

Ordinal-definable families in Cohen, random, and collapse extensions

Elliot Glazer

Version 24, September 28, 2026

Abstract

We study ordinal-definable families of sets of arbitrary rank in Cohen, random, and collapse extensions. Over L , countable OD families have OD enumerations after adding one Cohen or one random real. A single generalized Cohen subset gives the corresponding sharp theorem at every regular uncountable cardinal. When κ is singular strong limit and has uncountable cofinality, adding κ Cohen reals makes every member of a short-parameter definable family of size at most κ definable from a short parameter. Adding κ random reals gives a single short-parameter definable enumeration for every such family of size strictly below κ . The random bound is sharp. The proofs use coordinate amalgamation and small-index arguments. We also obtain arbitrary-rank descent for countable families after collapsing any infinite cardinal, with applications to choice in the relative Feferman–Levy model. In the full Solovay collapse extension of an arbitrary ZFC ground, every family definable from a real and ordinals that represents fewer than $2^{\mathfrak{c}}$ classes modulo null sets consists of measurable sets; the corresponding category assertion also holds.

In the extensions of L by ω_1 Cohen or random reals, no model of $I\Delta_0$ of size at most $\aleph_1$, definable from ordinals and a real, has full binary-coded standard system. $\Pi_1^1\text{-CA}_0$ proves a finite-fragment Borel obstruction to a full binary standard system for $I\Delta_0$. For regular uncountable κ , the generalized Cohen extension by $\text{Add}(\kappa, \Lambda)^L$, $\Lambda > \kappa$, has no ambiently κ^+ -saturated $I\Delta_0$ model in $\text{HOD}_{H_{\kappa^+}}$, regardless of its size. Consequently every infinite regular κ of L admits a cofinality-preserving GCH extension with no $\text{OD}_{\mathcal{P}(\kappa)}$ saturated arithmetic presentation of size κ^+ . ZFC also proves that every singular strong limit κ admits an OD κ^+ -saturated elementary extension of $\mathbb{N}$ of size 2^κ , hence of size κ^+ under GCH.

Contents

1	Introduction	2
2	Preliminaries	4
3	Countable families in a Cohen extension	5
4	One generalized Cohen subset	10
5	Countable families in a random extension	21
6	Countable families in collapse extensions	26
7	Definable families modulo null and meager sets	35
8	Small families in Cohen extensions	38
9	Higher random extensions	49
10	Singular Cohen extensions	62
11	Small index and small families at singular strong limits	70
12	Definable models of arithmetic	75
13	The relative Feferman–Levy model	104
14	Hereditary definability and choice	109
15	Saturation in generalized Cohen extensions	111

1 Introduction

An ordinal-definable set need not have ordinal-definable elements. Kanovei and Lyubetsky proved that countable OD sets of reals in several one-real extensions lie in the ground model, and asked about families of sets of reals [18]. At higher rank, the forcing matters. A Sacks extension of L can contain an OD pair of non-OD sets of reals [8]. We prove arbitrary-rank small-family theorems for Cohen, random, and collapse forcing, and apply them to arithmetic and choice in inner models of ZF.

Definability and cardinality are computed in the ambient extension. The notation $\text{OD}(a)$ permits the parameter a in addition to ordinals, and $\text{OD}_{\mathbb{R}}$ permits one real. For ordinary Cohen and random forcing, $\text{OD}_{<\kappa}$ permits fewer than κ reals or, equivalently, one subset of an ordinal of cardinality less than κ . For generalized Cohen forcing, $\text{OD}_{H_{\kappa^+}}$ permits one parameter of hereditary cardinality at most κ ; $\text{OD}_{\mathcal{P}(\kappa)}$ permits one subset of κ . Members of the families may have arbitrary rank unless stated otherwise.

Remark 1.1 (Ground models). Forcing lemmas are stated over arbitrary ZFC grounds, with their cardinal-arithmetic assumptions explicit. Every definability theorem stated over L remains valid over an original ZFC ground W satisfying $\text{GCH} + \text{GA} + (V = \text{HOD})$. These are the only features of L used. Here GA is the Ground Axiom. By Lemma 2.2, these assumptions give definitions of the ground and its objects that remain valid in further set-forcing extensions. Intermediate grounds need not satisfy GA or $V = \text{HOD}$. Statements over $L[a]$ similarly permit such a ground with a fixed ground parameter a . Several theorems explicitly weaken or dispense with GCH.

1.1 Small definable families

If c is Cohen over L , every countable $\text{OD}(a)$ family in $L[c]$ has an $\text{OD}(a)$ bijective enumeration, even for a new set of ordinals $a \in L[c]$ (Theorem 3.11). This also gives uniformization of countable sections with real first coordinates. One random real gives the corresponding enumeration theorem for ground parameters (Corollary 5.11). For regular uncountable κ in L , one Cohen subset of κ gives OD enumerations of OD families of size at most κ (Theorem 4.2).

For every infinite μ , a $\text{Coll}(\omega, \mu)$ -extension of $L[a]$ makes every countable $\text{OD}(a)$ family pointwise $\text{OD}(a)$. More precisely, it has an $\text{OD}(a)$ bijection from a ground cardinal $\nu \leq \mu$ (Corollary 6.13). There is no rank or cofinality restriction. The indexing ordinal may become countable; an OD enumeration indexed by ω need not exist.

The full Lévy collapse gives a different small-family theorem. Let $W \models \text{ZFC}$, let κ be inaccessible in W , and let $G \subseteq \text{Coll}(\omega, <\kappa)^W$ be generic. In $W[G]$, every $\text{OD}_{\mathbb{R}}$ family representing fewer than $2^{\aleph_1}$ classes modulo null sets consists of measurable sets; modulo meager sets, every member has the Baire property (Theorem 7.1). In particular, an $\text{OD}_{\mathbb{R}}$ class modulo either ideal has a Borel representative (Corollary 7.4). Friedman obtained these conclusions in a further extension and asked whether the single-class assertion already holds in the original Solovay extension [11]. Our theorem gives both conclusions there, over an arbitrary ground with an inaccessible cardinal.

After adding κ ordinary Cohen reals to L , for any uncountable κ ,

$$A \in \text{OD}_{<\kappa}, \quad |A| \leq \kappa \quad \implies \quad A \subseteq \text{OD}_{<\kappa} \quad (\text{Theorems 8.14 and 11.5 and Corollary 10.9}).$$

At singular κ , families of size less than κ have one $\text{OD}_{<\kappa}$ bijective enumeration, without any cardinal-arithmetic hypothesis (Theorem 10.11). At countable cofinality, OD families of size at most κ have OD enumerations (Corollary 10.9). At uncountable cofinality, the assertion for individual members requires only strong limitness in place of GCH.

For random forcing of uncountable Maharam type κ , the bounds differ. At singular countable cofinality, OD families of size at most κ have OD enumerations (Theorem 9.1). At uncountable cofinality, every $\text{OD}_{<\kappa}$ family of size strictly below κ has an $\text{OD}_{<\kappa}$ enumeration if κ is singular strong limit, or regular with $\kappa^{<\kappa} = \kappa$ (Theorem 11.6 and Corollary 9.20). An OD family of size $(\kappa^{\aleph_0})^L$ with no $\text{OD}_{<\kappa}$ member proves sharpness over L : its size is κ^+ at countable cofinality and κ at uncountable cofinality (Theorem 9.2).

For regular uncountable κ , in the extension by $\text{Add}(\kappa, \kappa^+)^L$, every $\text{OD}_{H_{\kappa^+}}$ family of size at most κ^+ is pointwise $\text{OD}_{H_{\kappa^+}}$. The uniform ultrafilters on κ form a counterexample of size κ^{++} ; this sharpness argument works over every ZFC+GCH ground. More general cardinal hypotheses for the positive result are given in section 8.

1.2 Arithmetic and inner models

In the extension of L by ω_1 Cohen or random reals, no $\text{OD}_{\mathbb{R}}$ model of $I\Delta_0$ of size at most $\aleph_1$ has full binary standard system (Theorems 12.30 and 12.32). For $I\Delta_0$, binary standard systems use residues modulo the standard numerals 2^{n+1} ; total exponentiation is not assumed. The forcing-theoretic proofs pass to canonical exponential cuts (section 12.2); EFA is a tool inside those proofs, while the metatheory is ZF. A direct Borel argument applies to a fixed finite subtheory $F_{\text{Bor}} \subseteq I\Delta_0$ such that $\Pi_1^1\text{-CA}_0$ proves that no Borel quotient F_{Bor} -model realizes all traces for one fixed bounded binary-digit formula (Theorem 12.28). Because F_{Bor} is fixed and finite, this is one sentence of second-order arithmetic; it uses no finite axiomatizability assumption for $I\Delta_0$.

Let κ be regular uncountable and $V = L[G]$, where $G \subseteq \text{Add}(\kappa, \Lambda)^L$ is generic and $\Lambda > \kappa$ is any ground cardinal. Then $S^* = \text{HOD}_{H_{\kappa^+}}^V$ is a ZF inner model, closed under ambient sequences of length at most κ , containing $L(\mathcal{P}(\kappa)^{\check{V}})$, and containing no uniform ultrafilter on κ . It has no ambiently κ^+ -saturated $I\Delta_0$ model, of any size (Theorem 15.2). Every completion of $I\Delta_0$ nevertheless has a κ -saturated model of size κ in the smaller inner model. At $\Lambda = \kappa^+$, the small-family theorem rules out definable presentations of κ^+ -saturated models of size at most κ^+ .

Consequently, for every infinite regular κ of L there is a cardinal- and cofinality-preserving GCH extension in which saturated arithmetic models of size κ^+ exist but have no $\text{OD}_{\mathcal{P}(\kappa)}$ copies (Corollary 15.9). ZFC proves that every singular strong limit κ admits an OD κ^+ -saturated elementary extension of $\mathbb{N}$ of size 2^κ (Theorem 12.41). Under GCH its size is κ^+ , so the regularity restriction is exact.

The collapse theorem also applies to the relative Feferman–Levy model $M_0 = W_0(\mathcal{R}, \mathcal{A})$, where $\mathcal{A}$ lists the finite-stage real sets and $\mathcal{R}$ is their union. It equals the corresponding ambient hereditary-definability model (Theorem 13.3). Its reals are exactly $\mathcal{R}$, a countable union of countable sets internally; it satisfies $\text{AC}_\omega^{\text{fin}}$, and every infinite set in it has a countably infinite subset.

1.3 Proof structure

To prove the small-family theorems, we first find minimal nonempty invariant subfamilies and then show that each has one member. For one Cohen real, this uses Kuratowski–Ulam and generic orbit comparison [30, 32]. For generalized Cohen forcing, we conjugate small automorphism groups into piecewise translations. For random forcing, we also compare equivalent probability measures. At singular countable cofinality, commuting coordinate automorphisms give invariant enumerations. At higher cofinality, we preserve the maps already constructed and use a fixed chain of proper factors to eliminate constant comparisons. At singular strong limits, the same small-index construction on the forcing coordinates works for Cohen and random forcing. The

collapse proof compares three factors and uses variable-depth prefix codes. For the full Lévy collapse, relative homogeneity lets us extend isomorphisms onto whole bounded initial algebras. Random or Cohen witnesses then give a binary tree of automorphisms whose branches produce 2^c inequivalent members of any definable family containing a member without the relevant regularity property.

Definition-code lemmas transfer the arithmetic problems to hereditary inner models. Category or probability traces and uniformization then produce small cointial families above arithmetic cuts, contradicting full standard systems or saturation. The positive saturation theorem uses all codes for small ultrafilter constructions in iterated finite-support ultrapowers. It does not require a definable choice of one such code. For Borel quotient presentations, analytic separation makes the remainder decoder Borel, while the Harrington–Marker–Shelah representation and a coded lexicographic argument supply the required countable cointial family.

2 Preliminaries

All Boolean algebras used for forcing are complete, and their embeddings preserve arbitrary joins. Computations involving names, automorphisms, and Boolean products take place in the ground model W . Write $\|\varphi\|_{\mathbb{B}}$ for a Boolean truth value and $\mathbb{B} \restriction b = \{a \in \mathbb{B} : a \leq b\}$, with unit b . For Boolean-valued forcing and the intermediate model theorem see [17]; our category conventions are those of [21].

Definition 2.1. A $\mathbb{B}$ -name τ is *invariant* if $1 \Vdash g\tau = \tau$ for every $g \in \text{Aut}^W(\mathbb{B})$. More generally, it is fixed by a subgroup if these equalities hold for that subgroup. Invariance always means forced equality, not literal equality of names.

Two elementary observations will be used repeatedly. First, if g is the identity on $\mathbb{B} \restriction b$, induction on names gives $b \Vdash g\tau = \tau$ for every name τ . Indeed $b \wedge g(a) = b \wedge a$ for every coefficient a , and the induction applies to the names attached to those coefficients. Second, if an invariant name is forced finite or countable and the full automorphism group has fixed algebra $\{0, 1\}$, its cardinality is decided by 1. The Boolean values of its possible cardinalities are fixed elements.

We use the same-extension theorem of Vopěnka–Hájek and Grigorieff: if G, H are W -generic filters on a complete Boolean algebra $\mathbb{B} \in W$ and $W[G] = W[H]$, then $H = gG$ for some $g \in \text{Aut}^W(\mathbb{B})$ [16, Theorem 3.5.1]; see also [26, Theorem 2.16]. Recall the covariance identity $(g\tau)^{gG} = \tau^G$.

Lemma 2.2 (Stable ground codes). *Suppose $W \models \text{ZFC} + \text{GA} + (V = \text{HOD})$. In every set-forcing extension V of W , the class W is parameter-free definable and every member of W is OD. If $W \subseteq W[a] \subseteq V$ is an intermediate model generated by a set of ordinals a , then $W[a]$ and a set-like well-order of its universe are definable in V from a and ordinals.*

Proof. Usuba’s downward-directed grounds theorem [31] implies that W is the mantle of V . Indeed, any other ground of V has a common ground below it and W ; the intermediate model theorem makes this a ground of W , and GA makes it equal to W . The mantle has a uniform parameter-free definition. Relativizing the canonical HOD well-order to this class gives ordinal codes for all members of W , including forcing notions and names.

For the intermediate model, fix a W -name for a and use the complete subalgebra generated by its Boolean membership values. Its generic trace is recoverable from a and the ground name; the intermediate model theorem identifies its extension with $W[a]$. Order its names by the ground

order and give each value its least name code. This defines the asserted well-order from a and ordinals. The same argument applies in every further set-forcing extension. No Ground Axiom or HOD assumption on $W[a]$ is required. $\square$

Lemma 2.3 (Invariant names give ordinal definitions). *Let $W \models \text{ZFC}$, let $\mathbb{B} \in W$ be a complete Boolean algebra, and let $V = W[G]$. Suppose that $a \in W$ is a set of ordinals, W is definable in V from a and ordinals, and $\mathbb{B}, \tau \in W$ are $\text{OD}(a)$ in V . If τ is invariant, then τ^G is $\text{OD}(a)$ in V .*

Proof. In V , use the given definitions to form the set of values

$$\{\tau^H : H \subseteq \mathbb{B} \text{ is } W\text{-generic and } W[H] = V\}.$$

This definition is first-order: genericity quantifies over dense sets in the definable class W , and the last condition says that every set is the value of a ground-model name under H . The displayed collection is a set, by Separation on $\mathcal{P}(\mathbb{B})$ followed by Replacement. The same-extension theorem gives $H = gG$, and invariance gives $\tau^H = (g\tau)^{gG} = \tau^G$. Thus its unique member is $\text{OD}(a)$. $\square$

The hypotheses hold for $W = L[a]$ by its canonical well-order. They also hold for the intermediate grounds of Lemma 2.2, with the indicated parameter.

We obtain invariant names from definitions as follows. Suppose the full automorphism group of $\mathbb{B}$ has fixed algebra $\{0, 1\}$, and in $W[G]$ a formula with fixed ground parameters and ordinals defines a unique nonempty countable set. The Boolean value of this assertion is fixed and belongs to G , so it is 1. Choose a name forced to satisfy the definition. Uniqueness makes that name invariant. The same argument applies to a finite family or to any other property expressed using fixed parameters.

Lemma 2.4 (Quotient invariance). *Let $C \subseteq B$ be complete in a ground W , let F be B -generic, and put $U = W[F \cap C]$. If a ground B -name τ is invariant under the pointwise stabilizer $G_C = \{g \in \text{Aut}^W(B) : g \restriction C = \text{id}\}$, its quotient interpretation over U is invariant under all automorphisms of the quotient completion computed in U .*

Proof. Use the two-step presentation of B as C followed by its quotient, and let $\dot{Q}$ name that quotient completion. Given a quotient automorphism $\pi \in U$, choose a C -name and a condition in $F \cap C$ forcing it to be an automorphism. Mix the name with the identity outside that condition, so $1_C \Vdash \dot{\pi} \in \text{Aut}(\dot{Q})$ and its actual value is π . On the dense iteration presentation, the map

$$(d, \dot{q}) \mapsto (d, \dot{\pi}(\dot{q}))$$

is an automorphism, with inverse defined by $\dot{\pi}^{-1}$. Complete it and identify the iteration with B . This gives a ground $g \in G_C$ whose quotient action is π . The equality $1_B \Vdash g\tau = \tau$ therefore yields the corresponding forced equality in the quotient. This covers every automorphism in U ; it does not assume that an old completion remains complete there. $\square$

3 Countable families in a Cohen extension

Theorem 3.1 (Invariant enumeration for Cohen forcing). *In any ground model W of ZFC, let $\mathbb{B} = \text{RO}(2^\omega)^W$. Every invariant name forced to be a nonempty countable set has an invariant bijective enumeration by some $I \in (\omega \setminus \{0\}) \cup \{\omega\}$. The ranks of its members are unrestricted.*

The proof has two steps. We first find minimal nonempty invariant subfamilies. We then show that each such subfamily has exactly one member. The first step uses Kuratowski–Ulam; the second uses the generic orbit-equivalence theorem of Sullivan–Weiss–Wright.

3.1 Category and orbit comparison

Lemma 3.2. *If a group H acts on a complete Boolean algebra R with a countable order-dense subset, then R^H has a countable order-dense subset. Moreover $R^{H_0} = R^H$ for some countable subgroup $H_0 \leq H$.*

Proof. Let $(u_n)_{n < \omega}$ be order-dense in R^+ and put $v_n = \bigvee_{h \in H} h(u_n)$. If $0 < d \in R^H$ and $u_n \leq d$, then $v_n \leq d$, so the v_n are order-dense in R^H . The algebra R is ccc. For each n choose countably many h whose images of u_n have join v_n , and let H_0 be the subgroup they generate. If $d \in R^{H_0}$, then $d = \bigvee \{v_n : u_n \leq d\}$, so $d \in R^H$. $\square$

Lemma 3.3 (A countable collection of membership patterns). *Let I be nonempty and countable, let $r_i : 2^\omega \rightarrow 2^\omega$ be Borel, and put $S(x) = \{r_i(x) : i \in I\}$. If $S(x \oplus a) = S(x)$ on a comeager set for each $a \in 2^\omega$, then $S(x)$ has a fixed countable value on a comeager set.*

Proof. The relation $E(x, y)$ asserting $S(x) = S(y)$ is Borel. By Kuratowski–Ulam, $E(x, x \oplus a)$ holds on a comeager subset of the product. The map $(a, x) \mapsto (x, x \oplus a)$ is a homeomorphism, so E itself is comeager. Another application of Kuratowski–Ulam gives x_* such that $S(x) = S(x_*)$ for comeager many x . $\square$

An isomorphism between principal ideals is *piecewise given* by a group H of automorphisms if it agrees with members of H on the pieces of a countable Boolean partition of its domain. Its inverse has the same property, and compositions do too: refine the first partition by inverse images of the second.

Lemma 3.4 (Generic orbit comparison). *Let R, S be atomless complete Boolean algebras with countable order-dense subsets. Let $H \leq \text{Aut}(R)$ and $K \leq \text{Aut}(S)$ be countable, with $R^H = \{0, 1\}$ and $S^K = \{0, 1\}$. There is an isomorphism $\alpha : R \rightarrow S$ such that every $\alpha h \alpha^{-1}$ is piecewise given by K .*

Proof. Choose countable invariant order-dense Boolean subalgebras. Their Stone spaces are Cantor spaces. The induced actions are by homeomorphisms and are generically ergodic: an invariant Borel set has a fixed class in the category algebra, hence is meager or comeager. Sullivan–Weiss–Wright [30, Theorem 1.8] gives an orbit equivalence through a homeomorphism between invariant dense G_δ subspaces. To obtain this two-space formulation, compare each action to the canonical finite-bit-flip action and intersect the two invariant comeager subspaces there.

Direct image induces a complete Boolean isomorphism α . Enumerate $K = (k_n)_{n < \omega}$. For each $h \in H$, partition the target subspace according to the least n for which its conjugate under the orbit equivalence agrees pointwise with k_n . These sets are Borel. Their nonzero category classes give the required partition. $\square$

3.2 The Boolean algebra of subfamilies

We first work with Boolean algebras alone. Put $G = \text{Aut}(\mathbb{B})$ and $C = \mathbb{B}^I$, the complete Boolean direct product. This is a lottery of copies of $\mathbb{B}$, not forcing with independent Cohen generics. Write $\delta(b) = (b)_{i \in I}$ and let e_i be the unit of its i th slice. For $z \in C$ put

$$s(z) = \bigvee_{i \in I} z_i, \quad \iota_z(b) = \delta(b) \wedge z \quad (b \leq s(z)).$$

The map $\iota_z : \mathbb{B} \upharpoonright s(z) \rightarrow C \upharpoonright z$ is a complete embedding. It is onto exactly when the coordinates of z are pairwise disjoint; in that case its inverse is s . For the converse, if $z \wedge e_i = \iota_z(b_i)$, then $z_i \leq b_i$ and $b_i \wedge z_j = 0$ for $j \neq i$.

Suppose $\Theta : G \rightarrow \text{Aut}(C)$ is an action satisfying

$$\Theta_g \delta(b) = \delta(gb), \quad (1)$$

$$g \upharpoonright (\mathbb{B} \upharpoonright b) = \text{id} \implies \Theta_g \upharpoonright (C \upharpoonright \delta(b)) = \text{id}. \quad (2)$$

Lemma 3.5 (Full quotient actions). *Let $A \subseteq B$ be complete Boolean algebras in a ZFC ground, and let I be any ground set. After forcing with A , a full local diagonal-equivariant action on B^I induces such an action on Q^I , where Q is the quotient completion computed in the intermediate extension. It includes every automorphism of that completion. For a product probability algebra and a coordinate factor A , the assertion also holds for the measure-preserving groups.*

Proof. Use $B = A * \dot{Q}$. Every A -name for a Q -value has a representative in B ; coordinatewise the same holds for Q^I and B^I . For $b, c \in B$ their relative equality has A -value

$$\bigvee \{a \in A : a \wedge b = a \wedge c\};$$

for tuples take the meet over I . Diagonal equivariance makes the lifted maps respect this equality.

The lifting in Lemma 2.4 represents every named quotient automorphism by a ground automorphism fixing A , mixing with the identity where necessary. If two operator names agree below $a \in A$, their lifts agree on $B \upharpoonright a$; locality makes their cover actions agree below $\delta(a)$. Thus these actions define one full relative group action. The same argument on a relative principal algebra proves its locality, and diagonal equivariance descends directly. For product probability algebras, a conditionally measure-preserving quotient map has a globally measure-preserving lift by integration. No bound on I or restriction to ground-model quotient operators was used. $\square$

The least diagonal element above z is $\delta(s(z))$. Therefore

$$s(\Theta_g z) = g(s(z)), \quad \Theta_g \iota_z(b) = \iota_{\Theta_g z}(gb).$$

In particular, the action preserves the property of having disjoint coordinates. Put $D = C^{\Theta[G]}$. Every nonzero $d \in D$ has $s(d) = 1$, since $\mathbb{B}^G = \{0, 1\}$. To see the latter assertion, finite bit flips can carry a basic cylinder below any $0 < b < 1$ to a cylinder below its complement.

