, a).$$

For nonempty compact subsets  $A, B \subseteq X$ , their Hausdorff distance is

$$d_H^X(A, B) = \max \left\{ \sup_{a \in A} d_X(a, B), \sup_{b \in B} d_X(b, A) \right\}.$$

The space of all nonempty compact subsets of  $X$ , equipped with this metric, is denoted by  $\text{Exp}(X)$ . When  $X$  is finite,  $\text{Exp}(X)$  is simply the set of all nonempty subsets of  $X$ .

A *correspondence* between compact metric spaces  $X$  and  $Y$  is a relation  $R \subseteq X \times Y$  whose coordinate projections are surjective. Its distortion is

$$\text{dis}(R) = \sup \{ |d_X(x, x_1) - d_Y(y, y_1)| \mid (x, y), (x_1, y_1) \in R \}.$$

Let  $R$  range over all correspondences between  $X$  and  $Y$ . We use the standard correspondence formula

$$d_{\text{GH}}(X, Y) = \frac{1}{2} \inf_R \text{dis}(R), \tag{1}$$

as in [1, Chapter 7].

The following result of Mikhailov controls the hyperspace distance from above.

**Proposition 2.1.** *Let  $X, Y$  be compact metric spaces. Then*

$$d_{\text{GH}}(\text{Exp}(X), \text{Exp}(Y)) \leq d_{\text{GH}}(X, Y).$$

*Proof.* This is the nonexpansion theorem for the Hausdorff mapping [4].  $\square$

A metric  $u$  on a set  $U$  is an *ultrametric* if, for every  $x, y, z \in U$ , we have

$$u(x, z) \leq \max\{u(x, y), u(y, z)\}.$$

If  $U$  is ultrametric, then the Hausdorff metric on  $\text{Exp}(U)$  is again ultrametric. Indeed, for nonempty compact subsets  $A, B, C \subseteq U$ , a point of  $A$  can be followed through a nearest point of  $B$  to one of  $C$ , and the strong triangle inequality yields

$$d_H^U(A, C) \leq \max\{d_H^U(A, B), d_H^U(B, C)\}.$$

For compact ultrametric spaces  $U, V$ , the Gromov–Hausdorff ultrametric is defined as follows. Let  $Z$  range over ultrametric spaces and let  $\varphi_U : U \rightarrow Z$  and  $\varphi_V : V \rightarrow Z$  range over isometric embeddings. We put

$$u_{\text{GH}}(U, V) = \inf_{Z, \varphi_U, \varphi_V} d_{\text{H}}^Z(\varphi_U(U), \varphi_V(V)),$$

This restriction on the ambient metric distinguishes  $u_{\text{GH}}$  from the ordinary distance  $d_{\text{GH}}$ .

For  $t \geq 0$ , let  $U_{\text{c}(t)}$  be the *closed quotient* of an ultrametric space  $U$ . Points  $x, x_1$  are identified when  $u(x, x_1) \leq t$ . If two equivalence classes are distinct, their distance is the distance between any pair of their representatives. The ultrametric inequality makes this well-defined. For finite ultrametric spaces  $U, V$ , Mémoli, Smith, and Wan proved the structural formula

$$u_{\text{GH}}(U, V) = \min \{t \geq 0 \mid U_{\text{c}(t)} \cong V_{\text{c}(t)}\}. \quad (2)$$

We refer to [3, Theorem 3] for (2).

### 3. STRONG RIGIDITY AND LOCAL DISTANCE PRESERVATION

A finite metric space  $X$  is *strongly rigid* if, for all  $x, x_1, y, y_1 \in X$ , we have

$$d_X(x, x_1) = d_X(y, y_1) > 0 \implies \{x, x_1\} = \{y, y_1\}.$$

Thus all unordered pairs of distinct points determine different distances. Strong rigidity first yields an exact reconstruction result and then, after a quantitative separation argument, local preservation of the ordinary Gromov–Hausdorff distance.

For a nonempty finite metric space  $M$ , put

$$\Sigma(M) = \{d_M(x, x_1) \mid x, x_1 \in M\}.$$

**Lemma 3.1.** *Let  $M$  be a nonempty finite metric space. Then*

$$\Sigma(\text{Exp}(M)) = \Sigma(M).$$

*Proof.* Take  $s \in \Sigma(M)$  and choose  $x, y \in M$  with  $s = d_M(x, y)$ . Since  $d_{\text{H}}^M(\{x\}, \{y\}) = s$ , we have  $s \in \Sigma(\text{Exp}(M))$ . Conversely, take  $s \in \Sigma(\text{Exp}(M))$  and choose  $A, B \in \text{Exp}(M)$  with  $s = d_{\text{H}}^M(A, B)$ . The defining maxima and minima are attained because  $A$  and  $B$  are finite. Thus  $s = d_M(a, b)$  for some  $a \in A$  and  $b \in B$ , and hence  $s \in \Sigma(M)$ .  $\square$

**3.1. Exact reconstruction.** We recognize singleton subsets by counting balls in the hyperspace. Let  $X$  be a finite metric space. For  $A \in \text{Exp}(X)$  and  $r \geq 0$ , define the closed-ball profile

$$\beta_A^X(r) = \#\{B \in \text{Exp}(X) \mid d_{\text{H}}^X(A, B) \leq r\}.$$

This profile recognizes the singleton locus intrinsically.

