The support conjecture for rank-two quantum triangular bases

Qiyue Tang

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Abstract

We prove Conjecture 11 of Lee, Li, Rupel, and Zelevinsky on the support of triangular basis elements in rank-two quantum cluster algebras.

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1 Introduction

The triangular basis of a quantum cluster algebra is characterized by bar invariance and triangularity with respect to a standard monomial basis. These conditions determine its elements algebraically, but do not directly describe their Laurent expansions. In rank two, a concrete form of this problem asks which lattice points occur in the expansion of a basis element and how its coefficient degrees depend on those points.

Berenstein and Zelevinsky constructed triangular bases for acyclic quantum cluster algebras and proved their seed independence in rank two [1]. Lee, Li, Rupel, and Zelevinsky formulated an explicit support conjecture in terms of a quadratic inequality [3, Conjecture 11]. Li proved that conjecture when the two exchange parameters are equal, using Nakajima's quiver varieties [4, Theorem 1.2]. The equal parameter case admits a geometric coefficient formula in which dimensions of graded multiplicity spaces supply the Laurent coefficients. Unequal parameters require two different degree scales, and the geometric construction must account for both.

Fix positive integers b, c and an invertible formal parameter $\mathbf{v}$. We work over $\mathbb{Z}[\mathbf{v}^{\pm 1}]$. Let $\mathcal{A}(b,c)$ be the coefficient-free rank-two quantum cluster algebra with initial quantum torus

$$X_2X_1 = \mathbf{v}^2 X_1X_2, \qquad X^{(i,j)} = \mathbf{v}^{ij} X_1^i X_2^j.$$

Here $i, j \in \mathbb{Z}$. Its cluster variables X_k, indexed by $k \in \mathbb{Z}$, satisfy

$$X_{k+1}X_{k-1} = 1 + \mathbf{v}^{d_k} X_k^{d_k}, \qquad d_k = \begin{cases} b, & k \text{ odd}, \\ c, & k \text{ even}. \end{cases}$$

The bar involution, denoted by an overline, fixes every normalized monomial $X^{(i,j)}$ and sends $\mathbf{v}$ to $\mathbf{v}^{-1}$; on products it reverses the order of the factors. For integers A, B, write $C[A,B]$ for the triangular basis element pointed at (A,B), so that

$$C[A,B] = \sum_{p,q\geq 0} e(p,q) X^{(bp-A,cq-B)}, \qquad e(0,0) = 1. \tag{1.1}$$

The sum runs over nonnegative integers p, q, is finite, and has $e(p,q) \in \mathbb{Z}[\mathbf{v}^{\pm 1}]$. Its support is the set of pairs (p,q) for which $e(p,q) \neq 0$. Here p, q index the normalized torus monomials,

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whereas each $e(p,q)$ is itself a Laurent polynomial in the formal variable $\mathbf{v}$. For a finite Laurent polynomial $f(\mathbf{v}) = \sum_{k\in\mathbb{Z}} a_k \mathbf{v}^k$, with $a_k \in \mathbb{Z}$, and an integer d, we write

$$[\mathbf{v}^d]f(\mathbf{v}) := a_d, \qquad \deg_{\mathbf{v}} f := \max\{k \in \mathbb{Z} : a_k \neq 0\} \quad (f \neq 0).$$

Thus $[\mathbf{v}^d]$ extracts the coefficient of $\mathbf{v}^d$; it returns zero if that power is absent. The degree is the largest exponent with nonzero coefficient, including when negative powers occur, and we set $\deg_{\mathbf{v}} 0 = -\infty$. For example, if $f(\mathbf{v}) = \mathbf{v}^2 + 3 + \mathbf{v}^{-2}$, then

$$[\mathbf{v}^2]f = 1, \qquad [\mathbf{v}^0]f = 3, \qquad [\mathbf{v}^1]f = 0, \qquad \deg_{\mathbf{v}} f = 2.$$

We use the same conventions for Laurent polynomials in an independent formal variable z, and omit the degree subscript when its variable is determined by the polynomial.

Theorem 1.1. *Let b, c, A, B be positive integers satisfying*

$$cA^2 - bcAB + bB^2 \leq 0.$$

For the coefficients in (1.1), set

$$D(p,q) = cAq + bBp - bp^2 - bcpq - cq^2.$$

Then

$$e(p,q) \neq 0 \quad \Longleftrightarrow \quad 0 \leq p \leq B, \quad 0 \leq q \leq A, \quad D(p,q) \geq 0.$$

At every point in this support,

$$\deg_{\mathbf{v}} e(p,q) = D(p,q), \qquad [\mathbf{v}^{D(p,q)}]e(p,q) = 1.$$

The coefficients are bar invariant. In particular, $e(p,q) = 1$ whenever $D(p,q) = 0$ in the stated rectangle.

To produce the two degree scales, put $g = \gcd(b,c)$, $m = c/g$, and $n = b/g$. We use a complete bipartite quiver with m sinks, n sources, and g arrows between each source and sink. An automorphism cycles the sinks and the sources. Its graded trace on Grassmannian cohomology replaces a parameter z by z^m and z^n, respectively; setting $z = \mathbf{v}^g$ gives precisely $\mathbf{v}^c$ and $\mathbf{v}^b$.

An equivariant family of transverse slices over a connected space of choices identifies the multiplicity representations. Their traces give the Laurent coefficients through triangular-basis uniqueness. Three intersection-cohomology bounds and a residual support restriction then reduce the leading-degree calculation to a convexity inequality. We use the rank-stratum and constant-local-system results of [5].

Use of generative AI

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2 Preliminaries

2.1 The triangular basis and the bipartite quiver

For a real number a, write $[a]_+ = \max(a, 0)$. The standard monomials in the initial seed are

$$M[A,B] = \mathbf{v}^{AB} X_3^{[A]_+} X_1^{[-A]_+} X_2^{[-B]_+} X_0^{[B]_+}.$$

The triangular basis is characterized by bar invariance and

$$C[A,B] - M[A,B] \in \mathcal{L} := \bigoplus_{i,j\in\mathbb{Z}} \mathbf{v}\mathbb{Z}[\mathbf{v}]M[i,j]. \tag{2.1}$$

We use this global positive-lattice form of the characterization [3, §4, P1–P2], as well as the seed independence [1, Theorem 1.6]. The standard triangular order implies that $C[A,B]$ is pointed at (A,B). Its expansion lies in the single exponent class $(-A,-B)+(b\mathbb{Z}\oplus c\mathbb{Z})$: the torus has this grading, the bar preserves it, and uniqueness in (2.1) preserves the class of the standard monomial.