Proposition 3.6. *The algebra D is atomic.*

Proof. By Lemma 3.2, choose an order-dense sequence (v_n) in D^+ . Represent $v_{n,i}$ by a Borel set $V_{n,i}$ and define $r_i(x)(n) = 1$ exactly when $x \in V_{n,i}$. For the translation $t_a(x) = x \oplus a$ write $m_{ij}^a = (\Theta_{t_a} e_i)_j$. Each row partitions $1_{\mathbb{B}}$ by the disjoint-coordinate observation, and each column partitions $1_{\mathbb{B}}$ because the e_i partition 1_C . Since v_n is fixed,

$$m_{ij}^a \wedge t_a(v_{n,i}) = m_{ij}^a \wedge v_{n,j}.$$

Taking all n shows that $\{r_i(x \oplus a) : i \in I\}$ equals $\{r_i(x) : i \in I\}$ on a comeager set. By Lemma 3.3 there is a countable set S_* of patterns such that this collection equals S_* for comeager many x .

For $t \in S_*$ let $a_t = \bigwedge_n v_n^{t(n)}$, where $v_n^1 = v_n$ and $v_n^0 = \neg_C v_n$. Coordinatewise this is the category class of $\{x : r_i(x) = t\}$. Hence the a_t are disjoint and have join 1_C . If $0 < z \leq a_t$ belongs to D , choose $0 < v_n \leq z$. The element a_t decides v_n , so it must lie below v_n . Thus $z = a_t$, and every nonzero a_t is an atom. $\square$

Atomicity uses only diagonal equivariance. We next use locality to show that each minimal invariant subfamily is a singleton.

Lemma 3.7. *If $0 < d \in D$ and $\alpha : \mathbb{B} \rightarrow C \restriction d$ is an isomorphism, there are $q \leq d$, whose coordinates partition $1_{\mathbb{B}}$, and an isomorphism $U : C \restriction q \rightarrow C \restriction d$, piecewise given by $\Theta[G]$, such that $\alpha = U\iota_q$.*

Proof. Split each nonzero $\alpha^{-1}(d \wedge e_i)$ into two positive pieces and enumerate the resulting partition as (p_n) . Let i_n be the slice containing $\alpha(p_n)$. The map $b \mapsto \alpha(b)_{i_n}$ is an isomorphism between principal ideals of $\mathbb{B}$. Both units are proper and nonzero, because of the splitting. Both complementary ideals are Cohen algebras, so the map extends to $g_n \in G$. Thus, for $b \leq p_n$,

$$\alpha(b) = e_{i_n} \wedge \delta(g_n b).$$

Put $u_n = \Theta_{g_n}^{-1}\alpha(p_n)$ and $q = \bigvee_n u_n$. Each u_n has disjoint coordinates and support p_n . Consequently the coordinates of q partition $1_{\mathbb{B}}$. Also $q \leq d$, since d is fixed. The formula $U(z) = \bigvee_n \Theta_{g_n}(z \wedge u_n)$ gives the required isomorphism. The displayed identity shows $U\iota_q = \alpha$ on each p_n , hence everywhere. $\square$

Theorem 3.8. *Every atom of D has coordinates partitioning $1_{\mathbb{B}}$. There are exactly $|I|$ atoms, and their matrix has every row and every column a partition of $1_{\mathbb{B}}$.*

Proof. Choose a countable $\Gamma \leq G$ with $C^{\Theta[\Gamma]} = D$. The diagonal embedding shows $\mathbb{B}^{\Gamma} = \{0, 1\}$. For an atom $d \in D$, the action on $C \restriction d$ also has trivial fixed algebra. Both algebras are atomless and countably order-dense. Apply Lemma 3.4 to obtain $\alpha : \mathbb{B} \rightarrow C \restriction d$ such that $\alpha\gamma\alpha^{-1}$ is piecewise given by $\Theta[\Gamma]$. By Lemma 3.7, write $\alpha = U\iota_q$. Then

$$V_{\gamma} = U^{-1}\alpha\gamma\alpha^{-1}U = \iota_q\gamma\iota_q^{-1}$$

is piecewise given by $\Theta[G]$.

On a piece $u = \iota_q(b)$ where V_{γ} agrees with Θ_g , take $a \leq b$. Taking supports in $\Theta_g\iota_q(a) = \iota_q(\gamma a)$ gives $g(a) = \gamma(a)$. Locality implies that Θ_g and Θ_{γ} agree below $\delta(b)$, hence below u . On all the pieces together, $V_{\gamma} = \Theta_{\gamma} \restriction (C \restriction q)$, so $\Theta_{\gamma}(q) = q$. Thus $q \in D$; since $0 < q \leq d$ and d is an atom, $q = d$.

There are countably many atoms because C is ccc. Their join is 1_C , so every column partitions $1_{\mathbb{B}}$, and the preceding argument gives the row partitions. These partitions imply the number of rows is $|I|$. If there are n columns, $n + 1$ rows are impossible: expand the meet of their row joins, and every term has two disjoint entries in one column. Transposing the matrix gives the reverse inequality when the number of rows is finite. The countably infinite case follows as well. $\square$

3.3 Applying the calculation to names

Proof of Theorem 3.1. Choose names $(\tau_i)_{i \in I}$ forced to enumerate the given family $\dot{A}$ bijectively. Set

$$m_{ij}^g = \|g\tau_i = \tau_j\|_{\mathbb{B}}, \quad (\Theta_g z)_j = \bigvee_i (g(z_i) \wedge m_{ij}^g).$$

The matrix has partition rows and columns, and

$$m_{ik}^{gh} = \bigvee_j (g(m_{ij}^h) \wedge m_{jk}^g).$$

Thus Θ is an action by complete automorphisms and satisfies (1). The rank induction on names from section 2 gives (2).

For a name σ forced to belong to $\dot{A}$, the element $q_\sigma = (\|\sigma = \tau_i\|)_i$ has coordinates partitioning 1, and $\Theta_g q_\sigma = q_{g\sigma}$. Enumerate the atoms of D as $(d_k)_{k \in I}$. Mix the names τ_i along the k th row: $d_{k,i} \Vdash \sigma_k = \tau_i$. Then $q_{\sigma_k} = d_k$. Since this atom is fixed, σ_k is invariant. Disjointness of the columns gives pairwise distinctness of the σ_k , and the column joins show that every member of the family occurs. The enumeration name is itself invariant. $\square$

In this calculation, $z = (z_i) \in \mathbb{B}^I$ codes the subfamily in which τ_i is present with Boolean value z_i . An atom of D is a minimal nonempty invariant subfamily. Its disjoint coordinates say that exactly one member is selected.

Corollary 3.9. *If a is a set of ordinals and c is Cohen over $L[a]$, every countable $\text{OD}(a)$ set in $L[a, c]$ has an $\text{OD}(a)$ bijective enumeration by a finite ordinal or by ω .*

Proof. More generally, use any ground with the definability and ordinal codes of Lemma 2.3; these are supplied by Lemma 2.2 in the grounds of the introduction. The definition gives an invariant name, to which we apply Theorem 3.1 and Lemma 2.3. The empty enumeration handles the empty set. $\square$

Lemma 3.10. *Let $W \models \text{ZFC}$, let c be Cohen over W , and let $a \in W[c]$ be a set of ordinals. Then either $W[a] = W[c]$ or $W[c]$ is a Cohen-real extension of $W[a]$.*

Proof. By the intermediate model theorem, $W[a] = W[H]$ for the generic trace on a complete subalgebra $\mathbb{B}_0$ of the Cohen algebra in W . Let $\pi : \mathbb{B} \rightarrow \mathbb{B}_0$ take an element to its least upper bound in $\mathbb{B}_0$, and fix a countable order-dense $P \subseteq \mathbb{B}^+$ in W . The quotient is represented by $Q = \{b \in \mathbb{B}^+ : \pi(b) \in H\}$. For $b \in Q$,

$$\pi(b) = \bigvee \{\pi(p) : p \in P, p \leq b\}.$$

Genericity for this ground-model join supplies $p \leq b$ in $P \cap Q$. Thus the quotient completion is countably order-dense. If its generic meets an atom it adds nothing. Otherwise the generic lies below the complement of the join of its atoms, a Cohen algebra. See also [19, Theorem 4]. $\square$

Theorem 3.11. *In $L[c]$, for c Cohen over L and every set of ordinals $a \in L[c]$, every countable $\text{OD}(a)$ set has an $\text{OD}(a)$ bijective enumeration. In particular all its members are $\text{OD}(a)$.*

Proof. Work more generally with $V = W[c]$, where W satisfies the hypotheses of Lemma 2.2. Put $N = W[a]$. If $N = V$, its $\text{OD}(a)$ well-order chooses an enumeration. Otherwise V is a Cohen extension of N by Lemma 3.10. The definition gives an invariant name for the family. Apply Theorem 3.1 to obtain an invariant enumeration. The ordinal-and- a codes in N , followed by Lemma 2.3, make that enumeration $\text{OD}(a)$ in V . The empty set has its empty enumeration. $\square$

Corollary 3.12 (Uniformization). *In $L[c]$, let a be a set of ordinals and let $R \in \text{OD}(a)$ be a set of ordered pairs with real first coordinates and countable vertical sections. There is an $\text{OD}(a)$ sequence $(f_n)_{n < \omega}$ of functions on $\text{dom}(R)$ with $R_x = \{f_n(x) : n < \omega\}$. In particular R has an $\text{OD}(a)$ uniformization.*

Proof. Each nonempty R_x is $\text{OD}(a, x)$, and (a, x) is coded by a set of ordinals. Choose the least $\text{OD}(a, x)$ surjection $e_x : \omega \rightarrow R_x$ in the canonical definable well-order of $\text{OD}(a, x)$, repeating entries for finite sections. This selection is uniform in x . Replacement gives $f_n(x) = e_x(n)$. $\square$

Remark 3.13. For families of reals, the invariant enumeration has an invariant real code. Each bit has Boolean value 0 or 1, so the code belongs to the ground. This recovers the rank-one Cohen theorem. At arbitrary rank, invariant values need not lie in the ground; the conclusion is ordinal definability, supplied by Lemma 2.3.

4 One generalized Cohen subset

Throughout this section, κ is regular uncountable and $\kappa^{<\kappa} = \kappa$ in the ZFC ground W . Fix a set Ω of cardinality κ , and write

$$P = \text{Fn}_{<\kappa}(\Omega, 2), \quad B = \text{RO}(P), \quad G = \text{Aut}^W(B).$$

We identify each condition with its nonzero Boolean value, which we call *basic*. All constructions in this section take place in W . The group Γ consists of translations $\tau_u(x) = x \triangle u$ for $u \in [\Omega]^{<\kappa}$, and $\Sigma = \text{Sym}(\Omega)$ acts by permuting coordinates. The group Γ has cardinality κ and acts weakly homogeneously on B . Write $[\Gamma]$ for the group of automorphisms which agree with members of Γ on the pieces of a Boolean partition of 1. Such partitions have size at most κ , since B has order density κ .

Theorem 4.1 (Invariant enumeration). *Every invariant B -name forced to be a nonempty set of cardinality at most κ has an invariant bijective enumeration by a ground cardinal $\nu \leq \kappa$. The ranks of its members are unrestricted.*

Theorem 4.2. *Let κ be regular uncountable in L and let $c \subseteq \kappa$ be $\text{Add}(\kappa, 1)^L$ -generic. In $L[c]$, every OD family of cardinality at most κ has an OD bijective enumeration by an ordinal at most κ . In particular, every member of the family is OD.*

We prove the invariant-name theorem first. For $0 < |I| \leq \kappa$, put $C = B^I$. Use the diagonal embedding δ , support s , and maps ι_z from section 3. A *selector* is $q \in C$ whose coordinates partition 1_B ; equivalently, $s(q) = 1$ and $\iota_q : B \rightarrow C \restriction q$ is an isomorphism. A *frame* is a partition of 1_C into selectors. Suppose $\Theta : G \rightarrow \text{Aut}(C)$ satisfies diagonal equivariance and locality, as in (1) and (2). For a subgroup $H \leq G$, write C^H for its fixed algebra under Θ . We will prove that the atoms of C^G form a frame.

Coordinate permutations and short translations first give a frame in C^Γ . We then compare small groups of Boolean automorphisms to show that all of G fixes this frame.

4.1 Atomicity of the fixed algebra

The following permutation-kernel calculation will also be used at singular width.

Lemma 4.3. *If a Boolean algebra D has order density at most an infinite cardinal λ , then $\text{Aut}(D)$ has at most λ pairwise elementwise-commuting nonabelian subgroups.*

Proof. Fix an order-dense $P \subseteq D^+$ of size at most λ . For noncommuting f, g , choose $d > 0$ with $gf(d) \wedge fg(d) = 0$, then choose successively

$$b \in P, \quad b \leq g(d), \quad p \in P, \quad p \leq d \wedge g^{-1}(b), \quad a \in P, \quad a \leq f(p), \quad q \in P, \quad q \leq g(a).$$

Thus $g(p) \leq b$, $a \leq f(p)$, $q \leq g(a)$, and $f(b) \wedge q = 0$. Assign such a quadruple to a noncommuting pair in each subgroup. If (f, g) and (f', g') receive the same quadruple, then $q \leq g'f(p)$ whereas $q \wedge fg'(p) = 0$, so f and g' do not commute. There are at most λ quadruples. $\square$

Lemma 4.4 (Branch permutation kernel). *Let G have at most λ pairwise elementwise-commuting nonabelian subgroups, where λ is infinite.*

- (i) *If $\text{cf}(\kappa) = \omega < \kappa$ and $\lambda \leq \kappa$, every homomorphism $\text{Sym}(\kappa) \rightarrow G$ killing finitary permutations kills countably supported permutations.*

- (ii) If $\lambda < 2^{\aleph_0}$, every homomorphism $\text{Sym}(\omega) \rightarrow G$ killing finitary permutations is trivial.
- (iii) If κ is regular uncountable, $\kappa^{<\kappa} = \kappa$, and $\lambda \leq \kappa$, every homomorphism $\text{Sym}(\kappa) \rightarrow G$ killing permutations of support less than κ is trivial.

These conclusions apply to $G = \text{Aut}(D)$ whenever D has order density at most λ , and to $G = \text{Sym}(J)$ when $|J| \leq \lambda$.

Proof. For the three cases use the trees $\kappa^{<\omega}$, $2^{<\omega}$, and $2^{<\kappa}$, respectively. Identify the coordinate set with five copies of the tree together with a reserve of the same cardinality. For each branch, repeat the same copy of A_5 on the five copies of every node. Distinct branch copies have commuting images: their commutators are supported on their common nodes, finitely many in (i)–(ii) and fewer than κ in (iii). The number of branches exceeds λ , by König's theorem in (i) and Cantor's theorem in the other cases. Simplicity of A_5 and the hypothesis on G therefore put one branch copy in the kernel.

Its double transposition is an involution with θ pairs and as many fixed points as the coordinate set, where $\theta = \omega$ in (i)–(ii) and $\theta = \kappa$ in (iii). The normal closure of any such involution contains every involution with at most θ pairs. For one with exactly θ pairs, divide the pairs into two equally large parts; both restrictions are conjugate to the given involution. For one with fewer pairs, multiply by a disjoint conjugate on its fixed region, leaving a reserve of fixed points; the product is another conjugate. Finally, every permutation is a product of two involutions, by reflections on its finite cyclic and bilateral infinite orbits. This proves all three assertions. For the final assertion use Lemma 4.3, taking $D = \mathcal{P}(J)$ in the permutation case. $\square$

Lemma 4.5. *Let κ be regular uncountable with $\kappa^{<\kappa} = \kappa$, let $B = \text{RO}(\text{Add}(\kappa, 1))$, and let $C = B^I$, where $0 < |I| \leq \kappa$. Let Γ be the translations with support less than κ and let $\Sigma = \text{Sym}(\kappa)$ act by coordinate permutations. Suppose Θ is a local diagonal-equivariant action of $\Gamma \rtimes \Sigma$ on C . Then C^Γ is atomic with at most κ atoms, and Σ fixes it pointwise.*

Proof. Identify the coordinate set Ω with κ . Write $\delta(b) = (b)_{i \in I}$ and $s(c) = \bigvee_{i \in I} c_i$. The scalar Γ action is weakly homogeneous, since a bounded translation makes any two basic conditions compatible. Thus every nonzero Γ -fixed element of C has support 1.

Put $D = C^\Gamma$. Choose a dense sequence $(u_\alpha : \alpha < \kappa)$ in C^+ and put

$$v_\alpha = \bigvee_{\gamma \in \Gamma} \Theta_\gamma(u_\alpha).$$

These elements are order-dense in D , since $0 < u_\alpha \leq d \in D$ implies $0 < v_\alpha \leq d$. Hence D has order density at most κ . The group Σ normalizes Γ , and therefore acts on D . If $\sigma \in \Sigma$ has short support, it is the identity on the positive cone assigning zero to its support. For $d \in D$, locality makes $d \triangle \Theta_\sigma(d)$ vanish on this cone. That difference is Γ -fixed, so its support cannot be 1 unless the difference is zero. Every such σ therefore fixes D pointwise. Lemma 4.4 makes the entire Σ action on D trivial.

To prove atomicity, we use only the fact that Σ fixes D pointwise. Set $X = 2^\kappa$ with its bounded topology. Here a κ -meager set is a union of at most κ nowhere-dense sets, and κ -Borel means generated from open sets by complements and unions of at most κ sets. These sets have the κ -Baire property. Their quotient modulo κ -meager sets is B , respecting joins and meets of at most κ elements.

Represent the i th coordinate of v_α by its regular-open set $V_{\alpha i} \subseteq X$, and define

$$r_i(x)(\alpha) = 1 \iff x \in V_{\alpha i}, \quad S(x) = \{r_i(x) : i \in I\}. \quad (**)$$

For each fixed $\sigma \in \Sigma$, diagonal equivariance means that Θ_σ acts by a Boolean-valued permutation of the sheets over the scalar action of σ . Its rows and columns partition 1. Since all v_α are fixed, these partitions identify the patterns in (**) at x and σx off a κ -meager set. Only κ partition and coordinate identities are involved for this fixed σ . Consequently

$$S(\sigma x) = S(x) \quad \text{for } \kappa\text{-comeager many } x. \quad (***)$$

This argument does not assert uniform regularity of the coefficients as a function of σ .

We show that (***), for all σ , makes S constant on a κ -comeager set. Give Σ its topology of short partial-bijection neighborhoods. This space, X , and their finite products have bases of size κ and are κ -Baire. A recursion of length κ meets κ dense open requirements. For a permutation, include requirements that put every coordinate in its domain and range. The union at each proper stage still has size less than κ .

The Kuratowski–Ulam argument applies with these bases. For a closed nowhere-dense subset of a product and each basic open set in the second factor, the first coordinates admitting a smaller rectangle disjoint from it form a dense open set. Intersecting these κ sets shows that its sections are nowhere dense off a κ -meager set. Taking κ unions gives the assertion for meager sets. In particular, if every section of a relation with the κ -Baire property is comeager, then the relation is comeager. Otherwise its complement is comeager on a nonempty rectangle, contradicting the section assertion.

Let X_{bal} consist of the points having κ zeros and κ ones. It is dense and κ -comeager. The map

$$\Phi : \Sigma \times X_{\text{bal}} \longrightarrow X_{\text{bal}} \times X_{\text{bal}}, \quad \Phi(\sigma, x) = (x, \sigma x)$$

is a continuous open surjection. In fact the image of $[t] \times ([p] \cap X_{\text{bal}})$, for a short partial bijection t , consists exactly of the balanced pairs (x, y) such that

$$x \supseteq p, \quad y(t(i)) = x(i) \quad (i \in \text{dom } t).$$

This set is open. Match the remaining zero and one coordinates to extend t to the required permutation. This proves both the description of the image and surjectivity.

The relation $E(x, y) \iff S(x) = S(y)$ is κ -Borel: each inclusion has the form $\forall i \in I \exists j \in I \forall \alpha < \kappa$ of equality tests from (**). Independently of the lift coefficients, define $R(\sigma, x) = E(x, \sigma x)$. It is κ -Borel by continuity of the coordinate action. By (***) and the section argument, R is comeager. On $\Sigma \times X_{\text{bal}}$ it is $\Phi^{-1}(E)$. An open continuous surjection pulls back meager sets to meager sets. Thus, if the complement of E were comeager on a nonempty open set, its inverse image would contradict comeagerness of R . Hence E is comeager on X_{bal}^2 , and therefore on X^2 . Applying the section argument once more, choose x_* with comeager E -section. Then $S(x) = S_* := S(x_*)$ on a comeager set, and $|S_*| \leq \kappa$.

For $t \in S_*$ put

$$a_t = \bigwedge_{\alpha < \kappa} v_\alpha^{t(\alpha)}, \quad v^1 = v, \quad v^0 = \neg v.$$

These elements belong to D . Their coordinate representatives are $\{x : r_i(x) = t\}$ modulo κ -meager sets, so they are disjoint and join to 1. Every nonzero a_t is an atom: if $0 < d \leq a_t$ in D , choose $0 < v_\alpha \leq d$. Since a_t decides v_α , this decision must be positive. Hence $a_t \leq v_\alpha \leq d \leq a_t$. This proves atomicity and the stated bound. $\square$

4.2 The initial frame

We retain the notation $P, B, C, \delta, s, \Gamma, \Sigma, \Theta$. A *partial selector* has pairwise disjoint coordinates and is a *selector* if its scalar support is 1. Below a partial selector u , the support map is a complete isomorphism

$$s : C \upharpoonright u \longrightarrow B \upharpoonright s(u), \quad s^{-1}(b) = \delta(b) \wedge u.$$

A *frame* is a partition of 1_C into selectors. We will show that the atoms supplied by Lemma 4.5 form a frame, also when $|I| = \kappa$.

We use the topology on $\Sigma = \text{Sym}(\Omega)$ generated by restrictions of permutations to fewer than κ coordinates. Inverse restrictions give the same topology. Its basis has size κ , and intersections of fewer than κ open sets are open. We use the κ -Borel and κ -meager conventions of the preceding proof; in particular κ -Borel sets have the κ -Baire property.

Lemma 4.6 (Uniform permutation factorization). *Allow also $\kappa = \omega$, with the usual Borel structure and finite-restriction topology. Fix an involution $r \in \Sigma$ having κ transposed pairs and κ fixed points. There are κ -Borel maps $a_m : \Sigma \rightarrow \Sigma$, $m < 4$, such that*

$$\sigma = \prod_{m < 4} a_m(\sigma) r a_m(\sigma)^{-1}, \quad (3)$$

with composition from right to left.

Proof. Identify Ω with κ . On each σ -orbit, choose its least element o and write the orbit as $\sigma^n(o)$, with indices in $\mathbb{Z}/d\mathbb{Z}$ for a finite orbit and in $\mathbb{Z}$ otherwise. The two reflections

$$\beta_\sigma(\sigma^n(o)) = \sigma^{-n}(o), \quad \alpha_\sigma(\sigma^n(o)) = \sigma^{1-n}(o)$$

are involutions with $\sigma = \alpha_\sigma \beta_\sigma$. For uncountable κ , both depend continuously on σ : the value at a coordinate is determined by its countable orbit, and fewer than κ such orbits still have size less than κ . For $\kappa = \omega$ they are Borel, since orbit membership and the choice of its least element use only countably many iterate tests.

Call an involution *balanced* if it has κ pairs and κ fixed points. We next factor every involution τ into two balanced involutions, uniformly κ -Borel. Put

$$L(\tau) = \{\xi < \kappa : \xi < \tau(\xi)\}, \quad F(\tau) = \{\xi < \kappa : \tau(\xi) = \xi\}.$$

The condition $|L(\tau)| = \kappa$ is κ -Borel, since it says that $L(\tau)$ is unbounded (infinite when $\kappa = \omega$). When this holds, enumerate $L(\tau)$ in increasing order and divide its index set into two fixed κ -sized parts. Restrict τ to the pairs in each part, fixing all other points. These restrictions are balanced, have disjoint supports, and multiply to τ .

In the other case $|F(\tau)| = \kappa$. Using its increasing enumeration, divide $F(\tau)$ into three indexed sets of size κ , pair the first two sets, and leave the third fixed. Let v exchange those pairs and fix the moved points of τ . Then v and τv are balanced and $(\tau v)v = \tau$. These constructions are continuous on their respective domains. Indeed, the η -th member of either increasing enumeration, and the rank of a specified member, are determined by a restriction of τ to a bounded initial segment. Thus both factor maps are κ -Borel.