**Theorem 3.2.** *Let  $X$  be a finite strongly rigid metric space with  $|X| \geq 3$ , and let  $A \in \text{Exp}(X)$ . Then  $A$  is a singleton if and only if  $\beta_A^X(r) + 1$  is a power of two for every  $r \geq 0$ .*

*Proof.* For a nonempty set  $S \subseteq X$  and  $r \geq 0$ , write

$$N_r(S) = \{x \in X \mid d_X(x, S) \leq r\}.$$

If  $A = \{a\}$ , then for every  $B \in \text{Exp}(X)$  we have

$$d_H^X(\{a\}, B) = \max_{b \in B} d_X(a, b).$$

Consequently, the sets  $B$  counted by  $\beta_{\{a\}}^X(r)$  are precisely the nonempty subsets of  $N_r(\{a\})$ , and hence

$$\beta_{\{a\}}^X(r) + 1 = 2^{|N_r(\{a\})|}.$$

Conversely, assume that  $|A| \geq 2$ . Among the positive distances having at least one endpoint in  $A$ , let  $R$  be the largest. Strong rigidity yields a unique pair  $\{u, v\}$  realizing  $R$ . There is at least one further distance having an endpoint in  $A$ , because  $|X| \geq 3$ . Let  $r < R$  be the next largest such distance. Every other distance with an endpoint in  $A$  is at most  $r$ .

Assume first that exactly one of  $u, v$  lies in  $A$ , say  $u \in A$ . Then  $N_r(A) = X$ . Among the nonempty subsets  $B \subseteq X$ , exactly  $B = \{v\}$  fails the other Hausdorff inclusion  $A \subseteq N_r(B)$ . Therefore

$$\beta_A^X(r) + 1 = 2^{|X|} - 1,$$

which is not a power of two. If both  $u$  and  $v$  lie in  $A$ , again  $N_r(A) = X$ , and exactly  $B = \{u\}$  and  $B = \{v\}$  fail  $A \subseteq N_r(B)$ . In this case

$$\beta_A^X(r) + 1 = 2^{|X|} - 2 = 2(2^{|X|-1} - 1),$$

which is not a power of two since  $|X| \geq 3$ . □

**Theorem 3.3.** *Let  $X, Y$  be nonempty finite strongly rigid metric spaces. If*

$$\text{Exp}(X) \cong \text{Exp}(Y),$$

*then  $X \cong Y$ .*

*Proof.* Hyperspace cardinality yields

$$2^{|X|} - 1 = |\text{Exp}(X)| = |\text{Exp}(Y)| = 2^{|Y|} - 1,$$

so  $|X| = |Y|$ . The one-point case is immediate. If the common cardinality is two, each hyperspace is an equilateral three-point space whose side length is the unique positive base distance, and the claim follows directly.

Assume now that the common cardinality is at least three, and let  $F : \text{Exp}(X) \rightarrow \text{Exp}(Y)$  be an isometry. It preserves every closed-ball profile. By theorem 3.2, it therefore bijects the singleton loci. Write for each  $x \in X$   $F(\{x\}) = \{f(x)\}$ . Then  $f : X \rightarrow Y$  is bijective. For all  $x, x_1 \in X$ , we have

$$d_Y(f(x), f(x_1)) = d_H^Y(F(\{x\}), F(\{x_1\})) = d_H^X(\{x\}, \{x_1\}) = d_X(x, x_1).$$

Thus  $f$  is an isometry. □

The preceding theorem is an injectivity statement on isometry classes. It does not assert that  $\text{Exp}$  preserves every ordinary Gromov–Hausdorff distance. section 5 yields strongly rigid pairs for which that stronger statement fails.

**3.2. Spectral-gap neighborhoods.** We next estimate the distortion under which singleton subsets are preserved. For a finite metric space  $M$  with at least two points, define

$$\eta(M) = \min\{|s - t| \mid s, t \in \Sigma(M), s \neq t\} > 0.$$

By theorem 3.1, this is also the minimum gap between distinct distance levels of  $\text{Exp}(M)$ .

**Theorem 3.4.** *Let  $X, Y$  be finite strongly rigid metric spaces with  $|X|, |Y| \geq 3$ , and put  $\eta = \min\{\eta(X), \eta(Y)\}$ . If*

$$d_{\text{GH}}(X, Y) < \frac{\eta}{4},$$

*then*

$$d_{\text{GH}}(\text{Exp}(X), \text{Exp}(Y)) = d_{\text{GH}}(X, Y).$$

*Proof.* Put

$$g = d_{\text{GH}}(X, Y), \quad h = d_{\text{GH}}(\text{Exp}(X), \text{Exp}(Y)).$$

By theorem 2.1,  $h \leq g < \eta/4$ . Since the hyperspaces are finite, there is an optimal correspondence  $R \subseteq \text{Exp}(X) \times \text{Exp}(Y)$  with

$$\delta = \text{dis}(R) = 2h < \frac{\eta}{2}.$$

If one source set were related to two distinct target sets, their positive Hausdorff distance would be at most  $\delta < \eta(Y)$ , contradicting the definition of  $\eta(Y)$ . The symmetric argument rules out two distinct source sets related to one target set. Thus  $R$  is the graph of a bijection  $F : \text{Exp}(X) \rightarrow \text{Exp}(Y)$ .

Fix a singleton  $A \in \text{Exp}(X)$ , and let  $r \in \Sigma(Y)$  be a nonmaximal distance level with successor  $r^+$ . For each  $C \in \text{Exp}(X)$ , if  $d_{\text{H}}^Y(F(A), F(C)) > r$ , then

$$d_{\text{H}}^Y(F(A), F(C)) \geq r^+ \geq r + \eta,$$

and the distortion bound yields

$$d_{\text{H}}^X(A, C) \geq r + \eta - \delta > r + \delta.$$

Together with the direct distortion implication, this shows that  $F$  satisfies

$$\begin{aligned} F(\{C \in \text{Exp}(X) \mid d_{\text{H}}^X(A, C) \leq r + \delta\}) \\ = \{D \in \text{Exp}(Y) \mid d_{\text{H}}^Y(F(A), D) \leq r\}. \end{aligned}$$

The same equality of cardinalities is automatic at the maximal level. Closed-ball counts are constant between consecutive spectrum levels, so the profile of  $F(A)$ , plus one, is a power of two at every radius. Hence theorem 3.2 shows that  $F(A)$  is a singleton. Applying the same argument to  $F^{-1}$  proves that  $F$  bijects the two singleton loci.