For integers $N \geq 0$ and $0 \leq k \leq N$, let

$$\begin{bmatrix} N \\ k \end{bmatrix}_z = \prod_{i=1}^{k} \frac{z^{N-i+1} - z^{-(N-i+1)}}{z^i - z^{-i}}$$

be the symmetric Gaussian binomial coefficient. Its degree is $k(N-k)$, and its highest coefficient is one. We set it equal to zero outside this range; the empty product for $k = 0$ is one.

Fix

$$g = \gcd(b,c), \qquad m = c/g, \qquad n = b/g. \tag{2.2}$$

Let Q have m sinks and n sources, with g arrows from each source to each sink. A sink is incident with b arrows and a source with c arrows; arrows are counted with multiplicity in all direct sums below. Let σ simultaneously cycle the sinks and the sources, preserving the labels of the parallel arrows. Since m, n are coprime, its order is mn. Its invariant dimension vectors are exactly the vectors constant on each side.

2.2 Rank varieties and their resolutions

For a general bipartite quiver, at each vertex choose complex vector spaces W_i, W_i', of dimensions w_i, w_i'. Write $w = ((w_i, w_i'))_i$ for these framing dimensions. All dimensions of varieties and framing spaces are over $\mathbb{C}$. For a sink α, a source β, and an arrow $h : \beta \to \alpha$, consider

$$x_\alpha : W_\alpha' \longrightarrow W_\alpha, \qquad x_\beta : W_\beta \longrightarrow W_\beta', \qquad y_h : W_\beta \longrightarrow W_\alpha.$$

Their affine parameter space is denoted by E_w. Let A_α be the horizontal concatenation of x_α and the incoming y_h, and let A_β be the vertical concatenation of x_β and the outgoing y_h. For nonnegative integer dimension vectors $\nu = (\nu_i)_i$ and $u = (u_i)_i$, define

$$E_{\nu,w} = \{p \in E_w : \operatorname{rank} A_i(p) \leq \nu_i \text{ for every } i\},$$
$$E_{u,w}^{\circ} = \{p \in E_w : \operatorname{rank} A_i(p) = u_i \text{ for every } i\}.$$

Vector inequalities and differences are componentwise; $u < \nu$ means $u \leq \nu$ and $u \neq \nu$. We write rank, im, ker, and coker for rank, image, kernel, and cokernel. The notation $\mathrm{Gr}(k,V)$ denotes the variety of k-dimensional linear subspaces of a vector space V. The flag variety $\mathscr{F}_{\nu,w}$ parametrizes

$$X_\alpha \subset W_\alpha, \quad \dim X_\alpha = \nu_\alpha, \qquad X_\beta \subset W_\beta' \oplus \bigoplus_{h:\beta\to\alpha} X_\alpha, \quad \dim X_\beta = \nu_\beta.$$

The incidence variety $\widetilde{\mathscr{F}}_{\nu,w}$ consists of such flags and matrices with $\operatorname{im} A_i \subset X_i$. Forgetting the flags gives a projective map

$$\pi_{\nu,w} : \widetilde{\mathscr{F}}_{\nu,w} \longrightarrow E_{\nu,w}.$$

Its domain is a vector bundle over the two-stage Grassmann bundle $\mathscr{F}_{\nu,w}$, and hence is smooth. The fibre over zero is $\mathscr{F}_{\nu,w}$.

We use the following geometric facts [5, Lemmas 3.3, 3.6 and Proposition 3.7]. The stratum $E^{\circ}_{u,w}$ is nonempty exactly when

$$u_i \le w_i, \qquad u_i \le w'_i + \sum_{h:i-j} u_j. \tag{2.3}$$

In the sum, $h : i - j$ runs over all arrows incident with i, and j is the other endpoint, counting parallel arrows separately. Such a vector is called dominant, and the set of these vectors is $\mathrm{Dom}(w)$. For a dominant vector, the stratum is smooth and irreducible, its closure is $E_{u,w}$, and $\pi_{u,w}$ is birational. All local systems in the decomposition theorem for $\pi_{\nu,w}$ are constant [5, Theorem 4.1]. The result numbers referring to [5] are those of its arXiv version 1.

For the quiver in (2.2) we use the four-parameter frame

$$w = (a, a', d, d'), \qquad (\dim W_\alpha, \dim W'_\alpha) = (a', a), \quad (\dim W_\beta, \dim W'_\beta) = (d, d').$$

All four entries are nonnegative integers. A uniform flag dimension is written $\nu = (x, y)$, with nonnegative integer entries x, y at each sink and source, respectively. The flag variety is nonempty precisely for

$$0 \le x \le a', \qquad 0 \le y \le d' + cx. \tag{2.4}$$

Its dimension and that of the incidence variety are

$$\begin{aligned} d_0(\nu, w) &= mx(a' - x) + ny(d' + cx - y), \\ \widetilde{d}(\nu, w) &= d_0(\nu, w) + mxa + nyd. \end{aligned} \tag{2.5}$$

The difference governing the normalized top fibre degree is

$$F_w(x, y) := 2d_0(\nu, w) - \widetilde{d}(\nu, w) = -mx^2 + mbxy - ny^2 + m(a' - a)x + n(d' - d)y. \tag{2.6}$$

We also write $F_w(\nu) = F_w(x, y)$ when $\nu = (x, y)$. When ν, w are fixed, d_0 and $\widetilde{d}$ denote the dimensions in (2.5).