Finally, for a balanced involution η , match in increasing order the smaller pair members of r with those of η , their partners with their partners, and the fixed points with the fixed points. The resulting permutation $c(\eta)$ satisfies $c(\eta)rc(\eta)^{-1} = \eta$. This choice is continuous on the balanced subspace, by the same enumeration argument. Apply the two-factor construction to α_σ and β_σ ,

then apply c to the resulting four balanced involutions. This gives (3). For uncountable κ , the coordinatewise continuity assertions give continuity in the stated topology by combining fewer than κ output requirements into one short input restriction. At $\kappa = \omega$, all the choices just described are ordinary Borel; continuity of the orbit reflections is not needed. $\square$

Lemma 4.7 (Borel permutation lifts). *Let $\kappa = \omega$, or let κ be regular uncountable with $\kappa^{<\kappa} = \kappa$. Put $\Omega = \kappa$, $P = \text{Fn}_{<\kappa}(\Omega, 2)$, $B = \text{RO}(P)$, $\Gamma = [\kappa]^{<\kappa}$ acting by bit translations, and $\Sigma = \text{Sym}(\kappa)$. Suppose a local diagonal action of $\Gamma \rtimes \Sigma$ on B^J , $0 < |J| \leq \kappa$, is Γ -ergodic. Assume also that every set action of Σ on at most κ points which kills short-support permutations is trivial. Then the actual map $\sigma \mapsto \Theta_\sigma$ is κ -Borel in Boolean-matrix coordinates. For $\kappa = \omega$, “ κ -Borel” means ordinary Borel.*

Proof. Write e_j for the coordinate selectors and δ, s for the diagonal and support maps. The four-conjugates factorization above applies in both cases.

The fiber centralizer. Let D be the group of complete automorphisms of B^J that fix the scalar diagonal pointwise and commute with every Θ_{τ_t} , $t \in [\Omega]^{<\kappa}$. For $d, e \in D$, the part of the cover where their Boolean permutations of the fiber labels agree is Γ -invariant. It is therefore 0 unless $d = e$. Consequently, for any selector q , the selectors $d(q)$, $d \in D$, are pairwise disjoint. Since B^J has order density at most κ , this proves

$$|D| \leq \kappa.$$

The same argument shows that two members of D which agree on any nonzero part of the cover agree everywhere.

Conjugation by Θ_σ gives an ordinary homomorphism $\Sigma \rightarrow \text{Sym}(D)$. If σ is supported on a short set E , it is the identity on the entire principal scalar algebra below the basic condition assigning zero on E . Locality makes its lift the identity there as well. Every member of D preserves this principal cover algebra, so conjugation by Θ_σ agrees with the identity on D , by the preceding agreement argument. The assumed set-action property makes the whole homomorphism trivial. Thus

$$\Theta_\sigma d = d \Theta_\sigma \quad (\sigma \in \Sigma, d \in D).$$

Call L a *scalar- σ intertwiner* if it covers σ on the diagonal and $L\Theta_{\tau_t}L^{-1} = \Theta_{\tau_{\sigma[t]}}$ for every t . Every such map has the form $d\Theta_\sigma$ for a unique $d \in D$. Hence, for every $\rho \in \Sigma$,

$$L\Theta_\rho L^{-1} = \Theta_{\sigma\rho\sigma^{-1}}. \quad (4)$$

Thus the conjugation does not depend on the choice of intertwiner. We do not assume that D is commutative.

Closed charts for intertwiners. Fix $i \in J$. A seed is a pair (p, j) with $p \in P$ and $j \in J$. For this seed put

$$e = \delta(p) \wedge e_i, \quad u_t = \Theta_{\tau_t}(e), \quad v_t(\sigma) = \Theta_{\tau_{\sigma[t]}}(\delta(\sigma p) \wedge e_j).$$

The two families $(u_t)_t$ and $(v_t(\sigma))_t$ consist of partial selectors and each has join 1, by ergodicity. Their supports satisfy $s(v_t(\sigma)) = \sigma s(u_t)$. Declare the seed valid at σ if

$$\sigma s(u_t \wedge u_z) = s(v_t(\sigma) \wedge v_z(\sigma)) \quad (t, z \in [\Omega]^{<\kappa}). \quad (5)$$

On the validity set there is a unique scalar- σ intertwiner taking e to $\delta(\sigma p) \wedge e_j$, namely

$$L(w) = \bigvee_t (\delta(\sigma s(w \wedge u_t)) \wedge v_t(\sigma)). \quad (6)$$

To verify this, identify each principal algebra below u_t or $v_t(\sigma)$ with its scalar support. Equation (5) says precisely that the canonical scalar- σ maps on these principal algebras agree on overlaps, and that their inverses agree on overlaps. As both families cover 1, they paste to a complete isomorphism given by (6). Translating the index t proves the intertwining identity. Because the families cover 1, the specified seed determines the intertwiner uniquely.

Each validity set is closed. For fixed t, z , the right side of (5) is locally constant: it depends only on σ on $t \cup z \cup \text{dom}(p)$. For fixed $b \in B$ and basic q , the condition $q \leq \sigma b$ is clopen, since it is equivalent to $\sigma^{-1}q \leq b$ and $\sigma^{-1}q$ is locally constant. Equality in (5) is therefore a closed condition, tested by all basic q . There are at most κ seeds, and their validity sets cover Σ . To see this, the actual lift Θ_σ takes some positive portion of e_i into an e_j . A basic restriction of its source then supplies a valid seed. A valid seed need not be realized by the actual lift.

Recovering the actual action. Well-order the seeds and choose the first valid one for each σ . The choice pieces are κ -Borel. Formula (6) gives a κ -Borel choice $\sigma \mapsto L_\sigma$ in Boolean-matrix coordinates. For the coding, represent $b \in B$ by $\{p \in P : p \leq b\}$ and an automorphism of the cover lying over a coordinate permutation by its scalar map and the coordinates of the images of the e_j . The maps $v_t(\sigma)$ are locally constant, and applying σ to a fixed Boolean element is coordinatewise continuous by the preceding inclusion test. Boolean meets and complements are κ -Borel on these codes. Joins of at most κ elements are κ -Borel as well, using

$$p \leq \bigvee_{\xi} b_{\xi} \iff (\forall q \in P, q \leq p)(\exists r \in P, r \leq q)(\exists \xi) r \leq b_{\xi}. \quad (7)$$

All quantifiers here range over sets of size at most κ . The joint scalar action is also κ -Borel, since $\sigma^{-1}p$ has at most κ possible, locally specified values. Matrix composition uses κ joins of meets and this scalar action; inversion uses the scalar inverse and transposed coefficients. They are consequently κ -Borel, and so is the choice of intertwiners. No regularity of Θ has been assumed.

Fix the actual lift Θ_r of the involution in Lemma 4.6. Equation (4) and the factorization there give

$$\Theta_\sigma = \prod_{m < 4} L_{a_m(\sigma)} \Theta_r L_{a_m(\sigma)}^{-1}. \quad (8)$$

Thus the *actual* map $\sigma \mapsto \Theta_\sigma$ is κ -Borel in its Boolean-matrix code. □

Theorem 4.8 (Initial frame). *Let κ be regular uncountable with $\kappa^{<\kappa} = \kappa$, and let $0 < |I| \leq \kappa$. For every local diagonal-equivariant action of $\Gamma \rtimes \Sigma$ on $C = B^I$, the atoms of C^Γ form a frame. Every member of this frame is fixed by $\Gamma \rtimes \Sigma$.*

Proof. By Lemma 4.5, C^Γ is atomic and Σ fixes it pointwise. Fix an atom U . Its scalar support is 1, because Γ is weakly homogeneous on B . Moreover, its fiber cardinality is forced to be a constant ground cardinal $0 < |J| \leq \kappa$. Indeed, the Boolean values of the possible cardinalities are Γ -invariant, and B preserves cardinals at most κ . Choose a Boolean enumeration of its fibers and identify $C \restriction U$ with B^J . Write e_j for the selectors of this chosen frame and 1 for U . This choice need not be invariant. The restricted Γ -action is ergodic, meaning that its fixed algebra is $\{0, 1\}$. The restricted Σ -action is defined because Σ fixes U .

By Lemma 4.4, the set-action hypothesis of Lemma 4.7 holds. Hence the actual permutation action on this ergodic cover is κ -Borel.

An open stabilizer. Fix $i \in J$. The κ -Borel sets

$$E_{p,j} = \{\sigma : \Theta_\sigma(\delta(p) \wedge e_i) = \delta(\sigma p) \wedge e_j\}$$

cover Σ . To see that Σ is κ -Baire, start in any basic open set and extend a short partial permutation through κ steps. At these steps avoid the prescribed nowhere dense sets and put every coordinate into its domain and range. At every proper stage the union is still a short partial permutation; the final union is a permutation. Hence some $E_{p,j}$ is nonmeager. Subdivide it by the at most κ possible values $q = \sigma p$, and choose a nonmeager piece $E_{p,j,q}$ and σ_0 in that piece. If $e = \delta(p) \wedge e_i$, then

$$\sigma_0^{-1} E_{p,j,q} \subseteq H_e := \{\sigma : \Theta_\sigma(e) = e\}.$$

The subgroup H_e is κ -Borel and nonmeager, so it is open. Indeed, its Baire property makes it comeager on some nonempty open set O . For all σ in a sufficiently small identity neighborhood, $O \cap \sigma O$ contains a nonempty open set. The two translates of H_e are comeager there, so they intersect; hence $\sigma \in H_e H_e^{-1} = H_e$.

It follows that H_e contains every permutation fixing some $E \subseteq \Omega$ pointwise, where $|E| < \kappa$. Enlarge E to contain $\text{dom}(p)$ and choose a full E -pattern $b \leq p$. The partial selector $w = \delta(b) \wedge e_i$ is fixed by all permutations of $\Omega \setminus E$.

A selector fixed by all bounded translations. Let τ_t be a bounded translation with $t \cap E = \emptyset$. Below any basic $q \leq b$, choose a short set $F \subseteq \Omega \setminus E$ containing $t \cup (\text{dom}(q) \setminus E)$, with two disjoint fresh sets of a common infinite size $\mu < \kappa$ at least as large as this required set. Extend q to a full pattern a on $E \cup F$, assigning zeros to one fresh set and ones to the other. Both $a|_F$ and $(\tau_t a)|_F$ then have exactly μ zeros and μ ones. There is a permutation ρ supported on F taking the first pattern to the second. The scalar maps ρ and τ_t agree on the entire principal algebra below a : they have the same output pattern and fix every undecided coordinate. Locality therefore makes their lifts agree below $\delta(a)$. Since ρ fixes w ,

$$\Theta_{\tau_t}(w) \wedge \delta(\tau_t a) = w \wedge \delta(\tau_t a).$$

Such a are dense below b , and $\tau_t b = b$. Hence $\Theta_{\tau_t}(w) = w$.

For each full E -pattern c , let $t_c \subseteq E$ take b to c . The join

$$v = \bigvee_{c \in 2^E} \Theta_{\tau_{t_c}}(w)$$

is a selector: its summands are partial selectors whose pairwise disjoint scalar supports c cover 1. Tail translations fix every summand by commutation, and translations on E permute the summands. Thus v is Γ -invariant. Ergodicity gives $v = 1$, showing that the original atom U is itself a selector.

Every atom of C^Γ is consequently a selector. These atoms form a frame, fixed also by Σ by Lemma 4.5. Their ground set has cardinality at most κ , by the density of C . Forcing with B evaluates the atom partition to a bijection between that ground set and I . Preservation of the relevant cardinals therefore makes its ground cardinality $|I|$, so the frame can be indexed by I . $\square$

4.3 Comparison of small automorphism groups

We continue to identify each condition of P with its basic Boolean value in B . The meet of a nonempty decreasing sequence of basic conditions of length less than κ is positive and basic, represented by the union of their partial functions. Every positive interval contains an antichain of size κ , obtained below a basic condition by specifying the position of the first 1 on a fresh coordinate set of size κ .

Lemma 4.9. *If $L \leq \text{Aut}(B)$ and $|L| < \kappa$, then*

$$Q_L = \{p \in P : (\forall h \in L) h(p) \in P\}$$

is dense in B^+ , is L -invariant, and is closed under nonempty decreasing meets of length less than κ .

Proof. Enumerate L in order type $\mu < \kappa$ and repeat this enumeration ω times. Start with a basic condition below any prescribed positive element. At a step assigned to h , with current basic condition p , choose basic conditions

$$q \leq h(p), \quad p' \leq p \wedge h^{-1}(q),$$

and continue with p' . At each proper limit take the union of the source conditions. The recursion has length $\mu \cdot \omega < \kappa$, so these unions and the final condition p_* are positive and basic. For a fixed h , the recorded target conditions q_n decrease, and the steps assigned to h are cofinal in the recursion. The inequalities at those steps and completeness give

$$h(p_*) = \bigwedge_{n < \omega} q_n \in P.$$

This proves density. The group law gives L -invariance. If $(p_\xi : \xi < \delta)$ decreases in Q_L , where $0 < \delta < \kappa$, then $p = \bigwedge_{\xi < \delta} p_\xi$ is basic and, for every $h \in L$,

$$h(p) = \bigwedge_{\xi < \delta} h(p_\xi)$$

is likewise a positive basic condition. □

For $h \in \text{Aut}(B)$, the positive p satisfying

$$h \upharpoonright (B \upharpoonright p) = \text{id} \quad \text{or} \quad p \wedge h(p) = 0 \tag{9}$$

form a dense downward-open class. To see this, if h is not the identity below b , choose $e \leq b$ with $h(e) \neq e$. If $e \setminus h(e) > 0$, this difference is disjoint from its image. Otherwise $e < h(e)$, and the same is true of the positive element $e \setminus h^{-1}(e)$.

Call p an *admissible L -root* if $p \in Q_L$ and (9) holds for every $h \in L$. Its distinct translates form a disjoint family of basic conditions, of size less than κ . Moreover,

$$h(p) \wedge k(p) > 0 \implies h(p) = k(p), \quad h \upharpoonright (B \upharpoonright p) = k \upharpoonright (B \upharpoonright p). \tag{10}$$

We call this orbit a *tower*, and its members its columns. Thus any group element that maps a column to itself fixes its whole principal algebra pointwise.

Lemma 4.10. *Let $L \leq \text{Aut}(B)$ have size less than κ . Admissible L -roots are dense. In choosing one, we may require each of its L -translates to lie below some member of each of fewer than κ specified Boolean partitions, and to decide fewer than κ specified Boolean elements.*

Proof. For a partition, the positive elements below one of its members form a dense downward-open class; the same is true of the class deciding a Boolean element. Pull back these classes by all members of L and add the requirements (9). There are fewer than κ requirements. Meet them by successive basic refinements, taking basic unions at proper limits, and then refine into Q_L using Lemma 4.9. All the earlier requirements persist under refinement. □

Theorem 4.11. *For every $H \leq \text{Aut}(B)$ with $|H| \leq \kappa$, there is $a \in \text{Aut}(B)$ such that*

$$aHa^{-1} \subseteq [\Gamma].$$

Proof. Work with a source and a target copy of B . Choose a continuous increasing sequence $(H_\alpha : \alpha < \kappa)$ of subgroups of size less than κ , with $H_0 = \{\text{id}\}$ and union H . Enumerate the source and target basic conditions as $(u_\alpha : \alpha < \kappa)$ and $(v_\alpha : \alpha < \kappa)$.

We construct refining basic partitions $\mathcal{A}_\alpha$ and $\mathcal{E}_\alpha$ of the source and target, together with compatible bijections $\psi_\alpha : \mathcal{A}_\alpha \rightarrow \mathcal{E}_\alpha$. We also construct homomorphisms $\pi_\alpha : H_\alpha \rightarrow \text{Aut}(B)$, each extending its predecessors. These maps satisfy:

- (i) H_α and $\pi_\alpha(H_\alpha)$ permute the respective partitions, and ψ_α is equivariant.
- (ii) On either side, every return stabilizer fixes its whole column pointwise.
- (iii) For $h \in H_\alpha$ and $q \in \mathcal{E}_\alpha$, the restriction of $\pi_\alpha(h)$ to $B|q$ is a translation whose mask is contained in $\text{dom}(q)$.

In particular, all source cells belong to Q_{H_α} . Target columns in one orbit have the same domain. At a successor $\alpha + 1$, all source cells will decide u_α and all target cells will decide v_α . At stage zero take the one-cell partitions and the trivial action. Each partition has at most κ cells, whereas each individual group orbit has fewer than κ columns.

Successor extension. Fix a stage, write $L = H_\alpha$, $M = H_{\alpha+1}$ and $\pi = \pi_\alpha$, and refer to the stage- α partitions and their L -orbits as old cells and old towers. In each old source tower choose a root column c and a reserve $r_c \in Q_L$ below it. Propagate the reserve to the other columns using L . This is well-defined by the pointwise return condition, and all propagated reserves are basic. In each paired target tower choose a basic root reserve and propagate it using $\pi(L)$; these reserves are basic by (iii).

An inner recursion of length κ will select disjoint paired new towers. The used source region will be M -invariant and the used target region $\pi(L)$ -invariant. Each old tower retains a positive coherent reserve in its unused region. On the source its root reserve is always in Q_L , and on the target it is always basic. Reserves only decrease. At an inner limit below κ , take their meets, separately for each old tower. These are positive basic conditions by Lemma 4.9; the old charts transport them coherently. We do not intersect reserves from different towers.

At each step we meet a request on the source or target side. A source-directed request is a positive basic condition in the unused source. Restrict it to an old source column. For a target-directed request, first restrict the unused target request to an old target column, and use the paired source column for the source choice; its reserve guarantees a positive unused part. By Lemma 4.10 choose an admissible M -root p there whose translates refine the old source partition and decide u_α . All translates remain unused, since the unused source is M -invariant.

We may choose this new source tower without exhausting any reserve. Take κ pairwise disjoint basic refinements of p and refine each further into Q_M , obtaining $(p_\xi : \xi < \kappa)$. Their saturations

$$F_\xi = \bigvee_{h \in M} h(p_\xi)$$

are pairwise disjoint: distinct translates of p are disjoint, and coincident translates have identical charts by (10). All these towers visit the same fewer-than- κ old L -towers. A given positive root reserve can be contained in at most one F_ξ , so fewer than κ indices are forbidden. Choose another index and replace p by that p_ξ . Shrink each affected source root reserve inside its positive

remainder, choosing it in Q_L . The selected saturation is L -invariant, so the propagated reserves remain coherent.

Let J be the set of columns of this selected M -tower. Split J into its L -orbits, and choose a representative j_ℓ in each; in a target-directed step take p itself as the representative of its orbit. If c_ℓ is the old source column containing j_ℓ , put $b_\ell = \psi_\alpha(c_\ell)$. Choose a basic representative $q_\ell \leq b_\ell$ on the target and propagate it by $\pi(L)$. Choose these representatives successively so their orbits are unused, pairwise disjoint, and leave every old target reserve positive. There are fewer than κ choices, even when an old tower occurs repeatedly. In a target-directed step choose the representative corresponding to the column p first, below the given target request.

For completeness, the reserve choice can be made inside any prescribed positive available region. Pull that region back to its old root column by the old translation chart. Split a basic subregion into two disjoint positive basic candidates. At most one contains the entire current root reserve. Choose the other and replace the reserve by a basic condition in its positive complement. Exclude previously selected new orbits when forming the available region. At a limit in this short recursion the decreasing basic reserves remain positive, so the next choice is still possible. Propagating through the old charts preserves these disjointness and reserve requirements.

The paired old columns have the same stabilizer. For an old source column c and $0 < d \leq c$,

$$\{h \in L : h(d) = d\} = \{h \in L : h(c) = c\}. \quad (11)$$

An element moving c sends it to a disjoint column. An element preserving c fixes its whole principal algebra. The paired target column has precisely the same stabilizer, also pointwise. Thus the orbit of q_ℓ under $\pi(L)$ has exactly the same indexing as the source orbit of j_ℓ .

Choose $D \subseteq \Omega$ of size less than κ containing all domains of the representatives q_ℓ , all old translation masks used to propagate them, and $\text{dom}(v_\alpha)$. Masks between propagated columns are differences of these root masks and are therefore also contained in D . Refine each q_ℓ to a full D -pattern. Its propagated images are full D -patterns as well. They remain disjoint and unused, and shrinking them preserves all reserves. Pair these patterns with J equivariantly under L , and denote the pattern paired with j by w_j . For $g \in M$ define its action on this target tower by

$$\pi'(g) \upharpoonright (B \upharpoonright w_j) = \tau_{\{\xi \in D : w_j(\xi) \neq w_{g(j)}(\xi)\}} \upharpoonright (B \upharpoonright w_j). \quad (12)$$

Pattern differences add modulo two, so these maps satisfy all group identities. For $h \in L$ the old mask on w_j is contained in D and equals the displayed pattern difference. Hence (12) agrees with the old $\pi(h)$ on the whole principal algebra below w_j . It therefore extends the old action and preserves its return stabilizers. All target columns decide v_α , and (iii) holds.

Enumerate all basic source and target requests in the inner κ recursion. Skip a request only if its unused part is zero; otherwise refine its unused part to a basic condition in an old cell and perform the appropriate directed step above. Before every inner stage fewer than κ columns have been used, and the coherent reserves are positive. At the end the source and target joins are both 1. Any positive omitted part would contain a basic request that could not have been skipped. The selected columns thus form refining paired partitions. Pasting (12) over their target towers gives complete automorphisms $\pi'(g)$, satisfying the group law and extending π on all of B . They belong to $[\Gamma]$. The source cells are in Q_M , all return stabilizers are pointwise identities, and the prescribed source and target tests have been decided. This completes the successor extension.

Proper limits. Let $0 < \lambda < \kappa$ be a limit. Form the common refinement of the partitions below λ by taking the meet along every coherent descending lineage. On either side every such meet is a positive basic condition, represented by the union of its short patterns. Distinct lineages have

disjoint meets, and there are at most $\kappa^{|\lambda|} = \kappa$ lineages. They cover 1: below any positive basic condition, refine successively through the earlier partitions, using basic unions at proper limits. The final positive condition lies below a lineage meet. The compatible earlier bijections pair the lineages, so they pair their nonzero meets and extend all earlier partition bijections.

Take the union of the previously fixed homomorphisms on $H_\lambda = \bigcup_{\alpha < \lambda} H_\alpha$. For $h \in H_\lambda$ choose $\beta < \lambda$ with $h \in H_\beta$. It and $\pi(h)$ permute all later partitions equivariantly, and hence permute the limit partitions. If either returns a limit cell to itself, it returns every sufficiently late containing cell and was pointwise the identity there. On a target limit cell its chart is the translation assigned on a containing stage- β cell; that mask is still contained in the enlarged domain of the limit condition. Completeness shows that each source image is the meet of the corresponding basic images. Thus all inductive properties hold at λ .

The complete isomorphism. Each basic partition generates the complete atomic subalgebra of all joins of its cells. Refinement and the compatible bijections give an isomorphism between the unions of these subalgebras on the two sides. Every source and target basic condition belongs to its respective union. At the successor stage assigned to it, all cells decided it, so it is the join of the cells below it. Both unions are therefore order-dense in their Boolean completions. Their isomorphism extends uniquely to a complete isomorphism $a : B \rightarrow B$.

For $h \in H$, equivariance holds on every sufficiently late partition algebra, so density and completeness give $aha^{-1} = \pi(h)$. The construction fixed $\pi(h)$ when h first entered a stage group and preserved it at every later stage. It belongs to $[\Gamma]$, as required. $\square$

4.4 Full invariance and ordinal definitions

Theorem 4.12. *Every local diagonal-equivariant action of G on B^I , $0 < |I| \leq \kappa$, has a frame fixed pointwise by G . The frame is exactly the atom set of C^Γ and of C^G .*

Proof. Let F be the frame supplied by Theorem 4.8. Let $[\Gamma]$ denote the automorphisms piecewise given by Γ , as in Theorem 4.11. Every $t \in [\Gamma]$ fixes F pointwise. Indeed, on each piece b where t agrees with a translation γ , locality makes Θ_t agree with Θ_γ below $\delta(b)$. The images of these pieces partition 1_C , and Θ_γ fixes each member of F .