The restriction of  $R$  to these loci is therefore a base correspondence of distortion at most  $\delta$ . By (1),  $g \leq \delta/2 = h$ . The reverse inequality is nonexpansion.  $\square$

For fixed cardinality the separation can be imposed on only one of the two spaces. The proof uses a second reconstruction mechanism.

**Theorem 3.5.** *Let  $Y$  be a finite strongly rigid metric space with  $n$  points. If an  $n$ -point subspace*

$$\mathcal{A} = \{A_0, \dots, A_{n-1}\} \subseteq \text{Exp}(Y)$$

*has pairwise distinct distances on its  $\binom{n}{2}$  unordered pairs, then  $\mathcal{A} \cong Y$ .*

*Proof.* The case  $n = 1$  is immediate, so assume  $n \geq 2$ . By theorem 3.1, the  $\binom{n}{2}$  distinct positive distances in  $\mathcal{A}$  exhaust the positive distance spectrum of  $Y$ . Let  $D = \text{diam}(Y)$ . A pair in  $\mathcal{A}$  is at Hausdorff distance  $D$ . This distance is realized by a directed nearest-point term, so after interchanging the two sets there are  $p \in A$  and  $B \in \mathcal{A}$  such that  $d_Y(p, B) = D$ . Every point of  $B$  is then at distance  $D$  from  $p$ . Strong rigidity yields a unique diameter pair  $\{p, q\}$ , whence  $B = \{q\}$ . Relabel so that  $A_0 = \{q\}$ .

For  $i > 0$ , put

$$r_i = d_H^Y(\{q\}, A_i) = \max_{a \in A_i} d_Y(q, a).$$

These  $n - 1$  distinct values exhaust the distances from  $q$ . Relabel the remaining sets and write

$$Y = \{q, y_1, \dots, y_{n-1}\}, \quad r_1 < \dots < r_{n-1}, \quad r_i = d_Y(q, y_i).$$

Strong rigidity makes  $y_i$  the unique farthest point from  $q$  in  $A_i$ , and hence

$$y_i \in A_i, \quad A_i \subseteq \{q, y_1, \dots, y_i\}. \quad (3)$$

For  $0 \leq k < n$ , set

$$\mathcal{A}_k = \{A_0, \dots, A_k\}, \quad Y_k = \{q, y_1, \dots, y_k\}.$$

Every distance inside  $\mathcal{A}_k$  is realized by a base edge inside  $Y_k$ , by (3). Both sets of positive distances have  $\binom{k+1}{2}$  members, so

$$\{d_H^Y(A_i, A_j) \mid 0 \leq i < j \leq k\} = \{d_Y(u, v) \mid u, v \in Y_k, u \neq v\}.$$

Subtracting the equality for  $k - 1$  shows that, for a permutation  $\pi$  of  $\{1, \dots, k - 1\}$ ,

$$d_H^Y(A_k, A_j) = d_Y(y_k, y_{\pi(j)}) \quad (1 \leq j < k).$$

The Hausdorff distance on the left is realized by a base edge. Strong rigidity and  $y_k \notin A_j$  force  $y_{\pi(j)} \in A_j$ . The inclusion (3) yields  $\pi(j) \leq j$ , so  $\pi$  is the identity. For  $1 \leq i, j < n$ , induction on  $k$  now yields

$$d_H^Y(A_i, A_j) = d_Y(y_i, y_j)$$

The distances involving  $A_0$  agree with those from  $q$  by the definition of  $r_i$ . Thus  $\mathcal{A} \cong Y$ .  $\square$

**Theorem 3.6.** *Let  $X, Y$  be finite strongly rigid metric spaces of the same cardinality, at least two, and put*

$$h = d_{\text{GH}}(\text{Exp}(X), \text{Exp}(Y)).$$

*If  $4h < \eta(X)$ , then*

$$d_{\text{GH}}(X, Y) = h.$$

*The same conclusion holds with  $X$  and  $Y$  interchanged. In particular, the equality holds whenever*

$$d_{\text{GH}}(X, Y) < \frac{\eta(X)}{4}$$

*or under the analogous condition on  $Y$ .*

*Proof.* Let  $Q$  be an optimal hyperspace correspondence and put  $\delta = \text{dis}(Q) = 2h$ . For each  $x \in X$ , choose  $B_x \in \text{Exp}(Y)$  with  $(\{x\}, B_x) \in Q$ . The sets  $B_x$  are distinct. For the sake of contradiction, suppose that  $B_x = B_{x_1}$  for  $x \neq x_1$ . Then  $d_X(x, x_1) \leq \delta < \eta(X)$ , a contradiction.

If two distinct unordered source pairs determined the same distance between the corresponding sets  $B_x$ , the distortion inequalities would yield

$$|d_X(x, x_1) - d_X(u, u_1)| \leq 2\delta = 4h < \eta(X),$$

contradicting strong rigidity and the definition of  $\eta(X)$ . Hence  $\{B_x \mid x \in X\}$  satisfies the hypotheses of theorem 3.5. By theorem 3.5, it is isometric to  $Y$ . Composing this isometry with  $x \mapsto B_x$  yields a bijection  $f : X \rightarrow Y$  of distortion at most  $\delta$ . Thus

$$d_{\text{GH}}(X, Y) \leq \frac{\delta}{2} = h.$$

The reverse inequality is theorem 2.1. Finally, if  $d_{\text{GH}}(X, Y) < \eta(X)/4$ , nonexpansion yields  $4h < \eta(X)$ .  $\square$

Thus, on every fixed-cardinality stratum of finite strongly rigid spaces,  $\text{Exp}$  preserves the distance from each point throughout an explicit spectral-gap neighborhood. Together with theorem 3.3, this yields local injectivity on that stratum. The quantitative assertion is stronger in that no distance from the chosen center is shortened. This pointwise local statement should not be confused with unrestricted global distance preservation, which is false.

