Strongly positive bases in rank-two quantum cluster algebras

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Abstract

We prove that, for arbitrary positive integers b, c, every strongly positive basis of the coefficient-free rank-two quantum cluster algebra $\mathcal{A}_v(b, c)$ has a unique element pointed at each integer pair, and these elements exhaust the basis. This proves Conjecture 12 of Lee, Li, Rupel, and Zelevinsky. The proof extracts a normalized corner term from a positive standard-monomial expansion and uses extremal support to obtain a finite spanning argument.

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1 Introduction

Distinguished bases of cluster algebras are often indexed by integer vectors that describe the leading terms of their Laurent expansions. For a basis constructed by prescribing such expansions, this indexing is part of the construction. For an arbitrary basis specified through positivity of multiplication and compatibility with cluster monomials, the existence and uniqueness of leading terms require a separate argument.

Lee, Li, Rupel, and Zelevinsky introduced strongly positive bases in their study of rank-two quantum cluster algebras [2, Section 5]. Such a basis is invariant under the bar operation, is independent of the initial cluster, contains every normalized cluster monomial, and has nonnegative Laurent polynomials in the quantum parameter as structure constants. Their Conjecture 12 asks whether every integer pair labels exactly one pointed basis element. Pointedness specifies both a support region and a coefficient equal to one at its corner; the precise conventions appear in Section 2.

We prove this assertion in Theorem 4.1, including the exhaustion of the basis by the resulting elements. The argument starts with the standard monomial basis. Each standard monomial has a prescribed corner and expands positively in any strongly positive basis. Laurent positivity prevents cancellation in this expansion, and bar invariance fixes the normalization of the element contributing at the corner. To pass from existence to uniqueness, we prove a support estimate for arbitrary, possibly signed, standard-basis expansions. The estimate gives a finite induction: the standard monomials outside the strictly positive quadrant are already cluster monomials, and each remaining reduction decreases the sum of the two positive coordinates.

The external algebraic inputs are the quantum Laurent phenomenon and the acyclic standard monomial basis theorem of Berenstein and Zelevinsky [1]. The normalization used here is the one in [2].

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2 Quantum Laurent expansions and standard monomials

Fix positive integers b, c and an indeterminate v. Set

$$R = \mathbb{Z}[v^{\pm 1}], \qquad R_+ = \mathbb{Z}_{\geq 0}[v^{\pm 1}].$$

Thus R_+ includes zero, and membership in R_+ means that every coefficient of every integer power of v is nonnegative. Let $\mathcal{T}$ be the quantum torus over R generated by the invertible elements X_1, X_2 with relation $X_2X_1 = v^2X_1X_2$. For $r, s \in \mathbb{Z}$, define the normalized monomial

$$X^{(r,s)} = v^{rs}X_1^rX_2^s.$$

These monomials form an R-basis of $\mathcal{T}$, and their multiplication is

$$X^{(r,s)}X^{(t,u)} = v^{st-ru}X^{(r+t,s+u)} \qquad (r, s, t, u \in \mathbb{Z}). \tag{1}$$

The bar operation is the additive anti-involution determined by $\overline{v} = v^{-1}$, $\overline{X_1} = X_1$, $\overline{X_2} = X_2$, and $\overline{FG} = \overline{G}\,\overline{F}$. In particular, $\overline{X^{(r,s)}} = X^{(r,s)}$.

Inside the skew field of fractions of $\mathcal{T}$, the cluster variables X_m, indexed by $m \in \mathbb{Z}$, are determined from X_1, X_2 by

$$X_{m+1}X_{m-1} = \begin{cases} v^bX_m^b + 1, & m \text{ odd}, \\ v^cX_m^c + 1, & m \text{ even}. \end{cases} \tag{2}$$

The coefficient-free quantum cluster algebra $\mathcal{A}_v(b, c)$ is the R-subalgebra generated by all X_m. Its clusters are the adjacent pairs (X_m, X_{m+1}), which satisfy $X_{m+1}X_m = v^2X_mX_{m+1}$. A normalized cluster monomial is

$$v^{ij}X_m^iX_{m+1}^j, \qquad i, j \in \mathbb{Z}_{\geq 0}, \quad m \in \mathbb{Z}.$$

These conventions agree with [2, Section 2]. The quantum Laurent phenomenon [1, Corollary 5.2] implies $\mathcal{A}_v(b, c) \subseteq \mathcal{T}$ and gives a finite Laurent expansion in every cluster.