Every Γ -fixed selector q is a member of F . Atomicity expresses q as a nonempty join of atoms in F . Two distinct atoms cannot lie below q . Both have scalar support 1, and the support map is injective below the selector q . Thus any Γ -fixed frame is exactly F , as a set.

Fix $g \in G$, and set $H = \langle \Gamma, g \rangle$. Since $|H| = \kappa$, Theorem 4.11 gives $a \in G$ with $aHa^{-1} \subseteq [\Gamma]$. Hence

$$F' = \{\Theta_{a^{-1}}(f) : f \in F\}$$

is a frame fixed pointwise by H . It is therefore Γ -fixed, so $F' = F$. In particular g fixes every member of F . As g was arbitrary, F is fixed pointwise by G . Since $C^G \subseteq C^\Gamma$ and every atom of the latter is fixed, the two fixed algebras coincide. $\square$

Proof of Theorem 4.1. The Boolean truth values of the possible cardinalities of the invariant family name $\dot{A}$ are invariant. Thus 1_B decides its cardinality. The forcing preserves cardinals at most κ , so choose a ground cardinal $0 < \nu \leq \kappa$ and names $(\tau_i)_{i < \nu}$ forced to enumerate $\dot{A}$ bijectively. Exactly as in the proof of Theorem 3.1, the matrices

$$m_{ij}^g = \|g\tau_i = \tau_j\|_B, \quad (\Theta_g z)_j = \bigvee_{i < \nu} (g(z_i) \wedge m_{ij}^g)$$

define a local diagonal-equivariant action on B'' . The argument is an induction on names and imposes no rank bound. By Theorem 4.12, the fixed algebra has a frame. Its atoms can be indexed by ν , since their row and column partitions force a bijection with ν and B preserves ground cardinals at most κ . Mixing the τ_i along each row gives pairwise distinct invariant names which exhaust $\dot{A}$. Their enumeration name is invariant as well. $\square$

Proof of Theorem 4.2. Work over a ground W satisfying $\text{GCH} + \text{GA} + (V = \text{HOD})$, as permitted by Remark 1.1. GCH gives $\kappa^{<\kappa} = \kappa$. The empty family has its empty enumeration. For a nonempty OD family of size at most κ , weak homogeneity turns its defining formula and ordinal parameters into an invariant name forced to have that size bound. Apply Theorem 4.1. The enumeration name and the forcing have ordinal codes by Lemma 2.2; Lemma 2.3 makes the enumeration OD in the extension. Each of its values is then definable from the enumeration and its ordinal index. Taking $W = L$ proves the stated theorem. $\square$

Corollary 4.13 (Sharpness). *In the extension of Theorem 4.2, there is an OD family of size κ^+ with no OD member.*

Proof. Take $A = \mathcal{P}(\kappa)^{L[c]} \setminus L$. Homogeneity makes every OD subset of κ belong to L , so A has no OD member. To see that its cardinality is κ^+ , note that $c \triangle a \in A$ for each $a \in \mathcal{P}(\kappa)^L$ and that these subsets are distinct. Conversely, P has size κ and is κ^+ -cc, so the number of nice names for subsets of κ is at most $(2^\kappa)^\kappa = \kappa^+$ in L . $\square$

5 Countable families in a random extension

Automatic continuity gives an enumeration invariant under probability-preserving automorphisms. By comparing equivalent probability measures, we then show that all nonsingular automorphisms fix it.

Work in a ZFC ground W on a standard atomless probability space (X, μ) , and let $\mathbb{B}$ be its measure algebra. Put $\mathcal{G} = \text{Aut}(\mathbb{B})$, $\mathcal{H} = \text{Aut}(\mathbb{B}, \mu)$, and $K = \text{Sym}(I)$, for nonempty finite or countably infinite I . The first group consists of all nonsingular transformations modulo null sets. The topology on K is pointwise convergence; $L^0(X, K)$ has convergence in measure.

Suppose $\dot{e} : I \rightarrow \dot{A}$ is a name for a bijective enumeration of an invariant family. For $g \in \mathcal{G}$, define its comparison permutation by

$$1 \Vdash g\dot{e} = \dot{e} \circ \dot{\pi}_g.$$

Permutation names are identified with measurable K -valued functions. Write $(g \cdot u)(x) = u(g^{-1}x)$. Then

$$\pi_{gh} = \pi_g(g \cdot \pi_h), \tag{13}$$

$$\pi_g = 1_K \text{ on } \text{fix}(g). \tag{14}$$

Here $\text{fix}(g)$ is the largest Boolean region on which g acts identically on every subregion. The first identity follows by comparing $gh\dot{e}$ to $\dot{e}$ in two steps; the second follows from induction on names. Replacing $\dot{e}$ by $\dot{e} \circ \dot{\beta}^{-1}$ changes the comparison to

$$\pi_g^\beta = \beta \pi_g(g \cdot \beta^{-1}).$$

It suffices to choose a relabeling that makes every comparison permutation the identity. We make all subsequent choices and measure calculations in W .

5.1 The probability-preserving subgroup

The following lemma gives a relabeling fixed by every probability-preserving automorphism.

Lemma 5.1 (Probability-preserving frames). *Let Q be a standard atomless probability algebra and $0 < |J| < 2^{\aleph_0}$. Every full local diagonal action of $\text{Aut}(Q, \mu)$ on Q^J has an invariant frame.*

Proof. For selectors set

$$d(u, v) = 1 - \mu\left(\bigvee_{j \in J} (u_j \wedge v_j)\right), \quad F_u(h) = d(\Theta_h u, u).$$

The action is isometric, and F_u is a seminorm: $F_u(1) = 0$, $F_u(h^{-1}) = F_u(h)$, and $F_u(hk) \leq F_u(h) + F_u(k)$. Its orbit pseudometric has density at most $\max(|J|, \aleph_0)$. Indeed, every selector has countably many nonzero entries. Truncate to finitely many labels and approximate their partition in a countable dense algebra of Q . Locality gives

$$d(\Theta_h u, \Theta_k u) \leq \mu(\text{supp}(k^{-1}h)).$$

The standard probability-algebra group has ample metric generics, and its uniform metric is equivalent to this support metric [2, Section 6.3]. Thus F_u is Baire measurable for that finer metric. The small-density seminorm theorem [3, Theorem 3.6] makes F_u continuous for the weak Polish topology. Every selector orbit is consequently weakly continuous.

We now use continuity to construct a frame. This argument also applies when J is uncountable. The counting measure over the fibers is

$$\nu(z) = \sum_{j \in J} \mu(z_j), \quad L^2(Q^J, \nu) = \bigoplus_{j \in J} L^2(Q, \mu).$$

The action preserves ν , since its Boolean permutation matrices have partition rows and columns. Also $\|\mathbf{1}_u - \mathbf{1}_v\|_2^2 = 2d(u, v)$. Continuity supplies a finite Boolean partition $\mathcal{P}$ such that its stabilizer $H_{\mathcal{P}}$ moves $\mathbf{1}_u$ by less than $1/8$ in norm. The minimum-norm vector v in the closed convex hull of this orbit is nonnegative, $H_{\mathcal{P}}$ -fixed, and within $1/8$ of $\mathbf{1}_u$. Its positive coordinates have countable support. Select the unique strictly positive maximum among its fiber coordinates wherever it exists. This defines an invariant partial selector w . It is nonzero because, outside a set of measure at most $16\|v - \mathbf{1}_u\|_2^2 \leq 1/4$, the coordinate chosen by u exceeds $3/4$ and all others are below $1/4$. The scalar support of w is fixed by $H_{\mathcal{P}}$, hence is a union of cells of $\mathcal{P}$. Restricting to one such positive cell p gives a partial selector of support p fixed by every automorphism supported on p .

Its translates agree on overlaps. For h, k , put $r = h(p) \wedge k(p)$, $b = h^{-1}(r)$ and $c = k^{-1}(r)$. The partial map $k^{-1}h : Q \restriction b \rightarrow Q \restriction c$ extends to an automorphism supported on p , by matching the equal-measure complements inside p . Locality and the invariance of w therefore identify $\Theta_h w$ and $\Theta_k w$ over r . Their join over all h is an invariant selector, since its scalar support is 1 by ergodicity. A maximal disjoint family of invariant selectors covers 1. Indeed, a nonzero invariant complement z would also have scalar support 1; disjointifying its coordinates supplies a selector $u \leq z$. Repeat the averaging and maximum selection within z , then join the translates as above. This gives another invariant selector, a contradiction. This is an invariant frame, of cardinality $|J|$ by ccc and the corresponding bijection in a generic fiber. $\square$

Proposition 5.2. *Let π be the comparison permutations of an enumeration as above. There is a permutation name β such that, after replacing $\dot{e}$ by $\dot{e} \circ \beta^{-1}$, the comparison permutations*

$$\pi^\beta(g) = \beta\pi(g)(g \cdot \beta^{-1})$$

satisfy $\pi^\beta(h) = 1_K$ for every $h \in \mathcal{H}$.

Proof. Apply Lemma 5.1 to the local diagonal action on $\mathbb{B}^I$ induced by the enumeration. Its invariant frame has cardinality $|I|$ and gives the required relabeling. Locality persists because $g \cdot \beta = \beta$ on every identity region. $\square$

For the remaining comparison argument, I may be any nonempty ground set. Let π_g be the Boolean permutation matrix of a full local diagonal $\mathcal{G}$ -action on $\mathbb{B}^I$ relative to a fixed frame. Products use Boolean joins and meets; the identities (13)–(14) remain valid, with 1 denoting the identity matrix. When I is countable these are the preceding measurable permutation functions. Every row and column has only countably many positive entries, by ccc. Assume the frame is probability-preserving invariant, so

$$\pi|\mathcal{H} = 1. \tag{15}$$

5.2 Equivalent probability measures

Let $\mathcal{P}$ be the set of probability measures on $\mathbb{B}$ equivalent to μ . For each $\nu \in \mathcal{P}$, choose $k_\nu \in \mathcal{G}$ such that $(k_\nu)_*\mu = \nu$, with $k_\mu = 1$, and put

$$b_\nu = \pi(k_\nu).$$

This does not depend on the choice of k_ν . If $k_\nu = \ell_\nu h$ for $h \in \mathcal{H}$, then (15) gives $\pi(k_\nu) = \pi(\ell_\nu)$. Define

$$C(\nu, \lambda) = b_\nu^{-1} b_\lambda.$$

Lemma 5.3. *For $\nu, \lambda, \rho \in \mathcal{P}$ and $g \in \mathcal{G}$,*

$$C(\nu, \lambda)C(\lambda, \rho) = C(\nu, \rho), \tag{16}$$

$$C(g_*\nu, g_*\lambda) = g \cdot C(\nu, \lambda), \tag{17}$$

$$\pi(g) = C(\mu, g_*\mu). \tag{18}$$

Proof. The first identity is immediate. Since gk_ν carries μ to $g_*\nu$,

$$b_{g_*\nu} = \pi(g)(g \cdot b_\nu).$$

The copies of $\pi(g)$ cancel in $b_{g_*\nu}^{-1} b_{g_*\lambda}$, giving (17). Taking $\nu = \mu$ gives (18). $\square$

Let $\lambda = e^Z \nu$. We say that the ambient probability algebra is *conditionally atomless over $\sigma(Z)$* if almost every conditional measure in the disintegration over Z is atomless.

Lemma 5.4. *Let K be a Polish space. Suppose the probability algebra is conditionally atomless over $\sigma(Z)$. If a measurable K -valued function is fixed by every ν -preserving automorphism which fixes Z , then it is $\sigma(Z)$ -measurable.*

Proof. The relative product theorem identifies the extension over $\sigma(Z)$ with a product with an atomless probability space. The full group of the fiber acts ergodically on almost every fiber, so its fixed algebra is the base algebra. Apply this to the inverse images of a countable basis of open subsets of K . Delbaen also characterizes conditional atomlessness by independent atomless subalgebras [4]. $\square$

Lemma 5.5 (Pexider rigidity). *Let J be nonempty and countable, and put $K = \text{Sym}(J)$. Let ξ be a probability measure on $\mathbb{R}$ which is not countably supported, and let η be equivalent to Lebesgue measure. Suppose Borel maps $A, B, D : \mathbb{R} \rightarrow K$ satisfy*

$$A(x)B(y) = D(x + y)$$

for $\xi \times \eta$ -almost every (x, y) . Then A is constant ξ -almost everywhere.

Proof. Replacing η by Lebesgue measure does not change its null sets. By Fubini, there is a ξ -conull set E such that the equation holds for every $x \in E$ and almost every y . For $x, x' \in E$, compare the equation at (x, y) and $(x', y + x - x')$ to obtain

$$B(y + x - x')B(y)^{-1} = A(x')^{-1}A(x) \quad (19)$$

for almost every y .

Let D_0 be the set of $h \in \mathbb{R}$ for which $B(\cdot + h)B(\cdot)^{-1}$ is almost everywhere constant. This is a subgroup of $\mathbb{R}$ containing $E - E$. The corresponding constants define a homomorphism $\chi : D_0 \rightarrow K$ with abelian range.

The group D_0 is dense. Otherwise its closure would be discrete, so D_0 would be countable. For any fixed $x \in E$, the countable coset $x + D_0$ would then contain E . Translation is continuous on $L_{\text{loc}}^0(\mathbb{R}, K)$, hence χ is continuous.

Every neighborhood of the identity in $K = \text{Sym}(J)$ contains an open subgroup. Its inverse image under χ is an open subgroup of D_0 , so it contains $D_0 \cap (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Given $h \in D_0$, choose $n \geq 2$ with $|h|/n < \varepsilon/2$ and $d \in D_0$ with $|d - h/n| < \varepsilon/(2n)$. Both d and $h - (n-1)d$ lie in that interval, so $h = (n-1)d + [h - (n-1)d]$ belongs to the subgroup. Thus every such inverse image is all of D_0 , and χ is trivial. Equation (19) gives $A(x) = A(x')$ for all $x, x' \in E$. $\square$

Proposition 5.6. *Let $\lambda = e^Z \nu$. Suppose the law of Z is not countably supported and the probability algebra is conditionally atomless over $\sigma(Z)$. Then $C(\nu, \lambda)$ is almost everywhere constant.*

Proof. An automorphism preserves both ν and λ exactly when it preserves ν and Z . By covariance and Lemma 5.4, each entry of $C(\nu, \lambda)$ is $\sigma(Z)$ -measurable. Conditional atomlessness supplies two atomless coordinates independent of Z . On the first choose Y with law equivalent to Lebesgue measure and $\mathbb{E}_\nu e^Y = 1$. Leave the second coordinate unused. Put $d\rho = e^Y d\lambda$. Independence makes ρ a probability. The unused atomless coordinate gives conditional atomlessness over Z, Y , and $Z + Y$ for the relevant measures. Entrywise, therefore,

$$C(\nu, \lambda) = A(Z), \quad C(\lambda, \rho) = B(Y), \quad C(\nu, \rho) = D(Z + Y).$$

The composition identity gives $A(Z)B(Y) = D(Z + Y)$.

To apply Lemma 5.5, close any label under the possible positive entries of rows and columns of these three matrices. The resulting ground component J is countable and is preserved by all three matrices. On J , choose simultaneous Borel representatives, replacing them by the identity on their exceptional null sets. They are $\text{Sym}(J)$ -valued functions of the displayed variables. The laws of Y

and $Z + Y$ are equivalent to Lebesgue measure. Changing between the three equivalent ambient probabilities preserves null sets, so we may use the natural laws of these variables. Independence of Z, Y and Lemma 5.5 make $A(Z)|J$ constant. Varying the initial label makes every entry of $C(\nu, \lambda)$ constant. The partition rows and columns therefore determine one ground permutation. This does not require an uncountable intersection of conull sets. $\square$

5.3 Arbitrary likelihood ratios

Proposition 5.7. *Let $\lambda = e^Z \nu$. If the law of Z is absolutely continuous with respect to Lebesgue measure, then $C(\nu, \lambda)$ is almost everywhere constant.*

Proof. Write $Z = n + r$, where $n \in \mathbb{Z}$ and $0 \leq r < 1$, and use the nonterminating binary expansion

$$r = \sum_{m \geq 1} \varepsilon_m 2^{-m}$$

off the dyadic rationals. Put

$$P = n + \sum_{m \text{ odd}} \varepsilon_m 2^{-m}, \quad Q = \sum_{m \text{ even}} \varepsilon_m 2^{-m}.$$

Then $Z = P + Q$, $Q \geq 0$, and $P \leq Z$.

Under Lebesgue measure, the joint law of (P, Q) is a product of two atomless measures, with a counting factor in the first coordinate. The law of Z is absolutely continuous, so the actual joint law of (P, Q) is absolutely continuous with respect to this product. The conditional law of Q given P is atomless, as is the conditional law of P given Q . Thus the ambient algebra is conditionally atomless over both $\sigma(P)$ and $\sigma(Q)$, and both marginal laws are nonatomic.

Let

$$d\eta = \frac{e^P}{\int e^P d\nu} d\nu.$$

This is well-defined because $P \leq Z$. The log-likelihood from ν to η is P plus a constant, and the log-likelihood from η to λ is Q plus a constant. Equivalent reweighting preserves conditional atomlessness. By Proposition 5.6, both $C(\nu, \eta)$ and $C(\eta, \lambda)$ are constant. Equation (16) then makes $C(\nu, \lambda)$ constant. $\square$

Lemma 5.8 (Uniform perturbation). *Let Z be a finite real-valued random variable on a standard atomless probability space. There is a uniform random variable $T : (X, \nu) \rightarrow (0, 1)$ such that $Z + T$ has an absolutely continuous law.*

Proof. Let $\{z_m : m \in J\}$ be the at most countable set of atoms of the law of Z , and put $E_m = \{Z = z_m\}$. On each positive-measure E_m , choose a conditionally uniform random variable T_m . Then $Z + T_m$ has a shifted uniform law on E_m .

On the complement E_* , the conditional law of Z is nonatomic. Let F be its continuous distribution function and Q its quantile function. Put $T_* = F(Z)$. Then T_* is uniform and $Z = Q(T_*)$ almost surely. The map

$$h(t) = Q(t) + t$$

is strictly increasing and satisfies $h(t) - h(s) \geq t - s$ for $s < t$. Its inverse on its range is 1-Lipschitz. Hence the pushforward of Lebesgue measure by h is absolutely continuous, so $Z + T_* = h(T_*)$ has an absolutely continuous law.

Patch the T_m and T_* . Conditional on each member of the resulting countable partition, T is uniform. Thus T is uniform, and the law of $Z + T$ is a countable mixture of absolutely continuous laws. $\square$

Theorem 5.9 (Nonsingular invariance). *For any ground set I , every probability-preserving invariant frame for a full local diagonal $\text{Aut}(\mathbb{B})$ -action on $\mathbb{B}^I$ is fixed by the whole group. No cardinality bound on I is required.*

Proof. With the preceding notation, we show that $C(\nu, \lambda)$ is constant for every $\nu, \lambda \in \mathcal{P}$.

Write $\lambda = e^Z \nu$. By Lemma 5.8, choose a uniform $T \in (0, 1)$ such that $U = Z + T$ has an absolutely continuous law. Put $V = -T$ and

$$d\eta = \frac{e^U}{\int e^U d\nu} d\nu.$$

The integral is finite because $e^U \leq ee^Z$. The first log-likelihood is U plus a constant, so Proposition 5.7 makes $C(\nu, \eta)$ constant. The second log-likelihood is V plus a constant. Its law under η is absolutely continuous because $\eta \ll \nu$ and the law of V under ν is uniform. Hence $C(\eta, \lambda)$ is constant, and (16) shows that $C(\nu, \lambda)$ is constant.

By (18), each $\pi(g)$ is a constant permutation. Thus π is an ordinary homomorphism to $\text{Sym}(I)$. Locality makes it trivial on any map with a positive identity region. Every member of $\mathcal{G}$ is a product of two such maps. For $g \neq 1$, choose $b > 0$ with $b \wedge g(b) = 0$, and $0 < a < b$. Let t exchange $a, g(a)$ by g, g^{-1} and fix their complement. Both t and tg have positive identity regions, and $g = t(tg)$. Hence every comparison permutation is the identity. $\square$

Theorem 5.10 (Invariant enumeration for random forcing). *In any ground model W of ZFC, let $\mathbb{B}$ be its standard atomless probability algebra. Every invariant name forced to be a nonempty countable set has an invariant bijective enumeration by a finite ordinal or by ω .*

Proof. Normalize a bijective enumeration by Proposition 5.2, then apply Theorem 5.9 to its fixed frame. $\square$

Corollary 5.11. *If a is a set of ordinals and r is random over $L[a]$, then every countable $\text{OD}(a)$ set in $L[a, r]$ has an $\text{OD}(a)$ bijective enumeration. In particular every countable OD set in $L[r]$ consists of OD elements, without a rank restriction.*

Proof. The argument also works over any ground W satisfying $\text{ZFC} + \text{GCH} + \text{GA} + V = \text{HOD}$, with $a \in W$ a set of ordinals. The ground and its names then have stable ordinal codes by Lemma 2.2; for the stated ground $L[a]$, use its canonical a -definable order instead. In either case the definition supplies an invariant name. Apply Theorem 5.10 and then Lemma 2.3 to the resulting enumeration name. No cardinal arithmetic is used in this argument. $\square$

Remark 5.12. This relative statement requires the parameter to belong to the ground $L[a]$. The conclusion for every set-of-ordinals parameter in $L[c]$ used the Cohen intermediate-extension lemma. We need no such extension of the parameter assertion here.

6 Countable families in collapse extensions

We compare three coordinate factors to obtain an enumeration, then factor automorphisms into recodings of finite words to prove its invariance. The enumeration is indexed by a ground cardinal, which may become countable in the extension.

Throughout this section, we work in a fixed ground model $W \models \text{ZFC}$. All algebras, names, groups, and choices belong to this model. Let μ be an infinite cardinal, put

$$\Omega = \mu \times \mu, \quad B = \text{RO}(\Omega^\omega), \quad G = \text{Aut}(B),$$

and identify the individual component coordinates with $I = \omega \times \{0, 1\}$. This is a presentation of $\text{Coll}(\omega, \mu)$, fixed before any enumeration is chosen. For $A \subseteq I$, let C_A be the complete subalgebra generated by the coordinates in A . For any complete subalgebra $C \subseteq B$, write

$$G_C = \{g \in G : g \restriction C = \text{id}\}.$$

Theorem 6.1. *Let $\dot{X}$ be a B -name invariant under G . If 1_B forces that $\dot{X}$ is nonempty and countable, there are a ground cardinal $\nu \leq \mu$ and invariant names $(\dot{y}_\xi : \xi < \nu)$ which are forced to enumerate $\dot{X}$ bijectively. If $1_B \Vdash |\dot{X}| = m < \omega$, then $\nu = m$.*

The members may have arbitrary rank. The theorem holds over any ZFC ground. We state the definability consequence after its proof. The fixed algebra of G is $\{0, 1\}$, so homogeneity decides whether $\dot{X}$ is infinite or has a particular finite size. For the proof, fix a nonempty finite ordinal or ω , denoted by J , such that $1_B \Vdash |\dot{X}| = |\check{J}|$.

6.1 Relative enumerations and three-factor comparison

Lemma 6.2. *Suppose $C \subseteq B$ is complete and forces that the quotient of B is a Cohen algebra. Then $\dot{X}$ has a bijective J -indexed enumeration by ground B -names invariant under G_C . Moreover,*

$$B^{G_C} = C.$$

Proof. In a C -generic extension $W[H]$, the quotient name for $\dot{X}$ is invariant by Lemma 2.4. Theorem 3.1, applied in that intermediate model, gives an invariant enumeration by quotient names. The maximum principle chooses C -names for such an enumeration. Flatten them to ground B -names. A ground member of G_C acts trivially on the intermediate model and descends to a quotient automorphism, so it fixes the flattened enumeration. This proves the first assertion.

For the second, an element of B^{G_C} becomes, in every C -generic extension, a Boolean element fixed by all automorphisms of the Cohen quotient. Its quotient value is therefore 0 or 1. The C -Boolean value deciding which case holds is the original element of B . Thus that element belongs to C . The reverse inclusion is immediate. $\square$

We use regular forcing products below. These differ from direct products of Boolean algebras. Represent a name forced to belong to a ground set S by the S -indexed Boolean partition of its possible values.