2.3 Multiplicity spaces and equivariant traces

All sheaves have rational coefficients. For a variety Z, $\mathbb{Q}_Z$ is its constant sheaf; we omit Z when the domain is clear. For a complex of sheaves K, $H^k(K)$ is its ordinary cohomology in degree k. The shift is normalized by $H^k(K[j]) = H^{k+j}(K)$. For a map f, Rf_* denotes derived direct image and f^* denotes pullback. The notation ${}^pH^k(K)$ denotes perverse cohomology, with respect to the middle perversity, rather than ordinary cohomology. For an irreducible variety Z, we normalize $\mathrm{IC}(Z)$ to restrict to $\mathbb{Q}[\dim Z]$ on its smooth locus; it is a perverse sheaf. The decomposition theorem expresses the proper direct image from a smooth variety as a sum of shifted intersection complexes. In our case it has the form

$$K_{\nu,w} := R\pi_{\nu,w*}\mathbb{Q}[\widetilde{d}] \simeq \bigoplus_{u,j} V^j_{\nu,u;w} \otimes \mathrm{IC}(E_{u,w})[j], \tag{2.7}$$

Here $u \in \mathrm{Dom}(w)$, $u \le \nu$, $j \in \mathbb{Z}$, and $V^j_{\nu,u;w}$ is the finite-dimensional rational multiplicity space of the indicated shifted intersection complex. The map $\pi_{\nu,w}$ is projective and its domain is smooth and quasi-projective. We apply the decomposition and semisimplicity theorem and relative hard Lefschetz in [2, Theorems 1.6.1 and 1.6.3].

For invariant framing dimensions, take each W_i, W_i' to be a labelled copy of the standard complex vector space of its dimension. Use the standard bases to identify copies on the same vertex orbit. The automorphism σ transports matrices through these identifications, preserving the parallel-arrow labels; it has order dividing mn. For invariant flag dimensions it also transports the flags. For invariant u, give $\mathrm{IC}(E_{u,w})$ the equivariant structure whose restriction preserves the constant section 1. The space $V^j_{\nu,u;w}$ is intrinsically the multiplicity of $\mathrm{IC}(E_{u,w})$ in the semisimple perverse sheaf ${}^pH^{-j}(K_{\nu,w})$, and thus acquires a σ-action. Let $i_0 : \{0\} \hookrightarrow E_{u,w}$ denote the inclusion of the zero representation. The pullback $i_0^* \mathrm{IC}(E_{u,w})$ is its stalk complex at zero, and H^k denotes the cohomology of this complex in degree k. For a finite-dimensional σ-representation V, $\mathrm{Tr}(\sigma \mid V)$ is the trace of the linear operator induced by σ on V. Define

$$\rho_{\nu,u;w}(z) = \sum_j \mathrm{Tr}(\sigma \mid V^j_{\nu,u;w}) z^j, \tag{2.8}$$
$$P_{u,w}(z) = \sum_k \mathrm{Tr}\big(\sigma \mid H^k(i_0^* \mathrm{IC}(E_{u,w}))\big) z^k.$$

The sums run over integer cohomological degrees and are finite. These Laurent polynomials have integral coefficients: a finite-order rational linear operator has trace both rational and an algebraic integer.

An invariant relatively ample class is obtained by summing an ample class over the finite group. Relative hard Lefschetz is therefore equivariant, giving

$$\rho_{\nu,u;w}(z) = \rho_{\nu,u;w}(z^{-1}). \tag{2.9}$$

Moreover,

$$P_{0,w} = 1, \qquad P_{u,w} \in z^{-1}\mathbb{Z}[z^{-1}] \quad (u \neq 0). \tag{2.10}$$

Here 0 is the zero dimension vector. For the second assertion the strict support condition for the intersection complex at the proper closed point 0 gives vanishing in degrees at least zero. A nonzero rank variety here is a positive-dimensional cone.

When a rank vector is not invariant, its orbit has length greater than one. On the sum of the corresponding isotypic terms, σ cyclically permutes distinct summands. Its trace on each stalk cohomology degree is zero. Thus only uniform ranks $u = (s, t)$ contribute to the trace identities.

3 Equivariant transverse slices

For uniform dominant $u = (s, t)$, put

$$w \ominus u = (a - s + bt,\ a' - s,\ d - t,\ d' - t + cs). \tag{3.1}$$

The symbol $\ominus$ denotes this residual-frame construction, not componentwise subtraction. Dominance makes all four entries nonnegative. The following comparison retains the action needed to compute traces.

Lemma 3.1. *Suppose $\mathscr{F}_{\nu,w} \neq \varnothing$, $u \in \mathrm{Dom}(w)$ is uniform, and $u \leq \nu$. With the equivariant structures fixed above, there are graded representation isomorphisms for the cyclic group $\langle\sigma\rangle$ generated by σ:*

$$V^j_{\nu,u;w} \cong V^j_{\nu-u,0;w\ominus u}.$$

Consequently $\rho_{\nu,u;w} = \rho_{\nu-u,0;w\ominus u}$.

Proof. Let $S = E^{\circ}_{u,w}$. At $p \in S$, set

$$U_\alpha = \operatorname{im} A_\alpha(p), \qquad K_\beta = \ker A_\beta(p), \qquad L_\beta = W_\beta / K_\beta.$$

The arrow maps take values in U_α and kill K_β. They induce a surjection and an injection

$$S_\alpha : W'_\alpha \oplus \bigoplus_{h:\beta\to\alpha} L_\beta \twoheadrightarrow U_\alpha,$$
$$T_\beta : L_\beta \hookrightarrow W'_\beta \oplus \bigoplus_{h:\beta\to\alpha} U_\alpha.$$

The residual spaces are

$$\begin{aligned} W_\alpha^\perp &= W_\alpha / U_\alpha, & (W'_\alpha)^\perp &= \ker S_\alpha, \\ W_\beta^\perp &= K_\beta, & (W'_\beta)^\perp &= \operatorname{coker} T_\beta. \end{aligned} \tag{3.2}$$

Their dimensions give (3.1). They form vector bundles over S, naturally transported by σ.

Let $\mathcal{P} \to S$ be the space of choices of splittings $W_\alpha = U_\alpha \oplus Z_\alpha$, $W_\beta = L_\beta \oplus K_\beta$, right inverses j_α of S_α, left inverses r_β of T_β, and frames of all four residual spaces in (3.2). Here Z_α is a chosen complement of U_α, and the chosen splitting identifies the quotient L_β with a subspace of W_β. It is a smooth surjection with connected fibres, built from affine bundles and general linear groups. Since S is smooth and irreducible, $\mathcal{P}$ is smooth and connected. Transporting all choices gives it a finite-order σ-action. A residual frame is an isomorphism from a fixed standard vector space to the corresponding residual space. Transport these isomorphisms using the same labelled standard copies. In the resulting coordinates, σ permutes vertices and arrows and preserves the matrix entries.