For $F \in \mathcal{T}$, write its finite expansion as $F = \sum_{r,s\in\mathbb{Z}} f_{r,s}X^{(r,s)}$, where $f_{r,s} \in R$. Its support is

$$\operatorname{supp} F = \{(r, s) \in \mathbb{Z}^2 : f_{r,s} \neq 0\}.$$

The coefficient at an exponent not in the support is understood to be zero. We call F Laurent positive in this cluster when all $f_{r,s}$ belong to R_+.

Definition 2.1. An element $F \in \mathcal{A}_v(b, c)$ is pointed at $(a_1, a_2) \in \mathbb{Z}^2$ if it has a finite expansion

$$F = \sum_{p,q\geq 0} e(p, q)X^{(bp-a_1,cq-a_2)}, \qquad e(p, q) \in R, \qquad e(0, 0) = 1, \tag{3}$$

where p, q range over nonnegative integers.

The corner of this expansion is $(-a_1, -a_2)$. In particular, the minimum of the first coordinate of $\operatorname{supp} F$ is $-a_1$, and the minimum of the second coordinate is $-a_2$. Thus a pointed element determines its label uniquely.

Definition 2.2. A strongly positive basis is an R-basis $\mathcal{B}$ of $\mathcal{A}_v(b, c)$ with the following properties:

1. every element of $\mathcal{B}$ is bar invariant;
2. $\mathcal{B}$ is independent of the choice of initial cluster;

3. $\mathcal{B}$ contains all normalized cluster monomials;

4. the product of any two elements of $\mathcal{B}$ is a finite R_+-linear combination of elements of $\mathcal{B}$.

Here independence of the initial cluster means that changing the chosen initial adjacent pair gives the same subset of the ambient cluster algebra, under its natural identification. This is the definition in [2, Section 5]; the proof below uses properties (1), (3), and (4) in the fixed initial cluster.

For an integer h, let $[h]_+ = \max(h, 0)$. Define the standard monomials, with the factors in the displayed order, by

$$M[a_1, a_2] = v^{a_1 a_2} X_3^{[a_1]_+} X_1^{[-a_1]_+} X_2^{[-a_2]_+} X_0^{[a_2]_+} \qquad (a_1, a_2 \in \mathbb{Z}). \tag{4}$$

Lemma 2.3. *The elements $M[a_1, a_2]$, for $(a_1, a_2) \in \mathbb{Z}^2$, form an R-basis of $\mathcal{A}_v(b, c)$. Each $M[a_1, a_2]$ is pointed at (a_1, a_2). If $a_1 \le 0$ or $a_2 \le 0$, it is a normalized cluster monomial.*

Proof. The basis assertion is the rank-two instance of [1, Theorems 7.3 and 7.5], stated in precisely the normalization (4) in [2, Section 4]. For the compatibility convention in [1], take $q^{1/2} = v$, exchange matrix $\left(\begin{smallmatrix} 0 & -b \\ c & 0 \end{smallmatrix}\right)$, and skew form matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The transpose of the former times the latter is $\mathrm{diag}(c, b)$. There are no frozen variables, and the exchange matrix is acyclic. Thus the standard monomials form a basis of the lower bound, the R-subalgebra generated by X_1, X_2, X_3, X_0, and the lower bound equals $\mathcal{A}_v(b, c)$. In the standard order used in [1], the factors are X_1, X_2, X_3, X_0, with X_1 and X_3, respectively X_2 and X_0, mutually exclusive. Moving X_3 past X_2 to obtain (4) only multiplies a monomial by a unit v^k, for an integer k.

The exchange relations give

$$X_0 = X^{(0,-1)} + X^{(b,-1)}, \qquad X_3 = X^{(-1,0)} + X^{(-1,c)}. \tag{5}$$

Consequently every exponent in (4) has the form $(bp - a_1, cq - a_2)$ with $p, q \ge 0$. The corner is obtained uniquely by choosing the first summand in each factor of (5). Before multiplication by $v^{a_1 a_2}$, this term is $X_1^{-a_1} X_2^{-a_2}$, so its normalized coefficient after that multiplication is one. This proves pointedness.