#### 4. THE NON-ARCHIMEDEAN THEOREM

The preceding section used distinct distance values to recover the base. For ultrametric spaces, we instead use the partition at the largest distance and the products it induces in the hyperspace. Let  $U$  be a finite ultrametric space of positive diameter  $D$ , and let

$$U = U_1 \sqcup \cdots \sqcup U_k$$

be its partition into open balls of radius  $D$ . Two points belong to the same block exactly when their distance is strictly smaller than  $D$ . Distinct blocks are at distance  $D$ . For every nonempty  $S \subseteq \{1, \dots, k\}$ , set

$$E_S(U) = \{A \in \text{Exp}(U) \mid A \cap U_i \neq \emptyset \iff i \in S\}.$$

These are precisely the open  $D$ -blocks of  $\text{Exp}(U)$ . If  $\boxtimes$  denotes the Cartesian product with the maximum metric, then

$$E_S(U) \cong \boxtimes_{i \in S} \text{Exp}(U_i). \quad (4)$$

Indeed, the isometry sends  $A$  to  $(A \cap U_i)_{i \in S}$ . For two sets with the same support, a nearest point can always be chosen in the same top block, and the Hausdorff distance is the maximum of the resulting coordinate distances.

The following elementary algebraic lemma allows us to invert the multiset of all nonempty products in (4) without assuming unique factorization.

**Lemma 4.1.** *Let  $\mathcal{M}$  be a commutative monoid with identity 1, and assume that there is a map*

$$\deg : \mathcal{M} \rightarrow \mathbb{N}_{\geq 1}$$

*such that, for every  $a, b \in \mathcal{M}$ ,*

$$\deg(ab) = \deg(a) \deg(b), \quad \deg(a) = 1 \iff a = 1.$$

*If finite multisets  $\{a_1, \dots, a_k\}$  and  $\{b_1, \dots, b_\ell\}$  satisfy*

$$\prod_{i=1}^k (1 + a_i) = \prod_{j=1}^{\ell} (1 + b_j) \quad \text{in } \mathbb{N}[\mathcal{M}],$$

*then the two multisets are equal.*

*Proof.* Let  $s$  be the number of factors  $a_i$  equal to 1. The coefficient of the identity monomial on the left is  $2^s$ , because no product containing a nonidentity factor can have degree one. The same coefficient on the right therefore recovers the same number of identity factors. Cancel the common scalar  $2^s$ .

Let  $d > 1$  be the least degree of a remaining factor. For every  $m \in \mathcal{M}$  of degree  $d$ , the coefficient of  $m$  in the product is exactly the multiplicity with which  $m$  occurs as a factor. A product of two or more remaining factors has degree at least  $d^2 > d$ . Hence the two sides have the same multiset of degree- $d$  factors.

These common factors may be cancelled even when  $\mathcal{M}$  is not a cancellative monoid. To see this, let  $F \in \mathbb{N}[\mathcal{M}]$  have identity coefficient one and all other monomials of degree greater than one. Let  $G, H \in \mathbb{N}[\mathcal{M}]$  satisfy  $FG = FH$ . For the sake of contradiction, suppose that  $G \neq H$ . Pass to  $\mathbb{Z}[\mathcal{M}]$  and choose the least degree  $n$  at which  $G - H$  has a nonzero coefficient. Choose a monomial  $m$  of degree  $n$  whose coefficient in  $G - H$  is nonzero. Then every contribution to the coefficient of  $m$  coming from a nonidentity monomial of  $F$  uses a coefficient of  $G - H$  of degree strictly smaller than  $n$ . Those

coefficients vanish, so the coefficient of  $m$  in  $F(G - H)$  equals its nonzero coefficient in  $G - H$ , a contradiction. Thus multiplication by  $F$  is injective.

Cancel all degree- $d$  factors and repeat. Finiteness terminates the procedure and recovers the complete factor multiset.  $\square$

**Theorem 4.2.** *Let  $U, V$  be nonempty finite ultrametric spaces. If*

$$\text{Exp}(U) \cong \text{Exp}(V),$$

*then  $U \cong V$ .*

*Proof.* We argue by induction on the common cardinality of  $U$  and  $V$ , which is determined by  $|\text{Exp}(U)| = 2^{|U|} - 1$ . The one-point case is immediate. Otherwise,

$$\text{diam}(U) = \text{diam}(\text{Exp}(U)) = \text{diam}(\text{Exp}(V)) = \text{diam}(V) = D > 0.$$

Write the open  $D$ -partitions as

$$U = U_1 \sqcup \cdots \sqcup U_k, \quad V = V_1 \sqcup \cdots \sqcup V_\ell.$$

An isometry of the hyperspaces preserves their open  $D$ -blocks and the isometry type of each block. By (4), equality of these block multisets, after adjoining the empty-support term, is

$$\prod_{i=1}^k (1 + [\text{Exp}(U_i)]) = \prod_{j=1}^{\ell} (1 + [\text{Exp}(V_j)]). \quad (5)$$

In (5), brackets denote isometry classes in the commutative monoid of nonempty finite ultrametric spaces under  $\boxtimes$ . Equip this monoid with the degree  $\deg[W] = |W|$ . Its identity is the one-point space, so the hypotheses of theorem 4.1 hold.