In these splittings the core matrices have the form

$$x_\alpha = \begin{pmatrix} Z^0_\alpha \\ 0 \end{pmatrix}, \qquad x_\beta = \begin{pmatrix} Z^0_\beta & 0 \end{pmatrix}, \qquad y_h = \begin{pmatrix} C_h & 0 \\ 0 & 0 \end{pmatrix}.$$

Here $Z^0_\alpha : W'_\alpha \to U_\alpha$, $Z^0_\beta : L_\beta \to W'_\beta$, and $C_h : L_\beta \to U_\alpha$ are the nonzero core blocks. A residual representation consists of $\xi_\alpha : (W'_\alpha)^\perp \to W_\alpha^\perp$, $\xi_\beta : W_\beta^\perp \to (W'_\beta)^\perp$, and $Y_h : W_\beta^\perp \to W_\alpha^\perp$. Let $\iota_\beta : \operatorname{coker} T_\beta \to \ker r_\beta$ be the isomorphism supplied by the splitting, and let 1 denote the identity on the relevant space. Extend and lift the framing maps by

$$R_\alpha = \xi_\alpha(1 - j_\alpha S_\alpha), \qquad R_\beta = \iota_\beta \xi_\beta.$$

Thus $R_\alpha j_\alpha = 0$ and $r_\beta R_\beta = 0$. Write

$$R_\alpha = (Z'_\alpha, C''_h)_h, \qquad R_\beta = (Z'_\beta, C'_h)^{\mathrm{vert}}_h.$$

The primes label blocks: $Z'_\alpha : W'_\alpha \to Z_\alpha$, $C''_h : L_\beta \to Z_\alpha$, $Z'_\beta : K_\beta \to W'_\beta$, and $C'_h : K_\beta \to U_\alpha$. A row concatenates the blocks horizontally, and the superscript vert concatenates them vertically, in the fixed arrow order. The map $\Phi : \mathcal{P} \times E_{w\ominus u} \to E_w$ is defined by

$$x_\alpha = \begin{pmatrix} Z^0_\alpha \\ Z'_\alpha \end{pmatrix}, \qquad x_\beta = \begin{pmatrix} Z^0_\beta & Z'_\beta \end{pmatrix}, \qquad y_h = \begin{pmatrix} C_h & C'_h \\ C''_h & Y_h \end{pmatrix}. \tag{3.3}$$

It is algebraic and equivariant.

We verify its rank formula. At a sink, decompose the core domain as $\operatorname{im} j_\alpha \oplus \ker S_\alpha$. The concatenated matrix takes the block form

$$\begin{pmatrix} I & 0 & (C'_h)_h \\ 0 & \xi_\alpha & (Y_h)_h \end{pmatrix}.$$

Here I is the identity block of size u_α. Eliminating $(C'_h)_h$ leaves the residual horizontal matrix. At a source, decompose the core target as $\operatorname{im} T_\beta \oplus \ker r_\beta$. The matrix has block form

$$\begin{pmatrix} I & 0 \\ 0 & \xi_\beta \\ D_\beta & (Y_h)_h^{\mathrm{vert}} \end{pmatrix}, \qquad D_\beta = (C''_h)_h^{\mathrm{vert}}.$$

In this display I has size u_β. Eliminating D_β leaves the residual vertical matrix; the mixed term vanishes because $r_\beta R_\beta = 0$. Let $A_i^\perp$ be the concatenated residual matrix, defined by the same rule as A_i. Hence

$$\operatorname{rank} A_i(\Phi(p,\xi,Y)) = u_i + \operatorname{rank} A_i^\perp(\xi, Y). \tag{3.4}$$

The differential of Φ is surjective along $\mathcal{P} \times \{0\}$. Indeed, the core blocks Z_i^0, C_h vary freely. An arbitrary remaining sink block decomposes as

$$R = (Rj_\alpha)S_\alpha + R(1 - j_\alpha S_\alpha).$$

Its first term comes from deforming the image subspace as a graph, and its second from ξ_α. At a source the corresponding decomposition is

$$R = T_\beta(r_\beta R) + (1 - T_\beta r_\beta)R,$$

using the kernel graph and ξ_β. The bottom-right arrow blocks are the freely varying Y_h. These are all tangent directions of E_w. Thus there is an invariant open neighbourhood $\mathcal{O}$ of $\mathcal{P} \times \{0\}$ on which Φ is smooth.

Its relative dimension follows from the rank-stratum dimension:

$$\dim S = \sum_i (w_i + w'_i)u_i - \sum_i u_i^2 + \sum_{h:\beta\to\alpha} u_\beta u_\alpha = \dim E_w - \dim E_{w\ominus u}.$$

Consequently the relative dimension is $e = \dim \mathcal{P} - \dim S$. Expanding the incidence dimensions also gives

$$\widetilde{d}(\nu, w) + e = \dim \mathcal{P} + \widetilde{d}(\nu - u, w \ominus u). \tag{3.5}$$

By (3.4), every restriction to a rank upper bound is a base change of this smooth map.

We next compare the flags. At a sink, $X_\alpha = U_\alpha \oplus X_\alpha^\perp$, where $X_\alpha^\perp = X_\alpha \cap Z_\alpha$ has dimension $\nu_\alpha - u_\alpha$. The incidence condition implies $C''_h L_\beta \subset X_\alpha^\perp$. In the source target

$$\Big(W'_\beta \oplus \bigoplus U_\alpha\Big) \oplus \bigoplus Z_\alpha$$

write a vector as (ζ, ζ') according to these two summands and apply the unipotent map

$$(\zeta, \zeta') \longmapsto (\zeta, \zeta' - D_\beta r_\beta \zeta). \tag{3.6}$$

It preserves the ambient space allowed for the flag, since $D_\beta L_\beta \subset \bigoplus X_\alpha^\perp$. It sends the image of L_β to $(T_\beta L_\beta, 0)$ and the image of K_β to $(R_\beta, (Y_h)_h^{\mathrm{vert}})$. Quotienting by $T_\beta L_\beta$ gives exactly a residual source flag. Conversely, take the full inverse image of a residual flag under this quotient and then apply the inverse of (3.6). These operations are mutually inverse and algebraic. They identify the pullback of $\pi_{\nu,w}$ with the restriction of $\mathcal{P} \times \pi_{\nu-u,w\ominus u}$. In particular, they form an equivariant Cartesian diagram.