Lemma 6.3. *Let P_0, P_1, P_2 be forcing notions and S a ground set. Suppose u is a $P_0 \times P_1$ -name and v a $P_1 \times P_2$ -name, both forced to belong to S . If the threefold product forces $u = v$, there is a P_1 -name w such that the respective twofold products force $u = w$ and $v = w$. The assertion remains true after restricting any unshared factor to a nonzero cone.*

Proof. Force first with P_1 . In the intermediate extension the remaining forcing is the product of the old posets P_0 and P_2 . A condition forcing $u = s$ and one forcing $v = t$ are compatible in this product, and therefore $s = t$. Conditions deciding a value are dense in each factor. Thus there is one value which both names are forced to take. The forcing theorem and maximum principle provide a P_1 -name for it. The argument also applies after restriction to a nonzero cone. When we use quotient completions, we take the new completions of the unchanged dense posets. $\square$

Lemma 6.4 (Effective descent). *Let J be a nonempty ground set. For $i < j < 3$, let t_{ij} be a $P_i \times P_j$ -name for a permutation of $\check{J}$ in the extension. Suppose*

$$1 \Vdash t_{02} = t_{12}t_{01}.$$

There are a ground set E , with $|E| \leq |P_0| \cdot |J|$, and P_i -names e_i for bijections $\check{E} \rightarrow \check{J}$ such that

$$1 \Vdash t_{ij} = e_j e_i^{-1} \quad (i < j < 3).$$

The bound may use any dense presentation of P_0 .

Proof. Call $(p, n) \in P_0 \times J$ *stable* if there is a P_1 -name σ for an element of $\check{J}$ such that

$$p \times 1 \Vdash t_{01}(n) = \sigma.$$

The witness is unique modulo P_1 -forced equality, since we can remove the nonzero P_0 -cone from the equality of two witnesses. Represent names for elements of $\check{J}$ by Boolean partitions, and choose one witness for each stable pair.

For every fixed $n \in J$, stable conditions are dense in P_0 . Indeed, below any given condition choose $p \in P_0$, $q \in P_2$, and $m \in J$ such that $p \times q \Vdash t_{02}(n) = m$. The triangle identity gives $t_{01}(n) = t_{12}^{-1}(m)$ on $p \times P_1 \times q$. Lemma 6.3, with shared factor P_1 , supplies a witness σ on the whole product $p \times P_1$. Only one value in J has been decided.

If σ witnesses stability of (p, n) , the unrestricted triangle identity, restricted only to p , gives $t_{12}(\sigma) = t_{02}(n)$. These two names share only P_2 . Separation supplies a P_2 -name ρ such that

$$1_{P_1 \times P_2} \Vdash t_{12}(\sigma) = \rho.$$

For two such witnesses σ, τ with counterparts ρ, η , injectivity of t_{12} gives

$$1_{P_1 \times P_2} \Vdash (\sigma = \tau) \longleftrightarrow (\rho = \eta).$$

The Boolean values on the two sides belong to independent factors, so their common value is 0 or 1. Thus distinct forced-equality classes of stable witnesses are forced unequal. Let E be their ground set of classes, and choose representatives σ_a, ρ_a for $a \in E$. There are at most $|P_0| \cdot |J|$ classes. Put $e_1(a) = \sigma_a$ and $e_2(a) = \rho_a$. Both names are forced injective, and $t_{12}e_1 = e_2$.

To prove that e_1 is onto, fix $l \in J$ and $q \in P_1$. Choose a rectangle $p_0 \times q'$, with $q' \leq q$, deciding $t_{01}^{-1}(l) = k \in J$. Refine p_0 to a stable condition p for this fixed k , and let $a \in E$ be its witness class. Then $p \times q' \Vdash e_1(a) = t_{01}(k) = l$. Because $e_1(a)$ is a P_1 -name, already $q' \Vdash e_1(a) = l$. These conditions are dense for every l , so e_1 is forced onto. The equation $t_{12}e_1 = e_2$ makes e_2 onto as well; a statement about its range can be read in P_2 alone.

Finally, for each $a \in E$, the triangle identity gives

$$t_{01}^{-1}(\sigma_a) = t_{02}^{-1}(\rho_a).$$

Separation over the shared factor P_0 gives a P_0 -name $e_0(a)$ for this value. Hence $t_{01}e_0 = e_1$ and $t_{02}e_0 = e_2$. These equations make e_0 a bijection and prove every asserted comparison. All chosen arrays of names are sets in the ground model. $\square$

Lemma 6.5. *There are a ground cardinal $\nu \leq \mu$ and a bijective enumeration $\bar{y} = \langle \dot{y}_\xi : \xi < \nu \rangle$ of $\dot{X}$ invariant under G_{C_A} for every infinite coinfinite ground $A \subseteq I$.*

Proof. Partition I into three infinite sets and denote their coordinate algebras by C_0, C_1, C_2 . Each makes μ countable, so the remaining coordinate forcing has a countable atomless dense poset. Its completion is a Cohen algebra. The same is true of the quotient over $C_{ij} = C_i \vee C_j$, where the join denotes the generated complete subalgebra.

Choose, by Lemma 6.2, enumerations $\bar{x}^i = \langle \dot{x}_n^i : n \in J \rangle$ invariant under G_{C_i} . Let t_{ij} be the comparison permutation, so that

$$1 \Vdash \dot{x}_n^i = \dot{x}_{t_{ij}(n)}^j \quad (n \in J).$$

Every automorphism fixing C_{ij} fixes both enumerations. Thus every coefficient $\|t_{ij}(n) = m\|$ lies in $B^{G_{C_{ij}}} = C_{ij}$. These are names over the genuine twofold coordinate products, and they satisfy $t_{02} = t_{12}t_{01}$. Apply Lemma 6.4. Each coordinate forcing has a dense presentation of size μ , so its ground index set E has size at most μ . Relabel E by its ground cardinal $\nu \leq \mu$, and retain the notation e_i for the resulting bijections $\nu \rightarrow J$. Mixing names, put

$$\dot{y}_\xi = \dot{x}_{e_i(\xi)}^i \quad (\xi < \nu).$$

This does not depend on i modulo forced equality, and it is invariant under all three groups G_{C_i} .

A transposition of component coordinates fixes at least one of the three coordinate sets pointwise. Hence $\bar{y}$ is invariant under all finitary coordinate permutations. For an arbitrary infinite coinfinite A , choose a G_{C_A} -invariant J -indexed enumeration $\bar{z}$ by Lemma 6.2. Compare it with $\bar{y}$. Both are invariant under finitary permutations supported on $I \setminus A$. The scalar fixed algebra of that group is C_A : over C_A , these finitary permutations act weakly homogeneously on the remaining finite-function forcing, since finite domains can be moved apart. The quotient truth values are therefore 0 or 1. Mixing them over C_A shows that the original invariant Boolean elements belong to C_A .

Apply this to each comparison coefficient $\|\dot{y}_\xi = \dot{z}_n\|$, for $\xi < \nu$ and $n \in J$. Every coefficient lies in C_A . Every member of G_{C_A} fixes these coefficients and $\bar{z}$, and therefore fixes $\bar{y}$. $\square$

Fix this enumeration for the rest of the proof, and let

$$K = \{g \in G : 1 \Vdash g\dot{y}_\xi = \dot{y}_\xi \text{ for all } \xi < \nu\}. \quad (20)$$

This subgroup contains all pointwise coordinate-factor stabilizers from Lemma 6.5. We will prove $K = G$ by factoring automorphisms into maps defined by finite words.

6.2 Prefix codes and whole-space factorizations

For a finite word $s \in \Omega^{<\omega}$, write $[s]$ for its cylinder and $s \frown t$ for concatenation. A *front* is a maximal prefix antichain of nonempty finite words. It is a *bar* if it meets every infinite branch. An internal node of a front is a proper prefix of one of its leaves. By maximality, every immediate child of an internal node is either a leaf or an internal node.

A history-dependent prefix code is a family of bijections

$$c_s : \Omega \longrightarrow S_s \quad (s \in \Omega^{<\omega}),$$

where the S_s are fronts. Its map replaces the next input letter by its c_s -word and concatenates the successive words. For each n , the union of the cylinders obtained after n complete input letters is dense open. The intersection of these sets is exactly the range of the map. Unique parsing gives the inverse, and the codewords are nonempty, so the output lengths tend to infinity. Thus the map is a homeomorphism onto a dense G_δ subset, and is onto the whole space if every S_s is a bar. It induces a complete Boolean automorphism. We use forward images for these Boolean maps and compose from right to left. If the code is independent of s , call it *stationary*.

Lemma 6.6. *Every level-preserving tree automorphism of $\Omega^{<\omega}$ belongs to K .*

Proof. Write its point map as $T(x)(n) = t_{x|n}(x(n))$, where each t_s permutes Ω . Let U keep the even paired positions unchanged and replace the odd paired positions by their T -images. With the even positions fixed, this is an invertible triangular recoding of the odd positions. Recover the old odd symbols successively, using the symbols already reconstructed. Both directions are continuous. The map $V = TU^{-1}$ fixes every odd paired position. Thus $T = VU$, and each factor fixes an infinite coinfinite set of component coordinates pointwise. Both belong to K . $\square$

Lemma 6.7 (Reservoirs for a bar). *Let S be a bar on $\Omega = \mu \times \mu$. There is a coloring $\lambda : S \rightarrow \mu$ such that, for each $b < \mu$, first-coordinate projection maps $\lambda^{-1}\{b\}$ bijectively onto a bar $P_b \subseteq \mu^{<\omega}$ of cardinality μ .*

Proof. Assign reservoirs $B_w \subseteq \mu$ to the nodes of the front tree, starting with $B_\emptyset = \mu$. Internal reservoirs have size μ and terminal reservoirs are singletons. At an internal node w , for each first-coordinate symbol $p < \mu$ separately, partition

$$B_w = \bigsqcup_{q < \mu} B_{w \smallfrown (p,q)},$$

giving a child reservoir size 1 if that child is terminal and size μ if it is internal. The sum of these μ prescribed nonzero cardinalities is μ , so the partition exists. The partitions for different p are independent partitions of the same reservoir. Recursion on finite node length defines all reservoirs. Color a leaf by the unique member of its reservoir.

Fix $b < \mu$. At an internal node containing b , each next first symbol determines exactly one second symbol whose child reservoir contains b . Every infinite first-coordinate branch therefore determines a unique paired path until it reaches a leaf. It must reach a leaf, since otherwise it would avoid the bar S . The resulting first-coordinate leaf words form a bar. Two leaves of color b with comparable first projections must follow the same second-coordinate choices on the shorter word. Neither can properly extend the other, so they are equal. This proves both prefix-freeness and injectivity of projection. Finally the projected bar has size μ : it has a leaf above every possible first symbol, and there are at most μ finite words. $\square$

Lemma 6.8. *Every history-dependent bar-front code map belongs to K .*

Proof. Apply Lemma 6.7 to each state front S_s . Choose bijections $e_{s,b} : \mu \rightarrow P_{s,b}$. There is a unique bijection

$$\theta_s : \mu \times \mu \longrightarrow S_s$$

whose value at (a, b) has color b and first projection $e_{s,b}(a)$. For an originally given labeling c_s , the next-letter bijections $\theta_s^{-1}c_s$ define a level-preserving input tree map T . Reindex the old states by the recursively recovered inverse of T . The original code map is then PT , where P uses the normalized labels θ_s . From now on the labels and their reservoirs are indexed by these transferred input histories. By Lemma 6.6, $T \in K$.

For an input $(a, b) \in \mu^\omega \times \mu^\omega$, let $s_i = (a, b) \upharpoonright i$. Write the two projections of $\theta_{s_i}(a(i), b(i))$ as u_i, v_i , and put

$$A = u_0 u_1 \cdots, \quad B' = v_0 v_1 \cdots.$$

Thus $P(a, b) = (A, B')$. Define $U(a, b) = (A, b)$. For fixed b , its inverse parses the next word in the bar $P_{s_i, b(i)}$ of the remaining A -stream, then uses $e_{s_i, b(i)}^{-1}$ to recover $a(i)$. The earlier reconstruction determines the state, and the unchanged second coordinate supplies the current $b(i)$. This procedure

works on every (A, b) . The codewords are nonempty, so the forward map is continuous. The inverse is jointly continuous because each finite reconstruction uses only finitely many letters of A, b .

Define $V(A, b) = (A, B')$ by first recovering a through U^{-1} and then producing the second projection words. To invert V on an arbitrary pair (A, B') , reconstruct a, b together. At the current state s_i , pair the remaining tails of A, B' coordinatewise and parse their next leaf $w \in S_{s_i}$. Since the front is a bar, this takes finitely many steps. Recover

$$b(i) = \lambda_{s_i}(w), \quad a(i) = e_{s_i, b(i)}^{-1}(\text{pr}_1 w).$$

The normalized code of this pair is exactly w . Continue after consuming its length in both streams. This constructs a unique inverse on the whole space. Both directions of V are jointly continuous, since each finite stage reads finitely many symbols and determines its next state. No uniform bound on that reading is needed.

Now $P = VU$, the map U fixes the second set of component coordinates pointwise, and V fixes the first. Both belong to K , so $PT \in K$. $\square$

Lemma 6.9. *Every individual prefix swap belongs to K . If h is a stationary prefix-code automorphism, then it permutes the names $\bar{y}$ by one ground permutation, and $hKh^{-1} = K$.*

Proof. For incompatible nonempty words s, t , take a finite-height bar containing them by filling the complement of $[s] \cup [t]$ with words at a common depth at least $\max(|s|, |t|)$. This bar has cardinality μ . Let F_1 expand the first input letter to a chosen enumeration of this bar and use identity letter codes thereafter. Let F_2 use the same enumeration with the leaves s, t exchanged. Both maps belong to K by Lemma 6.8. Their product $F_2F_1^{-1}$ is the desired prefix swap.

Let f swap two input words of the same length. Its conjugate hfh^{-1} swaps their concatenated stationary codewords and preserves the tails. The two output prefixes are incompatible, though their lengths may differ. Thus both f and hfh^{-1} belong to K . For every $\xi < \nu$, the name $h^{-1}\dot{y}_\xi$ is consequently invariant under all such f . Since $\dot{X}$ is invariant under G , these names form another enumeration of $\dot{X}$. Compare it with $\bar{y}$. Every comparison coefficient

$$b_{\xi\eta} = \|h^{-1}\dot{y}_\xi = \dot{y}_\eta\| \quad (\xi, \eta < \nu)$$

is fixed by all equal-length prefix swaps. This group is weakly homogeneous: any two basic conditions have appropriate equal-length extensions interchanged by a prefix swap. Hence the comparison values are all 0 or 1. Each row and column of the comparison matrix is a partition of unity. Thus each has exactly one nonzero entry, even if ν is uncountable in the ground. The matrix defines a ground permutation of ν . Thus h permutes $\bar{y}$ modulo forced equality and normalizes its pointwise stabilizer K .

If the stationary front is not a bar, the conjugacy holds on its dense parsing domain, which is invariant under the indicated prefix swap. It is therefore the asserted identity of complete Boolean automorphisms. $\square$

6.3 Truncation and alternating fronts

Lemma 6.10. *For every history-dependent maximal prefix-code map F , there are a bar-front code map R , a stationary prefix-code map h , and a level-preserving tree map v such that*

$$Rh = Fhv. \tag{21}$$

Proof. Let $c_s : \Omega \rightarrow S_s$ be the state codes of F . We first partition Ω into continuation and stopping sets C, D so that every internal raw node has μ witnesses of each color in distinct immediate-child cones.

Enumerate the requests (s, t, e, α) , where t is an internal node of S_s , $e \in \{C, D\}$, and $\alpha < \mu$, in order type μ . At a stage below μ , fewer than μ labels have been assigned. Discard every immediate child of t whose cone contains $c_s(a)$ for a previously assigned label a . Fewer than μ children are discarded. Choose a leaf above a remaining child and assign its label to e . Maximality supplies the leaf, and its label is fresh by the choice of child. Color all remaining labels arbitrarily.

Thus $|C| = |D| = \mu$. For each (s, t) we have disjoint sets of child symbols $A_{s,t}^C, A_{s,t}^D \subseteq \Omega$, both of size μ , and for $x \in A_{s,t}^e$ a chosen label $a_{s,t,x}^e \in e$ such that

$$t \frown x \subseteq c_s(a_{s,t,x}^e).$$

All chosen child symbols for this fixed (s, t) are distinct, including across the two colors. A previously chosen witness would have excluded that child at every later request.

Choose a stationary bijection $d : \Omega \rightarrow C^{<\omega}D$ and let h be its concatenation map. The composite Fh has maximal fronts obtained by encoding raw letters through F until the first stopping letter has been encoded. They are maximal because an unfinished raw code can be completed and, if necessary, followed by a stopping letter.

At any internal node q of a composite front, finite parsing determines a current raw state s and an internal raw node t , possibly empty. The suffix of $c_s(a_{s,t,x}^D)$ after t leads from q to a composite terminal node. The suffix of $c_s(a_{s,t,x}^C)$ leads to a composite internal node, since it finishes a continuation code. Denote these nonempty suffixes by $w_{q,x}^e$. Their first symbols are the distinct chosen child symbols.

Choose a positive finite cutoff for each immediate-child cone:

$$k_q(x) = \begin{cases} |w_{q,x}^e|, & x \in A_{s,t}^e, e \in \{C, D\}, \\ 1, & x \notin A_{s,t}^C \cup A_{s,t}^D. \end{cases}$$

Starting at q , stop at the first composite terminal or at distance $k_q(x)$, whichever comes first, where x is the first new symbol. The resulting relative front T_q is a bar. Each branch has a first symbol, which determines a finite cutoff. Each selected witness survives, since no composite terminal properly precedes it. Consequently T_q has exactly μ terminal pieces and exactly μ internal pieces. No bound uniform over its first-child cones is required.

At each state match C bijectively to the internal pieces and D bijectively to the terminal pieces of T_q . Reading a letter emits its matched piece. An internal piece advances within the current composite front; a terminal piece finishes it, recovers its composite input label, and starts the next front with that recovered history. This defines a history-dependent bar-front code R and hence a whole-space homeomorphism.

Each input d -word produces exactly one composite terminal leaf. Conversely, parse any composite leaf into successive T_q pieces. The terminal stopping rule stops parsing at the leaf. Since the pieces are nonempty, parsing terminates after finitely many steps. The corresponding input letters lie in C until the final letter in D , so form one unique d -word. Thus the matching of composite input labels is bijective at every state. These bijections define a level-preserving tree map v . For a source stream x , the successive composite labels emitted by $Rh(x)$ are $v(x)$; therefore $Rh(x) = Fhv(x)$. This is also the asserted identity of Boolean automorphisms. $\square$

Corollary 6.11. *Every history-dependent maximal prefix-code automorphism belongs to K .*

Proof. In (21), Lemmas 6.6 and 6.8 give $v, R \in K$, and Lemma 6.9 gives $hKh^{-1} = K$. Therefore $F = Rhv^{-1}h^{-1} \in K$. $\square$

Lemma 6.12. *Every $g \in \text{Aut}(B)$ has a factorization*

$$g = P b^{-1} o Q^{-1},$$

where P, Q are history-dependent maximal prefix-code automorphisms, b replaces every input letter by two letters using one fixed bijection $\Omega \rightarrow \Omega^2$, and o keeps the first input letter single and uses that bijection thereafter.

Proof. Let $\mathcal{A}_n$ be the complete atomic algebra with atoms the length- n cylinders, with $\mathcal{A}_0 = \{0, 1\}$. Construct increasing complete atomic algebras $\mathcal{D}_n$, starting at $\mathcal{D}_0 = \{0, 1\}$, such that every atom has exactly μ children at the next level. At even levels their atoms are ordinary cylinders; at odd levels they are g -images of cylinders.

Both cylinder bases are order dense and have cardinality μ . Below a current atom choose a maximal antichain from the next required basis. Its join is that atom and its size is at most μ . Replace one member by its μ immediate successors in that same basis. The resulting antichain has exactly μ members. Repeating this below every atom gives the next level. Each refinement is strict. Between consecutive even levels, every cylinder is thus replaced by proper subcylinders, so the atoms of $\mathcal{D}_{2n}$ have lengths at least n . Hence $\mathcal{A}_n \subseteq \mathcal{D}_{2n}$, and the union of the $\mathcal{D}_n$ is order dense. Similarly the cylinders whose images are atoms of $\mathcal{D}_{2n-1}$ have lengths at least n .

Choose bijections of corresponding sets of children. They define compatible atomic isomorphisms between $\mathcal{A}_n$ and $\mathcal{D}_n$. The isomorphism on their unions extends uniquely to a complete Boolean automorphism t , since both unions are order dense. Thus $t[\mathcal{A}_n] = \mathcal{D}_n$.

The stated reblockings satisfy

$$b[\mathcal{A}_n] = \mathcal{A}_{2n}, \quad o[\mathcal{A}_n] = \mathcal{A}_{2n-1} \quad (n \geq 1).$$

Put $P = tb$ and $Q = g^{-1}to$. Each sends input cylinders to actual cylinders, and sends immediate successors to a maximal antichain of proper subcylinders of the parent image. Their nonempty relative suffixes give the required state fronts and labels. At the root the images for Q are proper as well. The resulting code maps induce the prescribed Boolean maps, since they agree on every cylinder. Finally,

$$(tb)(b^{-1}o)(g^{-1}to)^{-1} = g.$$

$\square$

Proof of Theorem 6.1. Choose $\bar{y}$ by Lemma 6.5 and form K as in (20). For an arbitrary $g \in G$, apply Lemma 6.12. Its maps P, Q belong to K by Corollary 6.11. Both b and o are history-dependent bar-front code maps, so Lemma 6.8 gives $b, o \in K$. Every factor in the displayed factorization of g therefore belongs to K . Thus $K = G$. If $\dot{X}$ is forced to have finite size m , the ground cardinal ν is also forced to have size m and hence equals m already in W , since an infinite ground set cannot become finite. $\square$

Corollary 6.13. *Let a be a set of ordinals, $W = L[a]$, and let $V = W[H]$ be a $\text{Coll}(\omega, \mu)^W$ -extension. Every countable $\text{OD}(a)$ set in V admits an $\text{OD}(a)$ bijection from some ground cardinal $\nu \leq \mu$. In particular, all its members are $\text{OD}(a)$, without a restriction on their ranks. For a finite set of size m , one has $\nu = m$.*

Proof. The same proof works with any ground W satisfying $\text{ZFC} + \text{GA} + V = \text{HOD}$ and any ground set of ordinals a . For ground definability, we use only Lemma 2.2 or the canonical a -definable order when $W = L[a]$. No GCH assumption is needed. For the empty family, use the empty bijection with $\nu = 0$. Fix a formula and ordinal parameters uniquely defining the given nonempty countable family. Weak homogeneity makes this assertion forced by the top condition. A name for the unique family is invariant under all ground Boolean automorphisms. Theorem 6.1 supplies a bijective enumeration by invariant names indexed by a ground cardinal $\nu \leq \mu$. The name for the whole enumeration is invariant too, since its index ordinal is fixed. Lemma 2.3 makes this bijection $\text{OD}(a)$ in V . Each of its values is then $\text{OD}(a)$ using its ordinal index. $\square$

Remark 6.14. The theorem does not in general give an OD enumeration indexed by ω . If μ is uncountable in L , the ground set μ is an OD countable set in the collapse extension, but an OD bijection $\omega \rightarrow \mu$ would be a new OD set of ordinals, contrary to homogeneity over L . The ground-cardinal index in the conclusion is therefore necessary. The truncation fronts need not have bounded height. The reservoir factorization requires only that parsing terminate on each branch.

6.4 Rank-one consequences and invariant parameters

Corollary 6.15. *Let $W = L[z]$, where z is a real, let λ be an infinite cardinal of W , and let V be a $\text{Coll}(\omega, \lambda)^W$ -extension. Every countable $\text{OD}(z)$ family of sets of ordinals consists of $\text{OD}(z)$ members. The same holds for a countable $\text{OD}(z)$ family of open subsets of an $\text{OD}(z)$ set D of generic filters, with basic opens $N_p = \{H \in D : p \in H\}$.*

Proof. Both are instances of Corollary 6.13. Equivalently, an open set U is recovered from the ordinal code $c(U) = \{p : N_p \subseteq U\}$ by $U = \bigcup_{p \in c(U)} N_p$. The stable-ground generalization is the same as in that corollary. $\square$

The following variant permits a parameter fixed only by a subgroup. For an infinite cardinal λ in a ZFC ground W , use the dense presentation $\mathbb{P} = \lambda^{<\omega}$.