The lemma applied to (5) matches the factors  $\text{Exp}(U_i)$  and  $\text{Exp}(V_j)$  up to isometry. Every top block has smaller cardinality than its ambient space. The induction hypothesis therefore yields a matching isometry  $U_i \cong V_j$  for every factor. Their union is an isometry  $U \cong V$ , since distances between distinct top blocks equal  $D$ .  $\square$

We next compare the closed quotients of  $U$  and  $\text{Exp}(U)$ . For a finite ultrametric space  $U$  and  $t \geq 0$ , let  $q_t : U \rightarrow U_{c(t)}$  be the quotient map.

**Proposition 4.3.** *Let  $U$  be a finite ultrametric space and let  $t \geq 0$ . Then the map*

$$Q_t : (\text{Exp}(U))_{c(t)} \rightarrow \text{Exp}(U_{c(t)}), \quad [A] \mapsto q_t(A),$$

*is a well-defined isometry.*

*Proof.* For nonempty subsets  $A, B \subseteq U$ , we have

$$d_H^U(A, B) \leq t \iff q_t(A) = q_t(B). \quad (6)$$

The Hausdorff bound says exactly that each closed  $t$ -class met by one set is met by the other. Thus  $Q_t$  is well-defined and injective. It is surjective

because representatives may be chosen from the members of any nonempty subset of the finite quotient.

If  $q_t(A) \neq q_t(B)$ , at least one quotient class is met on only one side. Its Hausdorff contribution is greater than  $t$ , while the contribution of a class met on both sides is at most  $t$ . Distances between distinct classes are their quotient distances. Taking the maximum yields

$$d_H^U(A, B) = d_H^{U_{c(t)}}(q_t(A), q_t(B)).$$

Together with (6), this proves that  $Q_t$  is an isometry.  $\square$

**Theorem 4.4.** *Let  $U, V$  be finite ultrametric spaces. Then*

$$u_{GH}(\text{Exp}(U), \text{Exp}(V)) = u_{GH}(U, V).$$

*Consequently,  $\text{Exp}$  is an isometric embedding on the isometry classes of nonempty finite ultrametric spaces equipped with  $u_{GH}$ .*

*Proof.* The structural formula (2), theorem 4.3, and theorem 4.2 yield

$$\begin{aligned} u_{GH}(\text{Exp}(U), \text{Exp}(V)) &= \min \{t \geq 0 \mid (\text{Exp}(U))_{c(t)} \cong (\text{Exp}(V))_{c(t)}\} \\ &= \min \{t \geq 0 \mid \text{Exp}(U_{c(t)}) \cong \text{Exp}(V_{c(t)})\} \\ &= \min \{t \geq 0 \mid U_{c(t)} \cong V_{c(t)}\} \\ &= u_{GH}(U, V). \end{aligned}$$

The reconstruction theorem also shows directly that the map on isometry classes is injective.  $\square$

*Remark 4.5.* The theorem concerns the Gromov–Hausdorff ultrametric  $u_{GH}$ . It makes no claim that the ordinary distance  $d_{GH}$  is preserved on ultrametric bases. The next section shows that this stronger-looking interpretation is false.

## 5. STRICT CONTRACTION FOR THE ORDINARY DISTANCE

We now show that theorem 4.4 has no analogue for the ordinary Gromov–Hausdorff distance. The first example is ultrametric and has only two positive distance levels in each space.

**5.1. An exact ultrametric counterexample.** Let

$$X_0 = P_1 \sqcup P_2 \sqcup P_3, \quad |P_1| = |P_2| = |P_3| = 2.$$

Equip  $X_0$  with the ultrametric whose distance is 2 between distinct points of the same  $P_i$  and 10 between points in different components. Let

$$Y_0 = Q_1 \sqcup Q_2 \sqcup Q_3, \quad (|Q_1|, |Q_2|, |Q_3|) = (1, 1, 3).$$

The three distinct points of  $Q_3$  are at distance 3, and points in different  $Q_j$ 's are at distance 10.

**Theorem 5.1.** *The spaces  $X_0, Y_0$  defined above satisfy*

$$2d_{\text{GH}}(X_0, Y_0) = 3, \quad 2d_{\text{GH}}(\text{Exp}(X_0), \text{Exp}(Y_0)) = 2.$$

*In particular,*

$$d_{\text{GH}}(\text{Exp}(X_0), \text{Exp}(Y_0)) < d_{\text{GH}}(X_0, Y_0).$$

*Proof.* We first compute the base distance. Collapse  $P_1$  and  $P_2$  onto the singleton components  $Q_1$  and  $Q_2$ . If  $P_3 = \{a, b\}$  and  $Q_3 = \{u, v, w\}$ , use the component relation

$$\{(a, u), (a, v), (b, w)\}.$$

The first two component relations have distortion 2. The third has distortion 3, attained by the two relation edges with source  $a$ , and comparisons between different components compare 10 with 10. Thus there is a base correspondence of distortion 3.

For the sake of contradiction, suppose that a base correspondence  $R$  had distortion strictly less than 3. Any two targets related to source points in one  $P_i$  would have distance less than

$$2 + 3 = 5 < 10.$$

They must therefore lie in one target component  $Q_j$ . Coverage of  $Y_0$  induces a surjection from the three source components to the three target components, hence a bijection. One two-point component  $P_i$  must then cover all three points of  $Q_3$ . Some source point is related to two distinct points of  $Q_3$ , producing distortion 3, a contradiction. By (1), this proves  $2d_{\text{GH}}(X_0, Y_0) = 3$ .