Let $p_2 : \mathcal{O} \to E_{w\ominus u}$ be the second projection. Restrict Φ and p_2 to the appropriate rank varieties. Proper base change and (3.5) yield

$$\Phi^* K_{\nu,w}[e] \simeq p_2^* K_{\nu-u,w\ominus u}[\dim \mathcal{P}] \tag{3.7}$$

equivariantly. Both shifted smooth pullbacks are perverse exact. In perverse degree $-j$, consider the summands with support exactly $\mathcal{P}\times\{0\}$. On the left, only $\mathrm{IC}(E_{u,w})$ contributes such a summand. Indeed, (3.4) identifies the inverse image of $E_{v,w}$ with the residual rank bound $v-u$ whenever $v\geq u$. For a dominant rank vector $v\geq u$, the two inequalities

$$\begin{aligned} v_i-u_i &\leq w_i-u_i,\\ v_i-u_i &\leq \left(w'_i+\sum_{h:i-j} u_j-u_i\right)+\sum_{h:i-j}(v_j-u_j) \end{aligned}$$

are exactly the dominance inequalities for $v-u$ in the residual frame. Thus, for $v>u$, its residual rank variety is irreducible and positive-dimensional. Its product with $\mathcal{P}$, restricted to $\mathcal{O}$, is an open subset of that irreducible product. Shifted smooth pullback preserves the intermediate extension defining its intersection complex, so this term has that full support. For $v=u$, the support is $\mathcal{P}\times\{0\}$; for $v\not\geq u$, the inverse image is empty. On the right, precisely the zero-rank summand has the specified support.

The resulting isomorphism is

$$\mathrm{IC}(\mathcal{P})\otimes V^{j}_{\nu,u;w}\simeq \mathrm{IC}(\mathcal{P})\otimes V^{j}_{\nu-u,0;w\ominus u}.$$

The multiplicity local systems are constant by [5, Theorem 4.1]. Since $\mathcal{P}$ is smooth, $\mathrm{IC}(\mathcal{P})=\mathbb{Q}_{\mathcal{P}}[\dim\mathcal{P}]$. Removing this common shift gives an equivariant isomorphism of the constant local systems $\mathbb{Q}_{\mathcal{P}}\otimes V^{j}_{\nu,u;w}$ and $\mathbb{Q}_{\mathcal{P}}\otimes V^{j}_{\nu-u,0;w\ominus u}$. Connectedness gives $H^0(\mathcal{P},\mathbb{Q}_{\mathcal{P}})=\mathbb{Q}$; its generator 1 is fixed by σ. Hence

$$H^0(\mathcal{P},\mathbb{Q}_{\mathcal{P}}\otimes V^{j}_{\nu,u;w})=V^{j}_{\nu,u;w}$$

as $\langle\sigma\rangle$-representations, and the same identification on the other side proves the assertion. □

4 Trace formula and the triangular basis

4.1 The zero-fibre trace

Proposition 4.1. *For a uniform $\nu=(x,y)$ satisfying (2.4),*

$$\sum_{\substack{u\in\mathrm{Dom}(w),\,u\leq\nu\\ u\ \text{uniform}}}\rho_{\nu,u;w}(z)P_{u,w}(z)=z^{d_0-\widetilde{d}}\begin{bmatrix}a'\\ x\end{bmatrix}_{z^m}\begin{bmatrix}d'+cx\\ y\end{bmatrix}_{z^n}. \tag{4.1}$$

Proof. The relative Schur classes of the tautological bundles give a Leray–Hirsch basis for the two-stage Grassmann bundle $\mathscr{F}_{\nu,w}$. All degrees are even, and σ permutes the classes by permuting vertices. For a cycle of length ℓ on ℓ identical Grassmann factors, a tensor basis vector contributes to the trace exactly when all its indices agree. Its degree is then multiplied by ℓ. It follows that

$$\sum_k \mathrm{Tr}(\sigma\mid H^k(\mathscr{F}_{\nu,w}))z^k=z^{d_0}\begin{bmatrix}a'\\ x\end{bmatrix}_{z^m}\begin{bmatrix}d'+cx\\ y\end{bmatrix}_{z^n}.$$

The same argument applies to the second stage because its relative Schur classes are naturally transported under the arrow direct-sum permutations.

To take traces in (2.7), use the canonical perverse filtration of $K_{\nu,w}$. The stalk spectral sequence of this filtration degenerates: the ordinary decomposition theorem splits the object

into its shifted perverse cohomology, so its differentials vanish. The filtration itself is equivariant, and trace is additive on the resulting filtered stalk cohomology. This gives the sum of traces of the intrinsic isotypic terms. Non-invariant rank orbits contribute zero. A shift $[j]$ contributes z^{-j}, which can be replaced by z^{j} using (2.9). Finally the normalization of $K_{\nu,w}$ contributes $z^{-\widetilde{d}}$, proving (4.1). □

4.2 Triangular-basis identification

Set $z = \mathbf{v}^g$, and define finite sums over (2.4) by

$$G_w = \sum_{x,y} \rho_{(x,y),0;w}(\mathbf{v}^g) X^{(a-a'+by,\ d-d'-cx)}, \tag{4.2}$$

$$S_w = \sum_{x,y} \mathbf{v}^{-cxa-byd} \begin{bmatrix} a' \\ x \end{bmatrix}_{\mathbf{v}^c} \begin{bmatrix} d'+cx \\ y \end{bmatrix}_{\mathbf{v}^b} X^{(a-a'+by,\ d-d'-cx)}. \tag{4.3}$$

Equation (2.9) makes G_w bar invariant. Fix a uniform dominant $u = (s,t)$. The admissible pairs (x,y) with $u \le (x,y)$ satisfy

$$s \le x \le a', \qquad t \le y \le d' + cx.$$

Under $(x,y) \mapsto (x-s, y-t)$, these inequalities become

$$0 \le x-s \le a'-s, \qquad 0 \le y-t \le (d'-t+cs) + c(x-s).$$

They give exactly the flag range for $w \ominus u$, in both directions. The monomial exponents in (4.2) are unchanged by this correspondence. Summing Proposition 4.1 and using Lemma 3.1 therefore gives

$$S_w = G_w + \sum_{\substack{0 \ne u \in \mathrm{Dom}(w) \\ u \text{ uniform}}} P_{u,w}(\mathbf{v}^g) G_{w \ominus u}. \tag{4.4}$$