If $a_1, a_2 \le 0$, formula (4) is the initial cluster monomial $v^{a_1 a_2} X_1^{-a_1} X_2^{-a_2}$. In the two other boundary regions, adjacent quasi-commutation gives

$$M[a_1, a_2] = v^{-a_1 a_2} X_0^{a_2} X_1^{-a_1} \qquad (a_1 \le 0 \le a_2),$$
$$M[a_1, a_2] = v^{-a_1 a_2} X_2^{-a_2} X_3^{a_1} \qquad (a_2 \le 0 \le a_1).$$

These are normalized cluster monomials as well. □

3 Positivity and finite spanning

We first spell out the denominator-clearing argument from [2, Section 5].

Lemma 3.1. *Every element of a strongly positive basis is Laurent positive in the initial cluster.*

Proof. Let $Y \in \mathcal{B}$, and write $Y = \sum f_{r,s} X^{(r,s)}$. Since the support is finite, there exist nonnegative integers u, w such that $r + u, s + w \ge 0$ for every $(r, s) \in \operatorname{supp} Y$. By (1),

$$Y X^{(u,w)} = \sum_{r,s} v^{su - rw} f_{r,s} X^{(r+u, s+w)}. \tag{6}$$

The monomials on the right are distinct normalized initial cluster monomials, hence are elements of $\mathcal{B}$. On the other hand, Y and $X^{(u,w)}$ belong to $\mathcal{B}$, so their product has an R_+-linear expansion in this basis. Uniqueness of basis expansion shows that $v^{su - rw} f_{r,s} \in R_+$ for every r, s. Multiplication by an integer power of v preserves R_+, and therefore $f_{r,s} \in R_+$. □

The next support estimate applies even when the standard-basis coefficients have negative terms.

Lemma 3.2. *Let $a_1, a_2 \in \mathbb{Z}$, and suppose $F \in \mathcal{A}_v(b,c)$ satisfies*

$$\operatorname{supp} F \subseteq \{(r,s) \in \mathbb{Z}^2 : r \geq -a_1,\ s \geq -a_2\}.$$

In its finite standard-basis expansion

$$F = \sum_{t_1,t_2 \in \mathbb{Z}} d_{t_1,t_2} M[t_1,t_2], \qquad d_{t_1,t_2} \in R, \tag{7}$$

every nonzero coefficient has $t_1 \leq a_1$ and $t_2 \leq a_2$. Moreover, d_{a_1,a_2} is the coefficient of $X^{(-a_1,-a_2)}$ in F.

Proof. The assertion is immediate for $F = 0$. Otherwise, among the finitely many labels with nonzero coefficient in (7), choose one with largest first coordinate, and among those choose one with largest second coordinate. Denote it by (t_1^*, t_2^*). If another label has $t_1 < t_1^*$, every point in the support of $M[t_1,t_2]$ has first coordinate at least $-t_1 > -t_1^*$. If $t_1 = t_1^*$, its second coordinate satisfies $t_2 < t_2^*$, so every point in that support has second coordinate greater than $-t_2^*$. Thus only $M[t_1^*, t_2^*]$ contributes at $(-t_1^*, -t_2^*)$, and the coefficient there is $d_{t_1^*,t_2^*} \neq 0$. The assumed support bound gives $t_1^* \leq a_1$. It follows that all labels occurring in (7) have $t_1 \leq a_1$.

To bound the second coordinate, choose instead a label with largest second coordinate, breaking ties by the largest first coordinate. The same argument, with the roles of the coordinates interchanged, shows that its corner cannot cancel. The support bound then gives $t_2 \leq a_2$ for every occurring label.

For every such label different from (a_1, a_2), at least one of these inequalities is strict. Its standard monomial therefore does not contain $X^{(-a_1,-a_2)}$. Since the corner coefficient of $M[a_1,a_2]$ is one, the asserted coefficient identity follows. □

Lemma 3.3. *Suppose that, for every $(a_1,a_2) \in \mathbb{Z}^2$, an element $F[a_1,a_2] \in \mathcal{A}_v(b,c)$ is pointed at (a_1,a_2). Suppose also that*

$$F[a_1,a_2] = M[a_1,a_2] \quad \text{if } a_1 \leq 0 \text{ or } a_2 \leq 0.$$

Then the elements $F[a_1,a_2]$ span $\mathcal{A}_v(b,c)$ over R.