Lemma 6.16 (Generic graphs). *For $p \in \mathbb{P}$ and a family $\mathcal{E}$ of at most λ dense open subsets of $\mathbb{P}^2$, there is a ground tree automorphism h fixing p such that $\{q \supseteq p : (q, h(q)) \in E\}$ is dense below p for every $E \in \mathcal{E}$.*

Proof. In λ steps meet every requirement (E, s) with $s \supseteq p$, and put every node into the domain and range. Start by fixing the initial segments of p . For (E, s) , extend the partial tree isomorphism to $s \mapsto t$, choose unused successors $s \frown \alpha, t \frown \beta$, and find $(q, r) \in E$ extending this pair. Extend one coordinate to make their lengths equal, then match their fresh finite paths. Domain and range requirements use the same back-and-forth extension. At stage $\xi < \lambda$ the partial map is finite if $\xi < \omega$, and otherwise has size at most $\max(\aleph_0, |\xi|) < \lambda$. Fresh successors therefore remain, even when λ is singular. The union is the required total automorphism. $\square$

Lemma 6.17 (Invariant parameters). *Suppose every ground set is $\text{OD}(z)$ in every set-forcing extension of W , and $\Gamma \leq \text{Aut}(\mathbb{P})$ has the generic-graph property of Lemma 6.16 for every countable $\mathcal{E}$ and every p . If $\dot{C}$ is fixed by Γ , every countable $\text{OD}(z, C)$ family of sets of ordinals in a $\mathbb{P}$ -extension is pointwise $\text{OD}(z, C)$.*

Proof. Otherwise its non- $\text{OD}(z, C)$ subfamily is nonempty. Below some p choose names $(\tau_n)_{n < \omega}$ enumerating it. For each n, m , (p, p) forces $\tau_n^0 \neq \tau_m^1$. Otherwise a common value in two mutually generic extensions would belong to their intersection W and hence be $\text{OD}(z)$, a contradiction.

Since these are sets of ordinals, pairs deciding an ordinal oppositely for τ_n, τ_m form a dense open set below (p, p) . Extend these dense sets outside that cone and apply the stipulated property to obtain $h \in \Gamma$ fixing p . For the actual generic G containing p , it gives $\tau_n^G \neq \tau_m^{h[G]}$ for all n, m . But $W[G] = W[h[G]]$ and $C^G = C^{h[G]}$, so both enumerations describe the same nonempty definable family, a contradiction. $\square$

7 Definable families modulo null and meager sets

Let $\mathcal{N}$ and $\mathcal{M}$ denote the ideals of Lebesgue-null and meager subsets of $\mathbb{R}$, respectively. For either ideal $\mathcal{I}$, write $A \equiv_{\mathcal{I}} B$ when $A \triangle B \in \mathcal{I}$, and let $[A]_{\mathcal{I}}$ denote the resulting class in $\mathcal{P}(\mathbb{R})/\mathcal{I}$. Friedman [11] obtained the small-family conclusion below in a further extension of Solovay's model, and asked whether its single-class case already holds in the original collapse extension. We prove the full conclusion there.

Theorem 7.1. *Let $W \models \text{ZFC}$, let κ be inaccessible in W , and let $G \subseteq \text{Coll}(\omega, < \kappa)^W$ be generic. In $W[G]$, an $\text{OD}_{\mathbb{R}}$ family $K \subseteq \mathcal{P}(\mathbb{R})$ containing a non-Lebesgue-measurable member represents exactly 2^c classes modulo $\mathcal{N}$. Consequently, if it represents fewer than 2^c classes, all its members are Lebesgue measurable. The same statements hold for the Baire property and $\mathcal{M}$.*

The extension here is the full ZFC generic extension, rather than its inner Solovay model. No additional hypothesis on W is required. The same proof applies to both ideals.

7.1 Bounded collapse algebras and local separation

Work in W and use the finite-support presentation

$$P = \text{Coll}(\omega, < \kappa), \quad \mathbb{B} = \text{RO}(P), \quad \mathbb{B}_{\alpha} = \text{RO}(\text{Coll}(\omega, < \alpha)).$$

The forcing is κ -cc, $|\mathbb{B}| = \kappa$, and $|\mathbb{B}_{\alpha}| < \kappa$ for $\alpha < \kappa$. Every Boolean value and every name for a real is supported on some $\mathbb{B}_{\alpha}$. To see this, represent them by maximal antichains of size less than κ and bound their coordinates. There are κ nice names for reals. In the full extension, the reals of each bounded intermediate model $W[G_{\alpha}]$ form a countable set. There are fewer than κ nice $\mathbb{B}_{\alpha}$ -names for reals in W , and the full forcing makes this set of names countable.

We will repeatedly use two properties of the initial algebras. First, an automorphism of $\mathbb{B}_{\alpha}$ extends to every larger $\mathbb{B}_{\beta}$ by acting identically on the independent tail. Second, if $\langle \alpha_i : i < \eta \rangle$ is increasing with limit δ , then $\bigcup_{i < \eta} \mathbb{B}_{\alpha_i}$ is dense in $\mathbb{B}_{\delta}$, since it contains the initial forcing posets. Thus coherent automorphisms defined on whole initial algebras extend to their completion at a limit stage.

Lemma 7.2 (Bounded extension). *If E, F are complete subalgebras of $\mathbb{B}_{\gamma}$ and $h : E \rightarrow F$ is a complete isomorphism, then h extends to an automorphism of some $\mathbb{B}_{\delta}$, where $\delta < \kappa$ may be required to exceed any prescribed bound.*

Proof. Choose a regular uncountable cardinal $\lambda < \kappa$ above that bound and above $|\mathbb{B}_{\gamma}|$, and put $\delta = \lambda + 1$. In an E -generic extension, λ remains regular and the quotient $\mathbb{B}_{\delta}/E$ has density at most λ and collapses λ to countable. Its density is exactly λ , since a forcing of smaller density would preserve the regular cardinal λ . Its completion is therefore the collapse algebra $\text{RO}(\text{Coll}(\omega, \lambda))$. The same holds over F . Identify these quotients through h , then use the complete-subalgebra factorization to lift h to $\mathbb{B}_{\delta}$. For the collapse characterization and relative factorization used here, see [17, 22]. $\square$

Fix $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}\}$. For a set A failing the corresponding regularity property, there is a positive Borel region D on which both A and its complement meet every positive Borel set even after removal of any member of $\mathcal{I}$. In the measure case, first restrict to a bounded interval on which A is nonmeasurable. Choose a Borel inner core $I \subseteq A$ and outer hull $O \supseteq A$ attaining its inner and outer measures. Then $D = O \setminus I$ has positive finite measure, and $A \cap D$ and $D \setminus A$ both have full outer measure in D . In the category case, let U be the union of the basic open sets on which A is meager and V the corresponding union for its complement. The sets U, V are disjoint. If their union were dense, A would have the Baire property. Otherwise choose a nonempty basic open D disjoint from its closure. Both sides are nonmeager in every nonempty open subset of D , which gives the stated property for positive Borel sets.

Now suppose a basic condition p_0 forces this property of names A, D . Fix an initial stage $\mathbb{B}_{\alpha_0}$ supporting p_0 and the Borel code of D . All our automorphisms fix this algebra pointwise. Dots on names are suppressed.

Lemma 7.3 (Persistent pair separation). *Let $\theta_s, \theta_t \in \text{Aut}(\mathbb{B}_\alpha)$ fix $\mathbb{B}_{\alpha_0}$ pointwise. Suppose $0 < p \leq p_0$ belongs to $\mathbb{B}_\alpha$ and N is a $\mathbb{B}_\alpha$ -name for a Borel member of $\mathcal{I}$. There are extensions $\sigma_s, \sigma_t \in \text{Aut}(\mathbb{B}_\beta)$, for some $\beta < \kappa$, and $0 < f \leq p$ such that every pair of full automorphisms $\pi_s, \pi_t \in \text{Aut}(\mathbb{B})$ extending them satisfies*

$$f \Vdash (\pi_s(A) \triangle \pi_t(A)) \setminus N \neq \emptyset.$$

The two full extensions may be chosen independently.

Proof. We first construct a one-sided separation for an automorphism θ of $C = \mathbb{B}_\alpha$, a condition $0 < p \leq p_0$ in C , and a C -name N as above. Put $q = \theta^{-1}(p)$. We will extend θ to $\tau \in \text{Aut}(\mathbb{B}_\beta)$ and find conditions $e, r \in \mathbb{B}_\beta$ and real names x, y supported there such that

$$\begin{aligned} \tau(e) = r, \quad 0 < r \leq p, \quad e \Vdash x \in A, \quad r \Vdash y \notin A, \\ r \Vdash \tau(x) = y \text{ and } y \notin N. \end{aligned} \tag{22}$$

In a C -generic extension, let Q_D be the algebra of Borel subsets of D modulo $\mathcal{I}$. This is the complete ccc random algebra in the measure case and the complete ccc Cohen algebra in the category case. In the full extension, the union of the Borel members of $\mathcal{I}$ coded in this intermediate model belongs to $\mathcal{I}$, since its family of codes has become countable. Thus the reals generic for Q_D over the intermediate model form a conull, respectively comeager, subset of D . The property of $A \cap D$ therefore allows us to choose a name x such that q forces that $x \in A \cap D$ is generic over that model.

In a C -generic extension containing q , this name induces a unital complete homomorphism into the remaining forcing algebra:

$$j_x : Q_D \longrightarrow \text{RO}(\mathbb{B}/G_C), \quad [H]_{\mathcal{I}} \longmapsto \|x \in H\|.$$

Genericity makes the map well-defined, and Borel evaluation preserves countable joins. Since Q_D is ccc, each of its joins reduces to a countable join, so the map is complete. Its kernel is therefore a principal ideal. Let u be the complement of the largest element of the kernel. Then $u > 0$, $j_x(u) = 1$, and j_x is injective on $Q_D \restriction u$. Choose a C -name U for a positive Borel representative of u .

On the range side, below p , choose $y \in D \setminus A$ generic over the range intermediate model and belonging to $\theta(U)$. This is possible by the same property of the complement. Let V be a C -name representing the positive support of j_y ; thus $[V]_{\mathcal{I}} \leq [\theta(U)]_{\mathcal{I}}$. Restrict the domain side to

$$t = q \wedge \|x \in \theta^{-1}(V)\|.$$

Its Boolean projection onto C is q , because $\theta^{-1}([V]_{\mathcal{I}})$ is positive below u and j_x is injective there. The restricted random or Cohen homomorphism has exactly this smaller positive support.

All these real names and Boolean values occur at a bounded initial stage. Since membership in each Borel set coded in the fixed intermediate model is absolute, genericity over that model is absolute to further forcing. Hence the homomorphisms above can already be computed in the completion of a bounded quotient containing their real names, with the same kernels and supports. Beyond this stage choose fresh binary splits c, d , each of whose two values has projection 1 onto that stage, and put

$$e = t \wedge c, \quad r = p \wedge d.$$

Both e and $q \setminus e$ project to q in C ; both r and $p \setminus r$ project to p . Because the splits are fresh, these restrictions send no nonzero element of the respective support algebras to zero.

Form the complete subalgebra generated over C by e and the membership values of x restricted to e . Over a C -generic filter containing q , it is the lottery sum of $Q_D \restriction [\theta^{-1}(V)]_{\mathcal{I}}$ and a trivial branch. The first branch is embedded by $[H]_{\mathcal{I}} \mapsto e \wedge j_x([H]_{\mathcal{I}})$; the second is represented by $q \setminus e$. Outside q the quotient is trivial. Similarly, the range algebra has branches $Q_D \restriction [V]_{\mathcal{I}}$ and the trivial branch below $p \setminus r$. These descriptions give complete embeddings into a common bounded initial collapse algebra. They also give a complete isomorphism extending θ , sending e to r , and matching the membership values of x and y on those branches. Every extension to an algebra supporting x and y therefore sends x to y below r . After a fixed Borel coding of reals, the restricted values match each binary digit. Apply Lemma 7.2 to extend this isomorphism to $\tau \in \text{Aut}(\mathbb{B}_\beta)$.

Our choices give $e \Vdash x \in A$ and $r \Vdash y \notin A$. The genericity of y over a model containing the code of N also gives $y \notin N$. This proves (22). No bound on the support of A was used.

To separate the given pair, apply this construction to

$$\theta = \theta_t^{-1} \theta_s, \quad p' = \theta_t^{-1}(p), \quad N' = \theta_t^{-1}(N).$$

Obtain τ, e, r, x, y satisfying (22) with p', N' in place of p, N . Extend θ_t to $\eta \in \text{Aut}(\mathbb{B}_\beta)$ by the identity on the tail, and set

$$\sigma_s = \eta\tau, \quad \sigma_t = \eta, \quad f = \eta(r) \leq p.$$

These extend the required initial maps. For any independent full extensions π_s, π_t , the equations for the bounded names give

$$\pi_s(e) = \pi_t(r) = f, \quad f \Vdash \pi_s(x) = \pi_t(y).$$

Applying the full automorphisms to the two membership inequalities in (22), this common real belongs to $\pi_s(A) \setminus \pi_t(A)$ below f . It lies outside N , since $\pi_t(N') = N$. The equations matching the real names involve only bounded-stage Boolean values and so persist under further extensions; the membership inequalities are preserved by every full automorphism. Thus the separation holds for every independent choice of the two full extensions. $\square$

7.2 The maximal family of inequivalent images

Proof of Theorem 7.1. Fix $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}\}$. Suppose a basic condition p_0 forces that K is the family defined by a fixed formula from real and ordinal parameters and that $A \in K$ fails the corresponding regularity property. Strengthen p_0 and choose D as above. Take $\alpha_0 < \kappa$ supporting p_0 , the real parameters, and the code of D . We retain the original $A \in K$; the region D is used only to choose

the witnesses in Lemma 7.3. Every full automorphism fixing $\mathbb{B}_{\alpha_0}$ pointwise preserves the definition of K below p_0 . It therefore sends A to a member of K below that condition.

Enumerate in W all pairs (p_i, N_i) , $i < \kappa$, where p_i is a basic condition below p_0 and N_i is a name for a Borel member of $\mathcal{I}$. It suffices to use nice real names, replacing a name by the code of the empty set on the Boolean region where it fails to code a Borel member of $\mathcal{I}$. This list includes a name for every possible Borel $\mathcal{I}$ -cover.

We build a binary tree of partial automorphisms. For each $\xi < \kappa$, assign to every node $s \in 2^\xi$ an automorphism θ_s of the same whole initial algebra $\mathbb{B}_{\alpha_\xi}$. They fix $\mathbb{B}_{\alpha_0}$ pointwise and extend the maps at their predecessors. Require the stages to increase cofinally in κ . At a successor level initially give both children their parent's map. At a limit level extend the coherent union along each node to the completion of the common initial algebra.

At level ξ , first enlarge the common stage to support all (p_i, N_i) with $i < \xi$. For every pair of distinct nodes $s, t \in 2^\xi$ and every $i < \xi$, apply Lemma 7.3 to these two maps and (p_i, N_i) . Replace them by the resulting extensions and extend all other maps to the same larger initial stage by the identity on the tail. Earlier separations persist under these independent extensions. There are fewer than κ tasks at this level, since $2^{|\xi|} < \kappa$. At limits within the level take completions of the coherent maps on whole initial algebras. Regularity of κ keeps the common stage below κ . Finally enlarge it beyond ξ . The recursion is now complete. At each limit step, the union of the initial algebras is dense in the next initial algebra, as required.

Each ground-model branch $b \in (2^\kappa)^W$ now gives a full automorphism π_b of $\mathbb{B}$. If $b \neq c$ and $i < \kappa$, choose a level $\xi > i$ after their first split. The requirement for $b \restriction \xi$, $c \restriction \xi$, and i supplies $0 < f \leq p_i$ forcing a point of $\pi_b(A) \triangle \pi_c(A)$ outside N_i . Consequently,

$$p_0 \Vdash [\pi_b(A)]_{\mathcal{I}} \neq [\pi_c(A)]_{\mathcal{I}} \quad (b \neq c). \quad (23)$$

Indeed, a condition forcing equivalence would force the symmetric difference into a Borel member of $\mathcal{I}$. Take a name for that cover and a basic condition below the proposed condition. The corresponding requirement gives a contradiction.

It remains to count these ground-model branches in the extension. There $\mathfrak{c} = \kappa = \omega_1$ and

$$(2^\kappa)^{W[G]} = (2^\kappa)^W \quad \text{as cardinals.}$$

For the latter equality, $|\mathbb{B}|^W = \kappa$, so subsets of κ have at most $\kappa^\kappa = (2^\kappa)^W$ Boolean names. The κ -cc preserves this ground cardinal, and the ground subsets give the reverse inequality. Thus the branch set used in (23) has size $2^\mathfrak{c}$ in the extension. All images belong to K , so K represents at least that many classes, and it cannot represent more than $|\mathcal{P}(\mathbb{R})| = 2^\mathfrak{c}$. $\square$

Corollary 7.4. *In the extension of Theorem 7.1, if $[A]_{\mathcal{N}}$ is $\text{OD}_{\mathbb{R}}$, then A is Lebesgue measurable. If $[A]_{\mathcal{M}}$ is $\text{OD}_{\mathbb{R}}$, then A has the Baire property. In either case the class has a Borel representative.*

Proof. Apply the theorem to the family $K = [A]_{\mathcal{I}}$, which represents just one class. Every measurable set, respectively every set with the Baire property, agrees with a Borel set modulo the corresponding ideal. $\square$

8 Small families in Cohen extensions

We first prove a small-index theorem over any ZFC ground satisfying the cardinal-arithmetic assumptions below, then apply it to definable families. We distinguish the closure cardinal θ

from the width λ . Ordinary Cohen forcing is the case $\theta = \omega$. The construction follows the generic-automorphism and coset-tree method of [23, 12, 33].

Work in an arbitrary $W \models \text{ZFC}$. Let θ be regular infinite, let $\lambda > \theta$ be regular uncountable, and assume

$$\theta^{<\theta} = \theta, \quad \nu^{<\theta} < \lambda \quad (\nu < \lambda), \quad \lambda^{<\lambda} = \lambda. \quad (24)$$

Here ν ranges over cardinals. Put

$$P = \text{Add}(\theta, \lambda) = \text{Fn}(\lambda \times \theta, 2, <\theta), \quad B = \text{RO}(P), \quad G = \text{Aut}(B).$$

For $S \subseteq \lambda$, write $P_S = \text{Add}(\theta, S)$ and let B_S be its specified complete coordinate subalgebra of B . Write $b_{\alpha, \xi}$ for the Boolean value that coordinate bit (α, ξ) is 1. For a complete $C \subseteq B$, put $G_C = \{g \in G : g \restriction C = \text{id}_C\}$.

Theorem 8.1 (Cohen small index). *Every subgroup $H \leq G$ of index at most λ contains G_{B_S} for some $S \in [\lambda]^{<\lambda}$. Equivalently, it contains the pointwise stabilizer of a complete subalgebra having an order-dense subset of size less than λ .*

For the forward implication, B_S has an order-dense Boolean subalgebra of size at most $(\max(\theta, |S|))^{<\theta} < \lambda$ when $S \neq \emptyset$, obtained by closing its conditions under finite Boolean operations. Conversely, the supports of fewer than λ Boolean elements have union of size less than λ by the counting lemma below. Thus a complete subalgebra with a small dense subset is contained in a small B_S , and its pointwise stabilizer contains G_{B_S} . For $\theta = \omega$, the assumptions reduce to regular uncountable λ with $\lambda^{<\lambda} = \lambda$. In particular the ordinary theorem holds over every $\text{ZFC} + \text{GCH}$ ground. For uncountable θ , GCH implies the displayed assumptions when $\lambda = \theta^+$; for other widths we retain all three assumptions.

8.1 Coordinate factors and relative amalgamation

For complete Boolean algebras A, D , let $A \hat{\otimes} D$ denote the completion of the forcing product $A^+ \times D^+$, not their Boolean direct product. Put $\mathbb{C}_\nu = \text{RO}(\text{Add}(\theta, \nu))$, with $\mathbb{C}_0 = \{0, 1\}$. An isomorphism over C fixes its specified copy of C pointwise.

Definition 8.2. Let

$$\mathcal{F} = \{g[B_S] : g \in G, S \in [\lambda]^{<\lambda}\}.$$

For $C, D \in \mathcal{F}$, write $C \preceq D$ if some single $g \in G$ and $S \subseteq T$ in $[\lambda]^{<\lambda}$ satisfy $C = g[B_S]$ and $D = g[B_T]$.

Both $\mathcal{F}$ and $\preceq$ are invariant under G . The relation requires a simultaneous coordinate presentation, so it is stronger than inclusion of complete subalgebras.

Lemma 8.3. *Every $C \in \mathcal{F}$ has a factorization $B \cong C \hat{\otimes} \mathbb{C}_\lambda$ over C . Every complete isomorphism between members of $\mathcal{F}$ extends to an element of G .*

Proof. For $C = B_S$, the unused coordinates have cardinality λ . For an arbitrary member of $\mathcal{F}$, transport this product through its coordinate presentation. An isomorphism between two such factors combines with any isomorphism of their full Cohen complements to give an automorphism of the two product presentations of B . $\square$

Lemma 8.4. *For $C, D \in \mathcal{F}$, one has $C \preceq D$ exactly when, for some $\nu < \lambda$, there is an isomorphism $C \hat{\otimes} \mathbb{C}_\nu \cong D$ over C . In particular, $\preceq$ is transitive.*

Proof. A simultaneous presentation uses the coordinates in $T \setminus S$ as the relative complement. Conversely, write $C = u[B_S]$ and choose $U \subseteq \lambda \setminus S$ of cardinality ν . The relative product gives a complete isomorphism $B_{S \cup U} \rightarrow D$ restricting to u on B_S . Lemma 8.3 extends it to a full automorphism, which witnesses the required simultaneous presentation. Two relative Cohen complements combine to a Cohen complement on fewer than λ coordinates, proving transitivity. $\square$

Lemma 8.5. *Every Boolean element has a column support of size at most θ , and $|B| = \lambda$. There are at most λ members of $\mathcal{F}$, complete isomorphisms between them, and tuples of such isomorphisms of length less than λ .*

Proof. The forcing has size $\lambda^{<\theta} = \lambda$ and is θ^+ -cc. Indeed, the generalized delta-system lemma, using regularity and $\theta^{<\theta} = \theta$, gives a delta system of θ^+ domains from any such family of conditions. There are at most θ assignments on the common root, so two conditions agree there and are compatible. For $\theta = \omega$ this is the ordinary finite-set delta-system proof of ccc.

Every Boolean value is a join of a maximal antichain of conditions below it, of size at most θ . The union of their supports has size at most θ . Thus $|B| \leq \lambda^\theta = \lambda$, and the coordinate bits give the reverse inequality. A factor B_S has $|S| \cdot \theta < \lambda$ complete generators. An image factor and a complete isomorphism are determined by the images of these generators. The assumption $\lambda^{<\lambda} = \lambda$ gives the asserted bounds, including the bound for tuples. $\square$

Lemma 8.6 (Invariant coordinate enlargement). *Suppose $\mathcal{E} \subseteq \mathcal{F}$ and $\Gamma \subseteq G$ have cardinality less than λ , and $T \in [\lambda]^{<\lambda}$. There is $S \in [\lambda]^{<\lambda}$ containing T such that*

$$g[B_S] = B_S \quad (g \in \Gamma), \quad C \preceq B_S \quad (C \in \mathcal{E}).$$

The factor may also contain any prescribed set of fewer than λ Boolean elements.