It remains to compute the hyperspace distance. For every nonempty  $S \subseteq \{1, 2, 3\}$ , let  $E_S^X$  be the family of subsets of  $X_0$  whose set of met components is exactly  $S$ . Each  $\text{Exp}(P_i)$  is a three-point equilateral space of side 2, and

$$E_S^X \cong \boxtimes_{i \in S} \text{Exp}(P_i).$$

Thus  $E_S^X$  is equilateral of side 2, and the seven source-block cardinalities are

$$(3, 3, 3, 9, 9, 9, 27). \tag{7}$$

On the target side, the three support blocks not meeting  $Q_3$  are singletons. Each of the other four blocks is isometric to the seven-point equilateral space  $\text{Exp}(Q_3)$  of side 3. The target-block cardinalities are therefore

$$(1, 1, 1, 7, 7, 7, 7). \tag{8}$$

Pair the three size- 3 source blocks with the three singleton target blocks. Pair the remaining source blocks, of sizes 9, 9, 9, 27, with the four size- 7 target blocks. In each pair take the graph of a surjection from the source block onto the target block. A size- 3 block collapsed onto a singleton has distortion 2. For the other pairs, two distinct source points with the same image cost 2, while two source points with distinct images cost  $|2 - 3| = 1$ . Every block relation consequently has distortion at most 2.

Different support blocks are at Hausdorff distance 10 on both sides. The union of the seven block relations is therefore a correspondence  $\mathcal{Q}$  with  $\text{dis}(\mathcal{Q}) \leq 2$ .

For the reverse bound, the source and target hyperspaces have cardinalities 63 and 31, respectively, and the minimum positive distance in  $\text{Exp}(X_0)$  is 2. Given any hyperspace correspondence, choose one target partner for each of the 63 source points. Two distinct source points share a target partner, so those two relation edges have distortion at least 2. Hence every hyperspace correspondence has distortion at least 2, completing the proof.  $\square$

*Remark 5.2.* This example emphasizes the distinction between the two ambient geometries. The closed quotients of  $X_0$  and  $Y_0$  first become isometric at  $t = 3$ , so  $u_{\text{GH}}(X_0, Y_0) = 3$ . By theorem 4.4,

$$u_{\text{GH}}(\text{Exp}(X_0), \text{Exp}(Y_0)) = 3$$

even though their ordinary Gromov–Hausdorff distance drops from  $3/2$  to 1.

**5.2. Scale-quotient capacity.** The previous proof compares the cardinalities of entire support blocks. For multiscale targets, the exact invariant at a prescribed distortion is a quotient cardinality. If  $Z$  is a finite ultrametric space and  $a \geq 0$ , write

$$c_a(Z) = \#(Z/(d \leq a)).$$

The relation  $d \leq a$  is an equivalence relation by the ultrametric inequality.

**Lemma 5.3.** *Let  $P$  be a positive integer and let  $E_P(a)$  be the  $P$ -point equilateral space of side  $a > 0$ , and let  $Z$  be a nonempty finite ultrametric space with  $\text{diam}(Z) \leq 2a$ . Then there is a correspondence of distortion at most  $a$  between  $E_P(a)$  and  $Z$  if and only if*

$$P \geq c_a(Z).$$

*Proof.* In a correspondence of distortion at most  $a$ , all target partners of one source point have mutual distance at most  $a$ . They therefore lie in one closed  $a$ -class. Covering all classes requires  $P \geq c_a(Z)$ .

Conversely, map the  $P$  source points onto the closed  $a$ -classes of  $Z$ , and relate each source point to every point in its assigned class. Target distances within a class are at most  $a$ , while target distances between distinct classes belong to  $(a, 2a]$ . Comparison with source distances 0 and  $a$  yields distortion at most  $a$ .  $\square$

We apply the lemma support by support. Fix  $a > 0$  and a positive integer  $k$ . For each  $1 \leq i \leq k$ , let  $p_i$  be a positive integer. Let  $X$  have  $k$  top components

$$X_i = E_{p_i}(a),$$

at common mutual distance  $H$ . Let  $Y$  have  $k$  nonempty finite-ultrametric top components  $Y_j$ , also at mutual distance  $H$ , and assume

$$\max_{1 \leq j \leq k} \text{diam}(Y_j) \leq 2a, \quad H > 2a.$$

For nonempty supports  $S, T \subseteq \{1, \dots, k\}$ , put

$$P_S = \prod_{i \in S} (2^{p_i} - 1), \quad Q_T(a) = \prod_{j \in T} (2^{c_a(Y_j)} - 1).$$

**Theorem 5.4.** *Let  $X, Y, a, H, P_S, Q_T(a)$  be as defined above. Then there is a correspondence of distortion at most  $a$  between  $\text{Exp}(X)$  and  $\text{Exp}(Y)$  if and only if the nonempty supports admit a bijection  $\phi$  such that, for every nonempty  $S \subseteq \{1, \dots, k\}$ ,*

$$P_S \geq Q_{\phi(S)}(a).$$

*Proof.* The source support block indexed by  $S$  is the  $P_S$ -point equilateral space of side  $a$ . The target support block indexed by  $T$  is the maximum product of the spaces  $\text{Exp}(Y_j)$ ,  $j \in T$ . It has diameter at most  $2a$ .

For a finite ultrametric space  $Z$  and a nonempty subset  $A \subseteq Z$ , its closed  $a$ -class in  $\text{Exp}(Z)$  is determined by the collection of closed  $a$ -classes of  $Z$  met by  $A$ . Hence

$$c_a(\text{Exp}(Z)) = 2^{c_a(Z)} - 1.$$

Closed quotient cardinalities multiply under maximum products, so the target support block has  $Q_T(a)$  closed  $a$ -classes.