For each nonzero dominant $u = (s,t)$, the nonnegative integer $a' + d$ decreases by $s + t$. Thus (2.10) allows finite recursive inversion. Here η runs over the frames reached by successive nonzero residual-frame operations, and $q_{w\eta}$ are the coefficients of the resulting finite expansion:

$$G_w = S_w + \sum_{\eta} q_{w\eta} S_\eta, \qquad q_{w\eta} \in \mathbf{v}^{-1}\mathbb{Z}[\mathbf{v}^{-1}]. \tag{4.5}$$

For integers k_1, k_2, the standard monomials in the adjacent seed (X_1, X_0), with the second vertex preceding the first, are

$$M'[k_1,k_2] = \mathbf{v}^{k_1 k_2} X_2^{[-k_2]_+} X_0^{[k_2]_+} X_1^{[k_1]_+} X_{-1}^{[-k_1]_+}.$$

Proposition 4.2. *The element G_w is the triangular basis element represented by the standard monomial $M'[a-a', d'-d]$ in the adjacent seed (X_1, X_0). In particular, it depends only on $a-a'$ and $d'-d$.*

Proof. Put

$$M_w^* = \mathbf{v}^{(a-a')(d'-d)} X_2^d X_0^{d'} X_1^a X_{-1}^{a'}.$$

We first check

$$\overline{S_w} = M_w^*. \tag{4.6}$$

The quantum binomial theorem gives

$$X_{-1}^{a'} = \sum_{x=0}^{a'} \begin{bmatrix} a' \\ x \end{bmatrix}_{\mathbf{v}^c} \mathbf{v}^{-cxa'} X_0^{cx} X_1^{-a'}.$$

Move X_1^a past X_0^{cx} and expand

$$X_0^{d'+cx} = \sum_y \begin{bmatrix} d'+cx \\ y \end{bmatrix}_{\mathbf{v}^b} X^{(by,-d'-cx)}.$$

The exponent of $\mathbf{v}$ before the last normalization is $(a-a')(d'-d)-cxa'+2acx$. Multiplication on the left by X_2^d and on the right by $X_1^{a-a'}$ adds $dby+(d-d'-cx)(a-a')$. Their sum is $cxa+byd$, giving (4.6).

The exchange relations also yield

$$\begin{aligned} M^*_{(a,a',d,d')} &= M^*_{(a,a',d-1,d'-1)} \\ &\quad + \mathbf{v}^{b(d+d'-1)} M^*_{(a+b,a',d-1,d'-1)}, \quad d,d'>0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} M^*_{(a,a',d,d')} &= M^*_{(a-1,a'-1,d,d')} \\ &\quad + \mathbf{v}^{c(a+a'-1)} M^*_{(a-1,a'-1,d,d'+c)}, \quad a,a'>0. \end{aligned} \tag{4.8}$$

These identities follow by replacing one X_2X_0, respectively one X_1X_{-1}, by its exchange relation and commuting the remaining powers. In both recurrences $a'+d$ decreases. Their terminal terms are the adjacent-seed standard monomials defined above. The exponents multiplying the second terms of (4.7)–(4.8) are strictly positive. The first terms preserve $a-a'$ and $d'-d$. Define

$$\mathcal{L}' = \bigoplus_{k_1,k_2\in\mathbb{Z}} \mathbf{v}\mathbb{Z}[\mathbf{v}]M'[k_1,k_2].$$

It follows that $M^*_w \equiv M'[a-a',d'-d] \pmod{\mathcal{L}'}$, and every M^*_w belongs to the $\mathbb{Z}[\mathbf{v}]$-span of the $M'[k_1,k_2]$. Take the bar of (4.5). Equations (4.6) and (2.9) show that G_w is in the cluster algebra, is bar invariant, and is congruent to $M'[a-a',d'-d]$ modulo $\mathcal{L}'$. The triangular-basis characterization proves the assertion. □

Corollary 4.3. *For any four-parameter frame,*

$$x > [a'-a]_+ \quad\Longrightarrow\quad \rho_{(x,y),0;w} = 0.$$

Proof. By Proposition 4.2, subtracting $\min(a,a')$ from a,a' and $\min(d,d')$ from d,d' preserves G_w. It also preserves the monomial exponents in (4.2), which determine (x,y) uniquely. The reduced sink dimension is $[a'-a]_+$, so (2.4) gives the result. □

For the parameters of Theorem 1.1, set

$$w = (0, A, cA-B, 0). \tag{4.9}$$

The imaginary-root inequality implies $cA-B>0$. Indeed, if $B\ge cA$, its left side is $cA^2+bB(B-cA)>0$. Corollary 4.3 shows that every monomial of G_w has the form $X^{(bp-A,cq-B)}$, where $p=y\ge 0$, $q=A-x\ge 0$. At $(x,y)=(A,0)$ the incidence variety is a point, with coefficient one. Thus G_w is pointed at (A,B), and seed independence gives

$$G_{(0,A,cA-B,0)} = C[A,B], \qquad e(p,q) = \rho_{(A-q,p),0;w}(\mathbf{v}^g). \tag{4.10}$$

5 Intersection-cohomology degree bounds

From now on fix (4.9), and write

$$F_w(x,y) = -mx^2+mbxy-ny^2+mAx-n(cA-B)y, \qquad Q(s,t) = ms^2-mbst+nt^2.$$

Uniform dominance becomes

$$s \le A, \qquad t \le cA - B, \qquad s \le bt, \qquad t \le cs. \tag{5.1}$$

In particular, a nonzero dominant uniform vector has $s, t > 0$.