Proof. Lemma 3.2 gives a finite expansion

$$F[a_1,a_2] = M[a_1,a_2] + \sum_{\substack{t_1 \leq a_1,\ t_2 \leq a_2 \\ (t_1,t_2) \neq (a_1,a_2)}} d_{t_1,t_2} M[t_1,t_2]. \tag{8}$$

It suffices to put each standard monomial in the span of the $F[a_1,a_2]$. This holds by assumption if at least one coordinate is nonpositive. For $a_1, a_2 > 0$, use induction on $a_1 + a_2$. Each term in the sum in (8) either has a nonpositive coordinate or has both coordinates positive with $t_1 + t_2 < a_1 + a_2$. In the former case it is already in the span; in the latter it is in the span by induction. Solving (8) for $M[a_1,a_2]$ proves the claim. All expansions are finite, and the induction parameter decreases until it reaches the boundary. Lemma 2.3 now gives the required spanning assertion. □

4 Pointed parametrization

Theorem 4.1. *For arbitrary positive integers b, c, let $\mathcal{B}$ be a strongly positive basis of $\mathcal{A}_v(b,c)$. For every $(a_1,a_2) \in \mathbb{Z}^2$, there is a unique element $B[a_1,a_2] \in \mathcal{B}$ pointed at (a_1,a_2), and*

$$\mathcal{B} = \{B[a_1,a_2] : (a_1,a_2) \in \mathbb{Z}^2\}.$$

This is Conjecture 12 of [2], together with its stated exhaustion consequence. The conjecture is numbered 5.1 in the arXiv HTML rendering of that article.

Proof. Fix $(a_1, a_2) \in \mathbb{Z}^2$. The standard monomial (4) is an integer power of v times a finite product of cluster variables. Every factor belongs to $\mathcal{B}$, so repeated use of multiplication positivity gives a finite expansion

$$M[a_1, a_2] = \sum_{Y \in \mathcal{B}} h_Y Y, \qquad h_Y \in R_+. \tag{9}$$

By Lemma 3.1, all Laurent coefficients of every Y are in R_+. A product of two nonzero elements of R_+ is nonzero, and a sum of nonnegative Laurent polynomials cannot cancel. Consequently, for every Y with $h_Y \neq 0$,

$$\operatorname{supp} Y \subseteq \operatorname{supp} M[a_1, a_2] \subseteq \{(bp - a_1, cq - a_2) : p, q \in \mathbb{Z}_{\geq 0}\}. \tag{10}$$

Let $k_Y \in R_+$ be the coefficient of $X^{(-a_1,-a_2)}$ in Y. Taking that coefficient in (9) yields

$$1 = \sum_{Y \in \mathcal{B}} h_Y k_Y. \tag{11}$$

There is exactly one nonzero summand in this identity, and it equals one. For the corresponding element Y, we have $h_Y k_Y = 1$. Evaluating these two nonnegative Laurent polynomials at $v = 1$ shows that each has coefficient sum one. Hence, for some integer j, they are $k_Y = v^j$ and $h_Y = v^{-j}$.

Since Y is bar invariant and every normalized Laurent monomial is fixed by bar, each Laurent coefficient of Y is fixed by the substitution $v \mapsto v^{-1}$. In particular $v^j = v^{-j}$. As v is indeterminate, $j = 0$. Thus $k_Y = 1$, and (10) proves that this element is pointed at (a_1, a_2). Denote the selected element by $B[a_1, a_2]$.

When $a_1 \leq 0$ or $a_2 \leq 0$, Lemma 2.3 shows that $M[a_1, a_2]$ is itself a normalized cluster monomial, hence belongs to $\mathcal{B}$. Its expansion (9) therefore consists of itself with coefficient one. It follows that

$$B[a_1, a_2] = M[a_1, a_2] \quad \text{if } a_1 \leq 0 \text{ or } a_2 \leq 0.$$

Lemma 3.3 now implies that the selected elements $B[a_1, a_2]$ span the entire algebra. They are a subset of the basis $\mathcal{B}$, so this subset equals $\mathcal{B}$: otherwise an omitted basis element would be a finite linear combination of the selected ones, contradicting linear independence.

Finally, the label of a pointed element is determined by the two coordinate minima of its support. Thus different pairs give different selected elements. If $Y \in \mathcal{B}$ is pointed at (a_1, a_2), exhaustion gives $Y = B[t_1, t_2]$ for some pair (t_1, t_2). Comparing the two coordinate minima gives $(t_1, t_2) = (a_1, a_2)$, which proves uniqueness. □

References

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