A source support block cannot meet two target support blocks under a distortion-  $a$  correspondence. Its internal diameter is at most  $a$ , whereas distinct target supports are at distance  $H > 2a$ . Coverage induces a surjection on the two sets of  $2^k - 1$  nonempty supports, hence a bijection. Restriction to a paired block and theorem 5.3 yield the required inequalities.

Conversely, use theorem 5.3 on each paired pair of support blocks and take the union of the resulting correspondences. Distances between distinct support blocks are  $H$  on both sides.  $\square$

**5.3. Equal-cardinality counterexamples.** The quotient criterion permits low-scale target fibers without charging their full cardinality. This removes the cardinality imbalance in the first counterexample.

**Theorem 5.5.** *Let  $s, r$  be integers with  $2 \leq s \leq r$ , put  $n = r + s - 2$ , and assume*

$$0 < a < b \leq 2a, \quad H \geq 2b.$$

*Let  $X$  have three top components of sizes*

$$(s, r, n),$$

*all equilateral of side  $a$ . Let  $Y$  have top-component sizes*

$$(s, r - 1, n + 1),$$

*where the first two components are equilateral of side  $a$  and the last is equilateral of side  $b$ . Put distance  $H$  between distinct top components on both sides. Then  $|X| = |Y|$  and*

$$2d_{\text{GH}}(X, Y) = b, \quad 2d_{\text{GH}}(\text{Exp}(X), \text{Exp}(Y)) = a.$$

*Proof.* For every positive integer  $j$ , write  $A_j = 2^j - 1$ . A correspondence of distortion below  $b$  would induce a bijection of the three top components. The target component of size  $n + 1$  would then have to be covered by a source component of size at most  $n$ . Some source point would cover two of its target points, incurring distortion  $b$ . The reverse inequality follows by taking full relations between paired top components. Thus the base doubled distance is  $b$ .

For the hyperspaces, define the following bijection from source supports to target supports.

$$\begin{aligned} \{1\} &\mapsto \{1\}, & \{2\} &\mapsto \{2\}, & \{3\} &\mapsto \{1, 2\}, & \{1, 2\} &\mapsto \{3\}, \\ \{1, 3\} &\mapsto \{1, 3\}, & \{2, 3\} &\mapsto \{2, 3\}, & \{1, 2, 3\} &\mapsto \{1, 2, 3\}. \end{aligned}$$

The target blocks avoiding the third component have closed  $a$ -quotient cardinality one. Every target block meeting the third component has closed  $a$ -quotient cardinality  $A_{n+1}$ . The only nonautomatic source-capacity inequality is

$$A_s A_r \geq A_{n+1} = A_{r+s-1},$$

which follows from

$$A_s A_r - A_{r+s-1} = 2^{r+s-1} - 2^s - 2^r + 2 > 0.$$

The remaining source blocks mapped to supports meeting the third target component contain a factor  $A_n$  and at least one factor  $A_s$  or  $A_r$ . Since  $s, r \geq 2$ , their capacities are at least  $A_{n+1}$ . Theorem 5.4 yields hyperspace distortion at most  $a$ .

For the reverse inequality, the two hyperspaces have the same cardinality and minimum positive distance  $a$ . A correspondence of distortion below  $a$  would therefore be a bijection. It would preserve the seven  $H$ -support blocks and their cardinalities. Equality of the two support-block cardinality multisets would, by theorem 4.1, force equality of the top-component cardinality multisets. This is impossible because the largest target component has size  $n + 1$ , whereas every source component has size at most  $n$ .  $\square$

The minimal choice  $s = r = 2$  yields two six-point spaces.

$$X = E_2(a) \sqcup E_2(a) \sqcup E_2(a), \quad Y = E_2(a) \sqcup E_1(a) \sqcup E_3(b).$$

The components in these disjoint unions are at mutual distance  $H$ . Appending the same number of singleton top components to both spaces preserves the proof. Hence every cardinality at least six admits examples with equal cardinalities, and varying  $a/b$  realizes every ratio in  $[1/2, 1)$ . A strongly rigid example with strict triangle inequalities is constructed in the last subsection.

**5.4. Iteration and the half-ratio boundary.** The spaces in theorem 5.5 retain their first hyperspace distance under further iteration.

**Proposition 5.6.** *Let  $X, Y$  be the spaces in theorem 5.5. For each integer  $m \geq 0$ , put*

$$D_m = 2d_{\text{GH}}(\text{Exp}^m(X), \text{Exp}^m(Y)).$$

Then

$$D_0 = b, \quad D_m = a \quad (m \geq 1).$$

*Proof.* The upper bound  $D_m \leq a$  follows from the first-iterate construction and nonexpansion. The two iterated spaces have equal cardinality and minimum positive distance  $a$ . Thus a correspondence of distortion below  $a$  would be a bijection preserving their top support blocks.

For a vector  $\mathbf{v} = (v_1, \dots, v_k)$  of top-component cardinalities, the top-component cardinality multiset of its hyperspace is

$$F(\mathbf{v}) = \left\{ \prod_{i \in S} (2^{v_i} - 1) \mid \emptyset \neq S \subseteq \{1, \dots, k\} \right\}.$$

The graded subset-product inversion lemma says that  $F$  is injective. The initial top-size vectors of  $X$  and  $Y$  differ, so their  $F^m$ -images differ for every  $m$ . A block-preserving bijection is therefore impossible, proving  $D_m \geq a$ .  $\square$

Finally, the threshold  $1/2$  has a local explanation. Assume that a target ultrametric space  $V$  has diameter  $r > \delta$ , with open root blocks  $V_1, \dots, V_k$ . A distortion- $\delta$  correspondence between  $U$  and  $V$  partitions  $U$  into nonempty pieces

$$U = U_1 \sqcup \dots \sqcup U_k$$

such that each  $U_i$  corresponds to  $V_i$  at distortion  $\delta$ , and every distance between  $U_i$  and  $U_j$ ,  $i \neq j$ , belongs to  $[r - \delta, r + \delta]$ . Indeed, one source point cannot meet two target root blocks, because their mutual distance is  $r > \delta$ .