Lemma 5.1. *For $0 \neq u = (s,t) \in \mathrm{Dom}(w)$, set*

$$\begin{aligned} F_u &= -Q(s,t) + mAs - n(cA-B)t, \\ F_u^{\mathrm{sw}} &= -Q(s,t) - mAs + n(cA-B)t, \\ J_u &= -Q(s,t) - 2mbst + mAs + n(cA-B)t. \end{aligned}$$

Then

$$\deg P_{u,w} \le \min(-1, F_u, F_u^{\mathrm{sw}}, J_u). \tag{5.2}$$

Proof. If $f : Y \to E_{u,w}$ is a projective birational resolution with Y smooth, then $\mathrm{IC}(E_{u,w})$ is a direct summand of $Rf_*\mathbb{Q}[\dim Y]$. Therefore its stalk at zero has degree at most $2\dim f^{-1}(0) - \dim Y$. Taking a trace cannot increase this bound.

For $f = \pi_{u,w}$, (2.5) gives F_u. Dualizing every map and exchanging sources and sinks gives a second resolution. Its zero-fibre dimension is

$$nt(cA - B - t) + ms(bt - s),$$

whereas its domain dimension is $-Q(s,t) + mAs + n(cA-B)t$. Twice the first expression minus the second is F_u^{sw}.

For a third resolution choose an s-dimensional image subspace at every sink and a codimension-t kernel subspace at every source. Its base is

$$\prod_{\alpha} \mathrm{Gr}(s, W_\alpha) \;\times\; \prod_{\beta} \mathrm{Gr}(cA - B - t, W_\beta).$$

Let each arrow factor through the selected source quotient into the selected sink image. Since both primed spaces in (4.9) are zero, this is a vector bundle of rank $mbst$. The base dimension is $mAs + n(cA-B)t - ms^2 - nt^2$, and the total dimension is $-Q(s,t) + mAs + n(cA-B)t$. Over the full rank stratum the image and kernel are uniquely determined, so this projective map is birational. Its zero-fibre degree bound is J_u. Combining these three bounds with (2.10) proves the lemma. □

Lemma 5.2. *For every frame and every admissible ν,*

$$\deg \rho_{\nu,0;w} \le F_w(\nu), \qquad F_w(\nu) < 0 \implies \rho_{\nu,0;w} = 0.$$

For the special frame (4.9), a nonzero $\rho_{\nu,u;w}$, $u = (s,t)$, also satisfies

$$x - s \le [A - bt]_+. \tag{5.3}$$

Proof. For ordinary multiplicities, a point-supported summand shifted by $[j]$ contributes to normalized zero-fibre cohomology in degree $-j$. Relative hard Lefschetz gives its opposite shift as well. Thus every shift with nonzero multiplicity satisfies $|j| \le 2d_0 - \widetilde{d} = F_w(\nu)$. A negative value of $F_w(\nu)$ forces all point multiplicities to vanish. The trace polynomial satisfies the same bound.

For the second assertion apply Lemma 3.1 and Corollary 4.3 to the frame

$$w \ominus u = (-s + bt, A - s, cA - B - t, -t + cs).$$

The difference of its sink framing dimensions is $A - bt$, which gives (5.3). □

6 Degree comparison

For the frame $w = (0, A, cA - B, 0)$, the three bounds in Lemma 5.1 yield a strict degree comparison through the following inequality.

Lemma 6.1. *Suppose $A, B > 0$ and $cA - B > 0$, and assume either $bc > 4$ and $cA^2 - bcAB + bB^2 < 0$, or $bc = 4$ and $B = cA/2$. Let x, y, s, t be real numbers satisfying $F_w(x, y) \geq 0$ and*

$$\begin{gathered} 0 < s \leq x \leq A, \qquad 0 < t \leq y, \qquad s \leq bt, \quad t \leq cs, \quad t \leq cA - B, \\ x - s \leq [A - bt]_+. \end{gathered} \tag{6.1}$$

Define

$$\begin{aligned} \Delta &= Q(s,t) + m(A - 2x)s + n(2cx - 2y - cA + B)t, \\ L_2 &= 2[Q(s,t) - mxs + n(cx - y)t], \\ L_3 &= 2[Q(s,t) + m(A - x)s + n(cx - y - cA + B)t], \\ L_4 &= 2[ms^2 + nt^2 - mxs + n(cx - y - cA + B)t]. \end{aligned}$$

Then at least one of

$$\Delta \geq 0, \qquad L_2 > 0, \qquad L_3 > 0, \qquad L_4 > 0$$

holds.

Proof. First let $bc > 4$. On $0 \leq x \leq A$, define

$$h(x) = \frac{cx - cA + B + \sqrt{(cx - cA + B)^2 + 4(c/b)x(A - x)}}{2}.$$

The parameter inequality gives $b(cA - B) > A$, and

$$h(0) = 0, \qquad h(A) = B, \qquad h'(0) = \frac{cA}{b(cA - B)},$$

$$h''(x) = \frac{2c(-cA^2 + bcAB - bB^2)}{b^2((cx - cA + B)^2 + 4(c/b)x(A - x))^{3/2}} > 0.$$

Thus h is strictly increasing and strictly convex, and for $y \geq 0$, $F_w(x, y) \geq 0$ is equivalent to $y \leq h(x)$.

All four expressions have the common variable part $2m(bt - s)x - 2nty$. Consequently,

$$\begin{aligned} \max\{\Delta, L_2, L_3, L_4\}|_{y=h(x)} &= 2m(bt - s)x - 2nt\,h(x) \\ &+ \max\{Q(s,t) + mAs - n(cA - B)t,\ 2Q(s,t), \\ &\qquad 2Q(s,t) + 2mAs - 2n(cA - B)t,\ 2(ms^2 + nt^2 - n(cA - B)t)\}. \end{aligned} \tag{6.2}$$

The four constant terms are independent of x. The maximum therefore has second derivative $-2nt\,h''(x) < 0$ and is strictly concave.

If $x = s$, substitution gives

$$\begin{aligned} \Delta &= F_w(x, y) - n(y - t)(cs - cA + B - y + t), \\ L_4 &= 2nt(cs - cA + B - y + t). \end{aligned} \tag{6.3}$$

When $L_4 \leq 0$, we have $\Delta \geq F_w(x, y) \geq 0$; otherwise $L_4 > 0$. This also handles $A < bt$, because the last inequality of (6.1) then forces $x = s$.