Proof. Choose presentations $C = u_C[B_{T_C}]$ for $C \in \mathcal{E}$. Adjoin these u_C and all inverses to Γ , obtaining a family Γ^* of size less than λ . Let S_0 contain T , all T_C , and supports of size at most θ for the prescribed Boolean elements. From S_m , form S_{m+1} by adjoining supports of size at most θ for

$$g(b_{\alpha, \xi}) \quad (g \in \Gamma^*, \alpha \in S_m, \xi < \theta).$$

There are fewer than λ maps and bits at every stage. By regularity, $S = \bigcup_{m < \omega} S_m$ has size less than λ . The coordinate bits completely generate B_S , so each $g \in \Gamma^*$ maps it into itself; inverse closure gives equality. In particular $u_C[B_S] = B_S$ and $T_C \subseteq S$. The same u_C therefore witnesses $C = u_C[B_{T_C}] \preceq u_C[B_S] = B_S$. $\square$

The closure under the presentations u_C ensures the relative factor relation. Collecting only the supports of elements of C would not establish it.

Lemma 8.7 (Relative amalgamation). *Let $C \preceq D_0, D_1$ belong to $\mathcal{F}$. For i in an index set J , suppose $s_i \in \text{Aut}(D_0)$ and $t_i \in \text{Aut}(D_1)$ preserve C and agree there. There are $a \in G_C$, $E \in \mathcal{F}$, and $r_i \in \text{Aut}(E)$ such that $D_0 \preceq E$, $a[D_1] \preceq E$, and r_i extends both s_i and $at_i a^{-1}$ on their respective domains. The placement a and factor E are the same for all i .*

Proof. Choose $C = v[B_S]$, $D_0 = v[B_T]$, where $S \subseteq T$. Write $D_1 \cong C \hat{\otimes} \mathbb{C}_\nu$ over C , and choose $U \subseteq \lambda \setminus T$ of size ν . Set

$$D'_1 = v[B_{S \cup U}], \quad E = v[B_{T \cup U}].$$

An isomorphism $D_1 \rightarrow D'_1$ fixing C extends to $a \in G_C$. The three independent factors of E are C , the coordinates in $T \setminus S$, and those in U .

For each i , let $t'_i = at_i a^{-1} \upharpoonright D'_1$ and let β_i be its common restriction with s_i to C . Extend β_i independently to E . Removing this extension from s_i, t'_i leaves maps fixing C pointwise. Extend each independently over the other's complementary factor. These extensions commute. One fixes C and the entire third factor, and the other fixes C and the entire second factor. Each therefore fixes the images of the other's extra generators. Their product, followed by the independent extension of β_i , is the required r_i . The product factorizations make all these maps complete automorphisms. The placement a and factor E were chosen independently of i . $\square$

8.2 Coherent limits through coordinate factors

For this subsection only, let Ω be any set and θ any regular infinite cardinal. Put $P = \text{Add}(\theta, \Omega)$ and $B = \text{RO}(P)$, with the same notation P_S, B_S for $S \subseteq \Omega$. No cardinal-arithmetic assumption is needed.

Write $[p]$ for the Boolean value of a condition. For a complete subalgebra C , let $\pi_C(b)$ be the least element of C above b . Thus, for $c \in C$,

$$b \wedge c > 0 \iff \pi_C(b) \wedge c > 0. \quad (25)$$

If $C \subseteq D$ are complete, then $\pi_C \pi_D = \pi_C$. A complete isomorphism commutes with these projections onto the subalgebras it carries to one another. On conditions,

$$\pi_{B_S}([p]) = [p \upharpoonright (S \times \theta)]. \quad (26)$$

The projection identities follow from the definition of π_C , and the formula on conditions follows from the product on S and its complement. For increasing sets S_i , we also have

$$\bigwedge_i [p \upharpoonright (S_i \times \theta)] = [p \upharpoonright ((\bigcup_i S_i) \times \theta)]. \quad (27)$$

This meet is positive because all its prescribed bits belong to the domain of p , which has size less than θ .

Suppose increasing complete subalgebras satisfy $C_i \subseteq A_i \subseteq C_{i+1}$, where $A_i = B_{S_i}$ are literal coordinate factors. At a nonzero limit j , the complete closure of $\bigcup_{i < j} C_i$ is $B_{\bigcup_{i < j} S_i}$. This follows from complete generation by the individual coordinate bits. The interleaving inequalities between the projections, followed by (27), give

$$\bigwedge_{i < j} \pi_{C_i}([p]) = \bigwedge_{i < j} \pi_{A_i}([p]) = \pi_{B_{\bigcup_{i < j} S_i}}([p]). \quad (28)$$

Lemma 8.8 (Extension through coordinate factors). *Let δ be a nonzero limit ordinal. Suppose $\varphi_\xi : C_\xi \rightarrow D_\xi$, $\xi < \delta$, are coherent complete isomorphisms between increasing complete subalgebras of B . Suppose there are literal coordinate factors A_ξ and B'_ξ such that*

$$C_\xi \subseteq A_\xi \subseteq C_{\xi+1}, \quad D_\xi \subseteq B'_\xi \subseteq D_{\xi+1} \quad (\xi < \delta). \quad (29)$$

Let B_S, B_T be the complete closures of the respective unions. There is a unique complete isomorphism $\Phi : B_S \rightarrow B_T$ extending every φ_ξ .

Proof. If $\text{cf}(\delta) \geq \theta$, each condition on the union of the source supports belongs to some coordinate stage, because it uses fewer than θ columns. The same holds on the target side. The unions are therefore order-dense in their completions, and the coherent isomorphism extends uniquely.

This proves the case $\theta = \omega$, since every nonzero limit ordinal has infinite cofinality. Suppose instead that $\rho = \text{cf}(\delta) < \theta$. Pass to a strictly increasing cofinal sequence of length ρ . Between each pair of successive retained stages, keep an inserted coordinate factor on each side. Relabel so that (29) holds for $i < \rho$. In particular, every nonzero limit prefix of either chain still has a literal coordinate complete closure.

For $q \in P_S$, define

$$a_i = \varphi_i(\pi_{C_i}([q])), \quad F(q) = \bigwedge_{i < \rho} a_i. \quad (30)$$

The a_i decrease. We prove that their meet is positive by constructing a decreasing sequence of target conditions.

Take preimages of the target coordinate factors at the next stage. Set

$$E_i = \varphi_{i+1}^{-1}[B'_i], \quad r_i = \varphi_{i+1}|E_i : E_i \longrightarrow B'_i, \quad d_i = r_i(\pi_{E_i}([q])).$$

Then $C_i \subseteq E_i \subseteq C_{i+1}$, the E_i increase, and the r_i are coherent. Moreover

$$a_i \geq d_i \geq a_{i+1}, \quad F(q) = \bigwedge_{i < \rho} d_i.$$

Write $B'_i = B_{T_i}$. Construct conditions $t_i \in P_{T_i}$ extending all their predecessors and satisfying $[t_i] \leq d_i$. At stage i , put

$$t_{<i} = \bigcup_{j < i} t_j, \quad T_{<i} = \bigcup_{j < i} T_j,$$

with the empty condition at $i = 0$. Regularity of θ and $i < \rho < \theta$ ensure that $t_{<i}$ is a condition in $P_{T_{<i}}$.

Let U_i be the complete closure of $\bigcup_{j < i} E_j$, taking $U_0 = \{0, 1\}$. The map r_i sends U_i onto $B_{T_{<i}}$. We claim

$$\pi_{U_i}([q]) = \bigwedge_{j < i} \pi_{E_j}([q]). \quad (31)$$

At zero this is the empty meet. At a successor, the earlier E_j have a largest member. At a nonzero limit, the chains (E_j) and (C_j) have cofinal coordinate factors between them. For example, $E_j \subseteq C_{j+1} \subseteq A_{j+1} \subseteq C_{j+2} \subseteq E_{j+2}$. Thus U_i is the corresponding coordinate factor, and (28) proves the claim. Commuting projections with the complete map r_i now gives

$$\begin{aligned} \pi_{B_{T_{<i}}}(d_i) &= r_i(\pi_{U_i}([q])) \\ &= \bigwedge_{j < i} r_i(\pi_{E_j}([q])) = \bigwedge_{j < i} d_j. \end{aligned} \quad (32)$$

By induction, $[t_{<i}]$ lies below the last meet. Since this condition belongs to $B_{T_{<i}}$, equation (25) shows that $[t_{<i}] \wedge d_i > 0$. Choose $t_i \in P_{T_i}$ below that meet. Since its Boolean value lies below $[t_{<i}]$, it extends $t_{<i}$ as a partial function. This completes the recursion. The union $t = \bigcup_{i < \rho} t_i$ is still a condition, and

$$0 < [t] \leq \bigwedge_{i < \rho} d_i = F(q).$$

Thus $F(q)$ is positive.

Interchanging the two sides and applying the same argument to the inverse chain proves, for every $t \in P_T$,

$$F^-(t) = \bigwedge_{i < \rho} \varphi_i^{-1}(\pi_{D_i}([t])) > 0. \quad (33)$$

The map F preserves order and incompatibility. For the latter, a conflicting bit of two conditions belongs to some source stage; their projections there are incompatible, as are their images. Conversely, compatible q, q' have the common extension $q \cup q'$, whose positive image lies below both $F(q)$ and $F(q')$.

The image of F is order-dense in B_T . Given $t \in P_T$, choose $q \in P_S$ with $[q] \leq F^-(t)$. For every i ,

$$\pi_{C_i}([q]) \leq \varphi_i^{-1}(\pi_{D_i}([t])),$$

and consequently

$$F(q) \leq \bigwedge_i \pi_{D_i}([t]) = [t].$$

The last equality is (28) on the full target coordinate union. Order preservation, incompatibility preservation in both directions, and density give an isomorphism of Boolean completions with $\Phi([q]) = F(q)$.

If $b \in C_i$ and $[q] \leq b$, then $F(q) \leq \varphi_i(b)$. Taking the join of all conditions below b gives $\Phi(b) \leq \varphi_i(b)$. Apply the same argument to the complement of b to obtain equality. Finally, the C_i completely generate B_S , so a complete extension is unique. $\square$

8.3 Generic families and conjugacy

We return to the assumptions on θ, λ in (24).

Definition 8.9. A family $(g_i)_{i \in I}$ in G , with $|I| \leq \lambda$, is *generic* if the following holds for every $J \in [I]^{<\lambda}$. Whenever $C \preceq D$ belong to $\mathcal{F}$, all g_i for $i \in J$ preserve C , and $t_i \in \text{Aut}(D)$ extend $g_i \upharpoonright C$, there is $a \in G_C$ such that

$$g_i \upharpoonright a[D] = at_i a^{-1} \upharpoonright a[D] \quad (i \in J). \quad (34)$$

A generic family of length less than λ is called a generic tuple.

Genericity is preserved by restriction to a subfamily, by bijective reindexing, and by simultaneous conjugation. For conjugation, transport the two factors and their prescribed automorphisms, and use the invariance of $\mathcal{F}$ and $\preceq$.

Lemma 8.10 (Conjugacy of generic tuples). *Suppose $(g_i)_{i \in J}$ and $(h_i)_{i \in J}$ are generic tuples, $|J| < \lambda$, and $C, D \in \mathcal{F}$ are invariant under the respective tuples. If $p : C \rightarrow D$ is a complete isomorphism with*

$$p(g_i \upharpoonright C) = (h_i \upharpoonright D)p \quad (i \in J),$$

then p extends to $u \in G$ with $ug_i u^{-1} = h_i$ for all i .

Proof. Construct coherent intertwining isomorphisms $p_\alpha : C_\alpha \rightarrow D_\alpha$, for $\alpha < \lambda$, starting with p . Domains and ranges belong to $\mathcal{F}$, are invariant under the appropriate tuples, and increase in $\preceq$. At each successor, extend first the domain and then the range so that both contain coordinate α .

For the forth step, Lemma 8.6 gives a fixed coordinate factor A_α with $C_\alpha \preceq A_\alpha$, invariant under all g_i , and containing coordinate α . Extend p_α to $v \in G$. Then $D_\alpha \preceq v[A_\alpha]$, and $v(g_i \upharpoonright A_\alpha)v^{-1}$ extends $h_i \upharpoonright D_\alpha$. Genericity of (h_i) gives $a \in G_{D_\alpha}$ such that $p^* = av \upharpoonright A_\alpha$ intertwines the tuples. It extends p_α and has range R_α with $D_\alpha \preceq R_\alpha$.

Choose a fixed coordinate factor $D_{\alpha+1}$ strongly extending R_α , invariant under all h_i , and containing coordinate α . Apply the forth argument to $(p^*)^{-1} : R_\alpha \rightarrow A_\alpha$, now using genericity of (g_i) . Its extension maps $D_{\alpha+1}$ onto a factor $C_{\alpha+1}$ with $A_\alpha \preceq C_{\alpha+1}$. Invert it to obtain $p_{\alpha+1}$. Thus

$$C_\alpha \preceq A_\alpha \preceq C_{\alpha+1}, \quad D_\alpha \preceq R_\alpha \preceq D_{\alpha+1}, \quad (35)$$

where A_α and $D_{\alpha+1}$ are fixed coordinate factors.

At a nonzero limit $\delta < \lambda$, the inserted A_α are cofinal coordinate factors on the source side; the successor ranges $D_{\alpha+1}$ have the same property on the target side. Apply Lemma 8.8 to obtain coordinate closures C_δ, D_δ and the unique complete extension p_δ . Regularity of λ ensures that both supports have size less than λ . Invariance and intertwining extend by complete generation. Every earlier factor remains a strong subfactor of the limit: pass through A_α on the source side and $D_{\alpha+1}$ on the target side. At stage λ , every coordinate has been included. The same lemma therefore gives a full automorphism of B . $\square$

8.4 Simultaneous construction and the coset tree

Lemma 8.11 (Simultaneous multipliers). *Let $|I| \leq \lambda$, let $(q_{ij})_{i \in I, j < \lambda}$ be a matrix in G , and fix $C_i \in \mathcal{F}$ for each row. There are $u_i \in G_{C_i}$ such that, for every $f : I \rightarrow \lambda$, the selected family $(u_i q_{i, f(i)})_{i \in I}$ is generic.*

Proof. For each row construct automorphisms u_i^ξ of fixed coordinate factors E_i^ξ , for $\xi < \lambda$. On a fixed row the domains increase and the maps extend one another. Start with the identity on a coordinate factor containing C_i .

Schedule all requirements

$$(J, f_0, C, D, (t_i)_{i \in J}), \quad (36)$$

where $\emptyset \neq J \in [I]^{<\lambda}$, $f_0 : J \rightarrow \lambda$, $C \preceq D$ belong to $\mathcal{F}$, and $t_i \in \text{Aut}(D)$ preserve C . Also schedule coordinate coverage for each $(i, \alpha) \in I \times \lambda$. Lemma 8.5 and $\lambda^{<\lambda} = \lambda$ bound the total number of requirements by λ . Each is processed once in a recursion of length λ .

At a requirement (36), put $q_i = q_{i, f_0(i)}$ for $i \in J$ and extend the current row maps to $\hat{u}_i \in G$. Choose a fixed coordinate factor E containing the current E_i^ξ for $i \in J$, with $C \preceq E$, invariant under every q_i and $\hat{u}_i$. This follows from Lemma 8.6, since fewer than λ rows participate. Set $s_i = (\hat{u}_i q_i) \upharpoonright E \in \text{Aut}(E)$.

If $s_i \upharpoonright C \neq t_i \upharpoonright C$ for some i , replace the partial map on each participating row by $\hat{u}_i \upharpoonright E$ and leave the other rows unchanged. The disagreement persists under later extensions. Since $q_i[C] \subseteq E$, every later extension u_i satisfies

$$(u_i q_i) \upharpoonright C = (\hat{u}_i q_i) \upharpoonright C.$$

Thus the antecedent of this requirement fails for the eventual selected family. This argument also covers failure of s_i to preserve C .

Otherwise all s_i preserve C and agree there with t_i . Apply Lemma 8.7 to $C \preceq E, D$. It supplies $a \in G_C$, a common factor F with $E \preceq F$ and $a[D] \preceq F$, and $r_i \in \text{Aut}(F)$ extending both s_i and $at_i a^{-1}$. Extend r_i to $\hat{r}_i \in G$, and put $w_i = \hat{r}_i q_i^{-1}$. For $b \in E$, the invariance $q_i[E] = E$ gives

$$w_i(b) = r_i(q_i^{-1}b) = (\hat{u}_i q_i)(q_i^{-1}b) = \hat{u}_i(b). \quad (37)$$

Choose a fixed coordinate factor E' containing F and invariant under all $\hat{r}_i, q_i$. Then $w_i \upharpoonright E'$ is an automorphism of E' . Use it as the new partial map on row i ; (37) proves coherence.

The witness also persists. Since $q_i[a[D]] \subseteq E'$, every subsequent row extension satisfies, for $b \in a[D]$,

$$(u_i q_i)(b) = w_i(q_i(b)) = r_i(b) = (at_i a^{-1})(b). \quad (38)$$

Thus every later extension has the required restriction on $a[D]$.

A coordinate-coverage requirement is met by extending the current row map to G , choosing a coordinate enlargement invariant under that extension and containing the requested coordinate, and restricting the extension to it. At a nonzero limit $\delta < \lambda$, handle each row separately. Its increasing supports have a union of size less than λ , and Lemma 8.8 completes the union map to an automorphism of that coordinate factor. No union of the supports of the different rows is taken.

At the end, each row includes every coordinate and hence defines a full $u_i \in G$. It fixes C_i because all maps extend the initial identity. Fix any selector f and any nonempty short J with data to which Definition 8.9 applies. The requirement with $f_0 = f \upharpoonright J$ was processed. Had it been rejected, disagreement on C would persist. It was therefore accepted, and its witness in (38) is precisely (34). The empty-index case is immediate. Neither the construction nor the verification requires f to be injective. $\square$

To prove the small-index theorem, suppose that $[G : H] \leq \lambda$ but H contains no G_{B_S} with $|S| < \lambda$. It then contains no G_C with $C \in \mathcal{F}$, since invariant coordinate enlargement puts C inside a small coordinate factor. For $p \in \text{Aut}(C)$, its full extensions meet at least two left cosets of H : if q extends p and $v \in G_C \setminus H$, then q, qv both extend p , while $qH \neq qvH$.

Lemma 8.12. *Under this contrary assumption there is a generic family of length λ containing λ entries in H and, for each $C \in \mathcal{F}$ and $p \in \text{Aut}(C)$, λ entries outside H extending p . All entries are distinct.*

Proof. Partition the row set into two sets I_0, I_1 of cardinality λ . Assign every pair (C, p) to λ rows of I_0 . There are at most λ such pairs. On an assigned row choose matrix entries among the full extensions of p , including two from different left cosets, and prescribe $C_i = C$. On every row of I_1 include representatives of all left cosets and prescribe $C_i = \{0, 1\}$.

Apply Lemma 8.11. On a row in I_0 , left multiplication by u_i preserves distinctness of left cosets, so some product $u_i q_{ij}$ is outside H . It still extends p , since $q_{ij}[C] = C$ and u_i fixes C . On a row in I_1 , choose an entry in $u_i^{-1}H$; its product with u_i lies in H . The selected family is generic.

No generic family has two equal entries. Otherwise use those two indices, base $\{0, 1\}$, and a nontrivial small coordinate factor D , prescribing the identity for one index and a bit flip for the other. Definition 8.9 would make the same automorphism have two different restrictions on the same placed factor. Thus all the indicated multiplicities count distinct entries. $\square$

Proof of Theorem 8.1. Under the contrary assumption, fix the family of Lemma 8.12. We construct a tree indexed by $2^{<\lambda}$. Since $2^{<\lambda} \leq \lambda$, it has only λ nodes before its final level. At each node s , choose a fixed coordinate factor C_s and $p_s \in \text{Aut}(C_s)$. Along its path select source and target lists of family entries h_t, k_t , each without repetition. Require that these entries preserve C_s and

$$p_s(h_t \upharpoonright C_s) p_s^{-1} = k_t \upharpoonright C_s. \quad (39)$$

Factors and maps extend along branches, and C_s contains all coordinates below the length of s . The two outgoing edges satisfy

$$h_{s,0} \in H, \quad h_{s,1} \notin H, \quad k_{s,0} = k_{s,1}. \quad (40)$$

Choices on unrelated branches may reuse entries. Start with $C_\emptyset = \{0, 1\}$ and its identity.

At a node s of length α , the earlier lists are generic tuples of length less than λ . By Lemma 8.10, p_s extends to $v \in G$ intertwining those lists globally. Choose a fresh family entry $h_{s,0} \in H$. Enlarge to a fixed coordinate factor C' containing C_s and coordinate α , invariant under $v, h_{s,0}$ and all earlier source and target entries. Then $p' = v \upharpoonright C'$ is an automorphism of C' . Choose a fresh $h_{s,1} \notin H$ agreeing with $h_{s,0}$ on C' , and a further fresh family entry k extending

$$p'(h_{s,0} \upharpoonright C')(p')^{-1}.$$

Lemma 8.12 supplies λ possibilities for each required extension, while fewer than λ entries have been excluded along the path. Give both children C', p' and put $k_{s,0} = k_{s,1} = k$. The new entries preserve C' because their restrictions are automorphisms of it. The previous intertwining equations follow from v , and the new ones from the chosen restrictions.

At a node of nonzero limit length $\delta < \lambda$, take the coordinate factor on the union of the preceding supports. Regularity keeps it small. Lemma 8.8 completes the coherent union map to its automorphism. For each earlier edge, invariance and (39) hold on a cofinal tail, hence on the completion.

For every branch $\eta \in 2^\lambda$, coordinate coverage makes the complete closure equal to B . The limit lemma gives its complete extension $p_\eta \in G$. It satisfies $p_\eta h_t p_\eta^{-1} = k_t$ for each edge on that branch. If η, ζ first split after s , taking edges 0, 1, respectively, then

$$p_\eta h_{s,0} p_\eta^{-1} = k_{s,0} = k_{s,1} = p_\zeta h_{s,1} p_\zeta^{-1}.$$

If $p_\eta H = p_\zeta H$, then $z = p_\eta^{-1} p_\zeta \in H$ and this equation gives $h_{s,1} = z^{-1} h_{s,0} z \in H$, contrary to (40). Thus there are 2^λ distinct left cosets, contradicting $[G : H] \leq \lambda$. $\square$

The proof does not require H to be normal. It applies to index at most λ . To treat every index below 2^λ , one would need a different construction of the family, since the present construction places representatives of all cosets in λ columns.

8.5 Invariant names of small families

Proposition 8.13. *Under (24), suppose $\dot{A}$ is a B -name fixed modulo forced equality by every element of G , and $1_B \Vdash 0 < |\dot{A}| \leq \check{\lambda}$. Every name τ forced to belong to $\dot{A}$ is fixed by G_{B_S} for some $S \in [\lambda]^{<\lambda}$. For a generic $F \subseteq B$, its quotient interpretation over $W[F \cap B_S]$ is invariant under every automorphism of the quotient completion computed there. The ranks of the members are unrestricted.*

Proof. Coordinate permutations make P weakly homogeneous, so the full fixed algebra is $\{0, 1\}$ and the cardinality of the invariant family is decided. Choose names $(\tau_i)_{i < \eta}$ forced to enumerate it bijectively, where $0 < \eta \leq \lambda$. Every member name is equivalent to a mixture of these names: its possible index values form a maximal antichain, and the θ^+ -chain condition leaves at most θ nonzero pieces.

Fix a mixing construction and take its equivalence classes under forced equality. This set Z has size at most

$$|B|^\theta \lambda^\theta = \lambda.$$

There is an ordinary action $g[\sigma] = [g\sigma]$ on Z , representing the result again by an equivalent mixture. Invariance of $\dot{A}$ makes the action well-defined. A member-name stabilizer has index at most $|Z|$, so Theorem 8.1 puts G_{B_S} inside it for some $|S| < \lambda$. Apply Lemma 2.4 to this fixed-factor invariance. That lemma includes new quotient automorphisms named and mixed over B_S , and computes the completion in the quotient ground. $\square$

8.6 Definability in Cohen extensions of L

The name-reduction theorem gives the following two definability results. As in the introduction, $\text{OD}_{<\kappa}$ allows ordinal parameters and a sequence of fewer than κ reals. For any regular λ , OD_{H_λ} allows ordinal parameters and one parameter whose transitive closure has cardinality less than λ . Cardinalities and definability are computed in the extension.