**Proposition 5.7.** *Let  $U, V$  be finite metric spaces with  $V$  ultrametric,  $\text{diam}(U) \leq \delta$ , and  $r = \text{diam}(V) > \delta$ . If there is a correspondence between  $U$  and  $V$  of distortion at most  $\delta$ , then*

$$\delta < r \leq 2\delta.$$

*Consequently, the single-scale capacity-amplification mechanism used above cannot produce a contraction ratio below  $1/2$ .*

*Proof.* If the source piece covers two target root blocks, two of its points compare a source distance at most  $\delta$  with target distance  $r$ . The distortion is at least  $r - \delta$ , which exceeds  $\delta$  when  $r > 2\delta$ .  $\square$

The restriction in theorem 5.7 concerns a source piece of diameter at most  $\delta$ . It does not impose a lower bound on the contraction ratio for arbitrary finite ultrametric spaces.

**5.5. A strongly rigid perturbation.** We now perturb the spaces  $X_0, Y_0$  from theorem 5.1 so that both spaces are strongly rigid and every triangle inequality is strict.

Label the source points by  $0, \dots, 5$ . In lexicographic edge order

$$01, 02, 03, 04, 05, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45,$$

define the source distances to be

$$(2.001, 10.002, 10.003, 10.004, 10.005, 10.006, 10.007, 10.008, 10.009, 2.010, 10.011, 10.012, 10.013, 10.014, 2.015). \quad (9)$$

Label the target points by  $0, \dots, 4$ . In edge order

$$01, 02, 03, 04, 12, 13, 14, 23, 24, 34,$$

define the target distances to be

$$(10.001, 10.002, 10.003, 10.004, 10.005, 10.006, 10.007, 3.008, 3.009, 3.010). \quad (10)$$

**Theorem 5.8.** *Let  $X, Y$  be the spaces defined by (9) and (10). Then  $X$  and  $Y$  are strongly rigid and have only strict triangle inequalities. Moreover,*

$$d_{\text{GH}}(X, Y) = \frac{188}{125} = 1.504, \quad d_{\text{GH}}(\text{Exp}(X), \text{Exp}(Y)) \leq \frac{403}{400} = 1.0075.$$

*Proof.* All positive distances in either displayed list are distinct. To check the triangle inequalities for  $X$ , regard  $\{0, 1\}, \{2, 3\}, \{4, 5\}$  as its three displayed components. A triangle containing two points in one component has one edge at least 2.001 and two cross edges whose difference is at most 0.012. A triangle meeting all three components has all edges between 10.002 and 10.014. Every such triangle is strict. The verification for  $Y$  is similar. The component  $\{2, 3, 4\}$  has edge lengths 3.008, 3.009, 3.010, and all cross edges lie between 10.001 and 10.007. Hence both spaces are strongly rigid metrics in general position.

We next compute the base distance. For the sake of contradiction, suppose that a correspondence has distortion less than 3.008, all targets related to one of the source pairs  $\{0, 1\}, \{2, 3\}, \{4, 5\}$  would lie in a single target component, since

$$2.015 + 3.008 < 10.001.$$

Coverage again induces a bijection between the three displayed source and target components. One source pair must cover the three-point target component  $\{2, 3, 4\}$ . Repetition of a source point then incurs at least the target separation 3.008. Thus every base correspondence has distortion at least 3.008.

Equality is attained by the relation

$$\{(0, 0), (1, 0), (2, 1), (3, 1), (4, 2), (4, 3), (5, 4)\}.$$

The two edges with source 4 and targets 2, 3 incur distortion 3.008. The two collapsed pairs cost 2.001 and 2.010, the other within-component errors are smaller than one, and every cross-component error is at most 0.008. Therefore the exact base distortion is 3.008, and (1) yields

$$d_{\text{GH}}(X, Y) = \frac{3.008}{2} = 1.504.$$

For the hyperspace upper bound, use the same seven support-block pairing as in the proof of theorem 5.1. The three source blocks of cardinality 3

collapse onto the three singleton target blocks. Their diameters are at most 2.015. Each remaining source block has cardinality at least 9, diameter at most 2.015, and minimum positive distance at least 2.001. Each remaining target block has cardinality 7, diameter at most 3.010, and minimum positive distance at least 3.008. The graph of any surjection between a paired pair of these blocks has distortion at most

$$\max\{2.015, 3.010 - 2.001\} = 2.015.$$

For points in distinct support blocks, the source Hausdorff distance lies in  $[10.002, 10.014]$ , while the target Hausdorff distance lies in  $[10.001, 10.007]$ . The cross-block error is therefore at most 0.013. The union of the seven block correspondences has distortion at most 2.015, and hence

$$d_{\text{GH}}(\text{Exp}(X), \text{Exp}(Y)) \leq \frac{2.015}{2} = 1.0075.$$

This is strictly smaller than  $d_{\text{GH}}(X, Y)$ .  $\square$

The examples settle the unrestricted ordinary-distance isometry question negatively. They do not contradict hyperspace injectivity. Strict shortening of a positive distance does not identify two isometry classes, and theorem 5.5 shows that unequal base cardinality is not the source of the failure. What survives for  $d_{\text{GH}}$  is the general nonexpansion in theorem 2.1, equality on special pairs, and the open possibility of a coarse universal reverse inequality.

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