Assume $A \geq bt$. On the upper boundary with $y \geq t$, we have $t \leq h(A) = B$. The inverse h^{-1} is defined on $[0, B]$, and the admissible interval is

$$\max(s, h^{-1}(t)) \leq x \leq s + A - bt.$$

At a left endpoint with $x = s$, use (6.3). Otherwise $y = t$, and

$$L_2 = 2m(x - s)(bt - s).$$

This is positive when $x > s$ and $bt > s$. If $x = s$, then $\Delta = F_w(x, y) \geq 0$. If $bt = s$, then $L_3 = 2ntB > 0$. If the maximum of the four quantities at this endpoint is zero, these formulas also give $\Delta = 0$.

At the right endpoint $x = s + A - bt$,

$$\begin{aligned} \Delta &= F_w(x, y) - n(y - t)(cx - y + t - cA + B), \\ L_3 &= 2nt(cx - y + t - cA + B). \end{aligned} \tag{6.4}$$

If $L_3 \leq 0$, then $\Delta \geq F_w(x, y) \geq 0$; otherwise $L_3 > 0$. If the maximum at this endpoint is zero, the same formulas give $\Delta = 0$.

Strict concavity makes the maximum positive in the interior of every nondegenerate admissible interval. Degenerate intervals are covered by the endpoint calculations. Replacing $h(x)$ by a smaller y increases all four quantities by $2nt(h(x) - y)$, proving the assertion.

For $bc = 4$, we have $B = cA/2$ and

$$F_w(x, y) = n(cx/2 - y)(cA/2 - cx/2 + y).$$

The hypotheses imply $cx/2 \geq y$. If $cs = 2t$, then $\Delta = 2nt(cx/2 - y) \geq 0$. If $cs < 2t$, then

$$\frac{L_2}{2n} = (cs/2 - t)^2 + (cx/2)(t - cs/2) + t(cx/2 - y) > 0.$$

If $cs > 2t$, then

$$\frac{L_3}{2n} = (cs/2 - t)^2 + \frac{c(A - x)}{2}(cs/2 - t) + t(cx/2 - y) > 0. \qquad \square$$

Proposition 6.2. *For the frame (4.9), suppose $F_w(\nu) \geq 0$. Every nonzero uniform rank u satisfies*

$$\deg(\rho_{\nu,u;w} P_{u,w}) < F_w(\nu).$$

Proof. Suppose $\rho_{\nu,u;w} \neq 0$. Dominance, $u \leq \nu$, and (5.3) give (6.1). The imaginary-root condition is

$$cA^2 - bcAB + bB^2 \leq 0.$$

When $bc < 4$ there are no positive solutions; when $bc = 4$, we have $B = cA/2$. For an integer $bc > 4$, $bc(bc - 4)$ is not a square. It is 5 when $bc = 5$, and for $bc \geq 6$ it lies strictly between $(bc - 3)^2$ and $(bc - 2)^2$. Equality in the displayed quadratic would make B/A a rational root of a quadratic of discriminant $bc(bc - 4)$. Thus the inequality is strict when $bc > 4$, and Lemma 6.1 applies.

By (3.1),

$$\begin{aligned} \Delta &= F_w(\nu) - F_{w \ominus u}(\nu - u), \\ L_2 &= \Delta - F_u, \qquad L_3 = \Delta - F_u^{\mathrm{sw}}, \qquad L_4 = \Delta - J_u. \end{aligned}$$

Lemma 6.1 gives

$$\min(-1, F_u, F_u^{\mathrm{sw}}, J_u) < \Delta :$$

if $\Delta \geq 0$, use $-1 < \Delta$; otherwise one of the other three bounds is strictly smaller. Lemmas 3.1, 5.1, and 5.2 give

$$\begin{aligned} \deg(\rho_{\nu,u;w} P_{u,w}) &\leq F_{w \ominus u}(\nu - u) + \min(-1, F_u, F_u^{\mathrm{sw}}, J_u) \\ &< F_w(\nu). \end{aligned} \qquad \square$$

7 Proof of the support formula

Proof of Theorem 1.1. Use the frame $w = (0, A, cA - B, 0)$. Set $\nu = (x, y) = (A - q, p)$, so that the flag dimensions correspond to the coefficient indices in (4.10). Expansion of the quadratic form gives

$$gF_w(A - q, p) = D(p, q). \tag{7.1}$$

Equation (4.10) and Lemma 5.2 show that $D(p, q) < 0$ forces the coefficient to vanish whenever the flag dimensions are admissible; outside that range it is zero by definition.

Suppose now that $0 \le q \le A$, $0 \le p \le B$, and $D(p, q) \ge 0$. We first check that $0 \le y \le cx$. The nonnegative region is $0 \le y \le h(x)$, with h as in Lemma 6.1; when $bc = 4$, $h(x) = cx/2$. The imaginary-root condition gives $b(cA - B) > A$. For $x > 0$,

$$F_w(x, cx) = mx(A - b(cA - B) - x) < 0,$$

so the nonnegative root satisfies $h(x) < cx$. At $x = 0$, $F_w(x, y) \ge 0$ forces $y = 0$. Hence $y \le cx$, and $\mathscr{F}_{\nu,w}$ is nonempty.

The right side of (4.1) becomes

$$z^{-ny(cA-B)} \begin{bmatrix} A \\ x \end{bmatrix}_{z^m} \begin{bmatrix} cx \\ y \end{bmatrix}_{z^n}. \tag{7.2}$$

Its highest degree is

$$-ny(cA - B) + mx(A - x) + ny(cx - y) = F_w(x, y),$$

and its highest coefficient is one. Proposition 6.2 puts every contribution with $u \neq 0$ strictly below this degree. Since $P_{0,w} = 1$, the remaining term satisfies

$$\deg_z \rho_{\nu,0;w} = F_w(\nu), \qquad [z^{F_w(\nu)}]\rho_{\nu,0;w} = 1.$$

Substitute $z = \mathbf{v}^g$ and use (7.1). This proves nonvanishing, the degree formula, and the leading coefficient.

It remains to exclude the rest of the rectangle complement. The character construction and Corollary 4.3 already give $p \ge 0$ and $0 \le q \le A$. The function h is increasing, with $h(A) = B$. Therefore $p > B$ implies $F_w(x, p) < 0$, and Lemma 5.2 gives zero. Finally, (2.9) gives bar invariance. A bar-invariant Laurent polynomial of highest degree zero is constant, and the leading coefficient calculation makes that constant one. □

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Zhili College, Tsinghua University
Beijing 100084, China
Email: tangqy24@mails.tsinghua.edu.cn