Theorem 8.14 (Ordinary higher-Cohen definability). *Let κ be regular uncountable in L , and let $V = L[G]$ for G generic on $\text{Add}(\omega, \kappa)^L$. Then*

$$A \in \text{OD}_{<\kappa}^V, \quad |A|^V \leq \kappa \implies A \subseteq \text{OD}_{<\kappa}^V.$$

Theorem 8.15 (Generalized Cohen definability). *Let $\kappa < \lambda$ be regular uncountable cardinals of L satisfying (24) there with $\theta = \kappa$, and let $V = L[G]$ for $\text{Add}(\kappa, \lambda)^L$. Then*

$$A \in \text{OD}_{H_\lambda}^V, \quad |A|^V \leq \lambda \implies A \subseteq \text{OD}_{H_\lambda}^V.$$

Neither theorem imposes a rank bound on the members of A . We prove them together and treat the two parameter classes separately.

Proof of Theorems 8.14 and 8.15. We prove both conclusions with L replaced by an arbitrary original ground $W_0 \models \text{ZFC} + \text{GCH} + \text{GA} + V = \text{HOD}$, retaining the displayed cardinal assumptions. By Lemma 2.2, W_0 and its ground-name order are definable in all its set-forcing extensions. The intermediate grounds below need not satisfy GA or $V = \text{HOD}$. Write $(\theta, \lambda) = (\omega, \kappa)$ in the ordinary case, and retain $(\theta, \lambda) = (\kappa, \lambda)$ in the generalized case. Thus in both cases $V = W_0[G]$ for $\text{Add}(\theta, \lambda)^{W_0}$, where $\theta < \lambda$ are regular, λ is uncountable, and (24) holds in W_0 . In the ordinary case these assumptions follow from GCH and regularity of the width. Assume $A \neq \emptyset$.

Capturing the defining parameter. In the ordinary case, let $\vec{r} = \langle r_i : i < \delta \rangle$, $\delta < \lambda$, define A together with ordinals. Choose a name for the whole sequence and a condition in G deciding its length. Nice names for its entries each use countably many columns. The union of their supports, together with the support of the deciding condition, is contained in some $S \in W_0$ with $|S|^{W_0} < \lambda$. Consequently $\vec{r} \in W = W_0[G \restriction S]$.

In the generalized case, let $z \in H_\lambda^V$ and ordinals define A . Code the transitive closure of $\{z\}$ by a well-founded extensional relation on an ordinal $\delta < \lambda$, with a distinguished point representing z . Choose a name for this code and a condition in G deciding δ and the distinguished point. Each membership bit has a deciding antichain of size at most θ . There are fewer than λ bits, and every condition uses fewer than θ columns. Regularity of λ therefore places the code and the deciding condition on a ground set S of fewer than λ columns. The code belongs to $W = W_0[G \restriction S]$. Its well-foundedness is downward absolute, and its transitive collapse in W agrees with the ambient collapse; hence $z \in W$.

In either case enumerate S in W_0 in length $|S|^{W_0}$ and code the resulting sequence of generic columns by a set of ordinals a so that $W = W_0[a]$. For $\theta = \omega$, this code is recoverable from a sequence of fewer than λ reals and ordinals. For uncountable θ , choose the code with hereditary size at most $\max(\theta, |S|) < \lambda$, so $a \in H_\lambda^V$. The original defining parameter has an ordinal index in the a -definable ground-name order of $W_0[a]$ supplied by Lemma 2.2. Substituting this definition shows that A is $\text{OD}(a)$ in V .

The remaining forcing over W . The forcing on S is $<\theta$ -closed and θ^+ -cc. It preserves cardinals and adds no ordinal sequences of length less than θ . Thus W still satisfies $\theta^{<\theta} = \theta$ and $\nu^{<\theta} < \lambda$ for every cardinal $\nu < \lambda$. When $\theta = \omega$, this uses only the absoluteness of finite sequences. To

verify the remaining cardinal assumption, let $0 < \rho < \lambda$. A name for a function $\rho \rightarrow \lambda$ is specified by ρ antichains, each of size at most θ , with ordinal labels below λ . The forcing on S has size at most λ , so in W_0 there are at most

$$(\lambda^\theta)^\rho = \lambda$$

such specifications. These ground collections retain size at most λ in W , giving $W \models \lambda^{<\lambda} = \lambda$. The case $\rho = 0$ is immediate.

The untouched columns have cardinality λ . Since no short ordinal sequences were added, their conditions are unchanged; after reindexing, the residual forcing is $\text{Add}(\theta, \lambda)^W$. Its Boolean completion and automorphisms are now computed in W .

Descent of an individual member. Weak homogeneity gives an invariant residual name $\dot{A}$, forced nonempty and of size at most λ , from the definition using a and ordinals. Fix $x \in A$. Mix a name for x below a condition in the residual generic with a fixed member name off that condition. The resulting τ is forced to belong to $\dot{A}$ and has actual value x . By Proposition 8.13, there is a residual coordinate set $T \in W$, $|T|^W < \lambda$, such that τ is fixed by its pointwise factor stabilizer. Its quotient interpretation over $U = W[G \restriction T]$ is fully invariant by Lemma 2.4. Here T is a coordinate set in the new ground W ; it need not belong to W_0 .

Enumerate T in W and code the additional columns by a set of ordinals b , obtaining $U = W_0[a, b]$. The enumeration of T is specified by a and its ordinal index in the same relative ground-name order. Thus for $\theta = \omega$, the pair (a, b) is recoverable from fewer than λ reals and ordinals. For uncountable θ , both codes have hereditary size less than λ , as does their pair.

Code the pair as one set of ordinals and apply Lemma 2.3 to the quotient forcing over $W_0[a, b]$, whose names have ordinal-and- (a, b) codes by Lemma 2.2. It follows that x is $\text{OD}(a, b)$ in V . In the ordinary case this is $\text{OD}_{<\lambda}$, and in the generalized case it is OD_{H_λ} . Since $x \in A$ was arbitrary, both theorems follow. $\square$

Corollary 8.16. *In the extension of L by ω_1 Cohen reals, every $\text{OD}_{\mathbb{R}}$ family of cardinality at most $\aleph_1$ consists of $\text{OD}_{\mathbb{R}}$ elements.*

Proof. A sequence of fewer than ω_1 reals is coded by one real. Apply Theorem 8.14. $\square$

Corollary 8.17. *Let κ be regular uncountable in L , let $\lambda = (\kappa^+)^L$, and let $V = L[G]$ for $\text{Add}(\kappa, \lambda)^L$. Every $\text{OD}_{H_\lambda}^V$ family of cardinality at most λ consists of $\text{OD}_{H_\lambda}^V$ elements.*

Proof. GCH in L implies (24) with $\theta = \kappa$. Apply Theorem 8.15. $\square$

8.7 A chain without an order-dense union

A chain may have Cohen successor quotients and Cohen complements in the ambient algebra without having an order-dense union. In any ZFC ground let λ be regular uncountable and work in $B = \text{RO}(2 \times (2^\omega)^\omega \times (2^\omega)^\lambda)$, with coordinates $(t, (x_i)_{i < \omega}, z)$ and $r = [t = 0]$. Keep the tail of each x_i and replace its first bit by

$$y_i(0) = \begin{cases} 0, & t = 0, \\ x_i(0), & t = 1. \end{cases}$$

For $n \geq 1$, let C_n be the complete algebra generated by $y_0, \dots, y_{n-1}$. The output map is a continuous open surjection onto n reals; these maps commute with projection, so C_{n+1} is a Cohen product extension of C_n .

Each C_n also has a full λ -Cohen complement in B . Where all output first bits vanish, its fiber consists of one copy of the unused product for $t = 1$ and 2^n copies for $t = 0$; elsewhere only the $t = 1$ copy occurs. A finite disjoint sum of copies of $(2^\omega)^\lambda$ is homeomorphic to that space, using a finite clopen partition of a spare real. These fiber identifications give the required complement.

Let A_n be the image of the countable clopen algebra in C_n . The A_n are increasing and order dense there, but

$$p_n = \bigwedge_{i < n} [y_i(0) = 0] \in A_n, \quad \bigwedge_{n \geq 1} p_n = r > 0.$$

On $t = 1$ the infinite intersection has empty interior, since finite conditions leave some rows unrestricted. Every positive element of C_n , however, meets $t = 1$, where the output map is ordinary projection. Hence no positive element of $\bigcup_n C_n$ lies below r , and $\bigcup_n A_n$ is not order dense in its complete closure.

The coordinate condition (35) excludes this example. Any raw factor containing C_n contains t : otherwise its coordinate flip would fix the factor while changing $y_i(0)$. It therefore contains $r \notin C_{n+1}$, so no such factor lies between the consecutive stages.

9 Higher random extensions

Let κ be an uncountable cardinal in L and put

$$\lambda = (\kappa^{\aleph_0})^L.$$

Theorem 9.1. *Suppose $\text{cf}^L(\kappa) = \omega$. In the random extension of L of Maharam type κ , every OD family of cardinality at most κ has an OD bijective enumeration by a cardinal at most κ . In particular, it consists entirely of OD members. The members may have arbitrary rank.*

Theorem 9.2. *In the random extension of L of Maharam type κ , there is an OD family $\mathcal{A} \subseteq \mathcal{P}(\mathcal{P}(\lambda))$ of cardinality λ with no member definable from ordinals and fewer than κ reals.*

At singular κ of countable cofinality, these theorems give OD bijective enumerations for families of size at most κ and a counterexample of size κ^+ . At uncountable cofinality, the counterexample has size exactly κ . We first construct the counterexample and prove the countable-cofinality theorem. We then prove uniform enumeration below κ at regular width in Corollary 9.20. The corresponding singular strong-limit result is Theorem 11.6. The invariant-enumeration argument works over every ZFC ground at uncountable Maharam type of countable cofinality. Stable ground codes then give the OD conclusion.

9.1 Orbits of maximal random presentations

The construction and orbit calculations work over any ZFC ground W , with $\lambda = (\kappa^{\aleph_0})^W$. For the OD conclusion, take $W \models \text{ZFC} + \text{GCH} + \text{GA} + V = \text{HOD}$; Lemma 2.2 makes W and its internal HOD order definable in the extension. This includes the stated case $W = L$. We use GCH only to evaluate λ in the final corollary.

Let $\mathbb{R}_\kappa$ be the measure algebra of the product probability space $(2^\omega)^\kappa$ computed in W . Counting coordinate supports and Borel codes gives $|\mathbb{R}_\kappa| \leq \kappa^{\aleph_0}$. For the reverse bound, write r_α for the α th coordinate and, for each countably infinite $S = \{\alpha_n : n < \omega\} \subseteq \kappa$, put

$$b_S = \bigvee_{n < \omega} [r_{\alpha_n} \upharpoonright (n+2) = 0^{n+2}].$$

For each n , the other cylinders are independent of r_{α_n} and their union has measure at most $1/2$. Flipping the first bit at α_n therefore changes b_S on a positive set. Thus b_S cannot be supported on a set omitting any member of S , and distinct S give distinct elements. A single coordinate factor already has $2^{\aleph_0}$ elements. Consequently $|\mathbb{R}_\kappa| \geq \max(2^{\aleph_0}, |[\kappa]^{\aleph_0}|) = \kappa^{\aleph_0} = \lambda$.

Fix a bijection $e : \lambda \rightarrow \mathbb{R}_\kappa$ in W , using the first one in its internal HOD order when proving the OD conclusion. For a filter $F \subseteq \mathbb{R}_\kappa$, put

$$x_F = \{\xi < \lambda : e(\xi) \in F\} \subseteq \lambda.$$

Let G be $\mathbb{R}_\kappa$ -generic over W , put $M = W[G]$, and define

$$D = \{F \subseteq \mathbb{R}_\kappa : F \text{ is } \mathbb{R}_\kappa\text{-generic over } W \text{ and } W[F] = M\}.$$

Let

$$K = \text{Aut}^W(\mathbb{R}_\kappa), \quad H = \text{Aut}^W(\mathbb{R}_\kappa, \mu).$$

For $F \in D$, set

$$[F]_H = \{x_{h[F]} : h \in H\} \subseteq \mathcal{P}(\lambda),$$

and let

$$\mathcal{A}_\kappa = \{[F]_H : F \in D\}. \quad (41)$$

For the specified definable grounds, this family is OD in M , since the ground, the algebra, e , and the two ground groups have ordinal definitions there. The assertion $W[F] = M$ says that every ambient set is the value of a ground name under F .

Lemma 9.3. $D = \{\pi[G] : \pi \in K\}$.

Proof. This is the arbitrary-ground same-extension theorem stated in section 2, applied directly to $\mathbb{R}_\kappa$. The fixed coding gives $W[F] = W[x_F]$. $\square$

Lemma 9.4 (Fixing a generic filter). *For a complete Boolean algebra B and $g \in \text{Aut}(B)$, the Boolean value $b = \|g\dot{G} = \dot{G}\|$ is the largest region on whose principal ideal g is the identity.*

Proof. For every $a \in B$, the assertions $a \in \dot{G}$ and $g(a) \in \dot{G}$ are equivalent below b . Thus $b \wedge a = b \wedge g(a)$. Taking $a = b$ and $a = g^{-1}(b)$ gives $b \leq g(b)$ and $b \leq g^{-1}(b)$, hence $g(b) = b$. For $a \leq b$ the same identity now gives $g(a) = a$. Conversely, an identity region forces the two filters to agree, by the rank induction for the generic name. $\square$

Proposition 9.5. $M \models |\mathcal{A}_\kappa| = \lambda$.

Proof. For $\pi \in K$, define $\nu_\pi(b) = \mu(\pi b)$. The assignment $H\pi \mapsto \nu_\pi$ is injective. Equivalent probability measures have strictly positive L^1 densities, and every measurable real-valued function depends on countably many coordinates. There are therefore at most λ cosets, so $|\mathcal{A}_\kappa| \leq \lambda$.

For the reverse inequality, fix in W a bijection between $\omega \times \kappa$ and κ . Let $U_{n,\alpha}$ be a uniform $[0, 1]$ random variable read from the corresponding product coordinate. For $s \in (\kappa^\omega)^W$, put

$$X_s = \sum_{n < \omega} 2^{-n-1} U_{n,s(n)}, \quad f_s = Z^{-1} e^{X_s},$$

where $Z = \int e^{X_s} d\mu$ is independent of s . If $s \neq t$, choose n with $s(n) \neq t(n)$. Conditional on all coordinates except $U_{n,s(n)}$, the equation $X_s = X_t$ determines at most one value of that continuously distributed variable. Thus

$$\mu(f_s = f_t) = 0. \quad (42)$$

The probability algebra $(\mathbb{R}_\kappa, f_s \mu)$ is homogeneous of Maharam type κ . By Maharam's theorem, choose $\pi_s \in K$ such that

$$\mu(\pi_s b) = \int_b f_s d\mu$$

for every $b \in \mathbb{R}_\kappa$. Suppose $\pi_s[G]$ and $\pi_t[G]$ lie in the same H -orbit. Then, for some $h \in H$, the automorphism $\delta = \pi_t^{-1} h \pi_s$ fixes G . By Lemma 9.4, δ is the identity on a nonzero principal ideal $\mathbb{R}_\kappa \restriction c$. For every $b \leq c$,

$$\int_b f_s d\mu = \mu(\pi_s b) = \mu(h \pi_s b) = \mu(\pi_t b) = \int_b f_t d\mu.$$

Hence $f_s = f_t$ almost everywhere on c , contradicting (42). Thus the ground family of λ presentations gives distinct orbit classes; random forcing preserves this cardinal. $\square$

Proposition 9.6. *No member of $\mathcal{A}_\kappa$ is $\text{OD}_{<\kappa}$ in M .*

Proof. Fix a maximal presentation F and let $\dot{O}$ be the canonical name for its H -orbit class. Suppose this class is defined from a sequence $\vec{z}$ of $\delta < \kappa$ reals and ordinal parameters. Nice names for the entries of $\vec{z}$ use fewer than κ coordinates in total. Choose a condition $p \in F$ forcing uniqueness of the definition, and include the countably many coordinates of p in the support. Since κ is uncountable, choose a coordinate β outside this support.

On coordinate β , choose a nonsingular involution whose Radon–Nikodym derivative is 2 on a set of measure $1/3$ and $1/2$ on its complement. Extend it by the identity on all other coordinates, obtaining $k \in K$. Then k fixes p , every name in $\vec{z}$, and all ordinal parameters. Symmetry gives

$$p \Vdash k(\dot{O}) = \dot{O}.$$

But if this equality held below a nonzero condition, some measure-preserving $h \in H$ would agree with k or k^{-1} on a nonzero principal ideal. This contradicts their Radon–Nikodym derivatives. The derivative of h is 1, while those of k and k^{-1} are nowhere 1. $\square$

Proof of Theorem 9.2. The family in (41) is OD, has size λ by Proposition 9.5, and has no $\text{OD}_{<\kappa}$ member by Proposition 9.6. $\square$

Corollary 9.7. *Assume κ is uncountable in L .*

- (i) *If $\text{cf}^L(\kappa) > \omega$, then the extension by κ random reals has an OD family of size κ with no $\text{OD}_{<\kappa}$ member.*
- (ii) *If $\text{cf}^L(\kappa) = \omega$, the construction gives such a family of size κ^+ .*

Proof. More generally, in every ZFC+GCH ground W ,

$$(\kappa^{\aleph_0})^W = \begin{cases} \kappa, & \text{cf}^W(\kappa) > \omega, \\ \kappa^+, & \text{cf}^W(\kappa) = \omega. \end{cases}$$

Apply the construction over L , or over the definable grounds used in the proof of Theorem 9.2. $\square$

9.2 An initial invariant enumeration

We work in a ZFC ground W . Let κ be an uncountable cardinal of cofinality ω . Write $\mathbb{B} = \mathbb{B}_\Omega$ for the fair product probability algebra on 2^Ω , where $|\Omega| = \kappa$, and put

$$H = \text{Aut}(\mathbb{B}, \mu), \quad K = \text{Aut}(\mathbb{B}), \quad C = \mathbb{B}^I, \quad 0 < |I| \leq \kappa.$$

The full K -action Θ on C is assumed to satisfy diagonal equivariance and locality, as in (1)–(2). A *selector* is an element $u \in C$ whose coordinates partition $1_\mathbb{B}$; a *frame* is a partition of 1_C into selectors. An invariant frame amounts to an invariant bijective enumeration. Let Γ be the finite coordinate translations and let $\Sigma = \text{Sym}(\Omega)$ act by coordinate permutations. Write K_{cs} for the subgroup of K consisting of automorphisms fixing $\mathbb{B}_{\Omega \setminus S}$ pointwise for some countable $S \subseteq \Omega$; their action on S may depend on the complementary coordinates.

We first obtain a frame fixed by Γ and K_{cs} . We then use diagonal actions of standard probability-algebra groups to prove that Σ fixes this frame. The following subsections extend its invariance to the full group.

Lemma 9.8. *Let D be a complete Boolean subalgebra of $\mathbb{B}^I$. Every homomorphism $\Sigma \rightarrow \text{Aut}(D)$ which kills the finitary permutations also kills the countably supported permutations.*

Proof. The space $\mathcal{M}(D)$ of normal finite signed measures on D , with the total-variation norm, has density at most κ . A measure is normal if it preserves arbitrary increasing joins. To verify the bound, restrict the coordinate measures $\nu_i(d) = \mu(d_i)$ to D . If e_i is the complement of the join of the zero sets of ν_i , then ν_i is strictly positive on $D \restriction e_i$, and these e_i cover 1. That interval embeds in $\mathbb{B}$ by its i th coordinate, so its measure-metric density is at most κ . Disjointify the e_i into a partition (f_i) with $f_i \leq e_i$. Every normal finite measure is concentrated on countably many f_i ; on each, the Radon–Nikodym theorem represents it by an $L^1(\nu_i)$ function. Truncating to finitely many intervals and approximating by simple functions gives the asserted density bound.

Pushforward gives a faithful isometric representation of $\text{Aut}(D)$ on $\mathcal{M}(D)$. Faithfulness follows by restricting a coordinate measure to a nonzero part of a moved Boolean element. The strong topology on this isometry group has weight at most κ , since a dense set of that many vectors determines it. A Hausdorff group of weight at most κ has at most κ pairwise commuting nonabelian subgroups: cover its open noncommutation relation by κ basic rectangles and assign to each subgroup a rectangle containing a noncommuting pair. Two commuting subgroups cannot receive the same rectangle.

Now apply Lemma 4.4(i). □

Lemma 9.9. *The algebra $D = C^\Gamma$ is atomic with at most κ atoms.*

Proof. Locality makes every finitary coordinate permutation fix D : on each pattern of its finite support it agrees, on the whole principal algebra, with a finite translation. Since Σ normalizes Γ , Lemma 9.8 gives

$$D \subseteq C^{\Sigma_c}, \quad \Sigma_c = \{\pi \in \Sigma : |\text{supp}(\pi)| \leq \aleph_0\}.$$

We first show that a Σ_c -invariant name for a family of at most κ reals is forced to consist of ground members. Suppose otherwise and choose a member name $\dot{r}$ which is nonground with positive Boolean value. Weak homogeneity of Σ_c forces the family to be nonempty everywhere, since countable event supports can be moved apart. Let η be the ground distribution of $\dot{r}$, and remove its countable set of atoms, leaving a set E of positive η -measure. Copy the countable input support of $\dot{r}$ onto the nonempty initial segments of each $s \in \kappa^\omega$. Each copy is induced by a

countably supported coordinate permutation; a countable reserve extends the prescribed support bijection. Denote these member names by $\dot{r}_s$.

For $s \neq t$, condition on their finitely many common input bits. The remaining inputs are independent, and their laws on E are nonatomic, being absolutely continuous with respect to $\eta \upharpoonright E$. Hence

$$\|\dot{r}_s = \dot{r}_t \text{ and } \dot{r}_s \in E\| = 0.$$

The set of branches s for which $\dot{r}_s \in E$ is therefore forced to have size at most κ . By ccc covering, it lies in a ground set of at most κ branches: use countable antichains deciding the values of a κ -term enumeration. Yet below any positive condition there are κ^ω branches whose supports avoid that condition's countable support. Choose a first branch symbol unused by that support. For each of these branches, membership in E has the same positive conditional probability $\eta(E)$. This contradicts the existence of the ground cover.

Now suppose C^{Σ_c} has a nonatomic part. Restrict to a nonzero interval a carrying a strictly positive coordinate measure ν_i , as in the preceding proof, and split it into a binary tree $(a_t : t \in 2^{<\omega})$ with $\nu_i(a_t) = 2^{-|t|}\nu_i(a)$. Every branch has zero meet. In a generic fiber, the labels belonging to a determine a set of at most $|I|$ branches through this invariant tree. That real-family name is Σ_c -invariant, so all its members must be ground. On the positive event a_i , however, the branch selected by label i differs from every ground branch, because its Boolean equality value is a coordinate of the corresponding zero meet. This is a contradiction.

Thus C^{Σ_c} is atomic. A complete subalgebra of a complete atomic Boolean algebra is atomic: the least upper bound in the subalgebra of an ambient atom is again an atom. Consequently D is atomic. Every antichain in $\mathbb{B}^I$ has size at most $\max(|I|, \aleph_0) \leq \kappa$, by ccc in each coordinate, which bounds the number of atoms. $\square$

Lemma 9.10 (Standard factors below the continuum). *Let Q be a standard atomless probability algebra and let $0 < |J| < 2^{\aleph_0}$. Every full local diagonal action of $\text{Aut}(Q, \mu)$ on Q^J has an invariant frame. If the action extends to $\text{Aut}(Q)$, it has a frame fixed by that full group.*

Proof. Use Lemma 5.1 for the probability-preserving frame and Theorem 5.9 for its full invariance. $\square$

Lemma 9.11 (Full relative actions). *In any ZFC ground, let $\mathbb{B} = \mathbb{B}_\Omega$ be a product probability algebra, without a restriction on $|\Omega|$ or cardinal arithmetic. Let $\Omega = S \sqcup T$, let $A = \mathbb{B}_S$, and force first with A . In that extension let Q be the recomputed probability algebra on 2^T . A full local diagonal action on $\mathbb{B}^I$ descends to a full local diagonal action on the original Q^I , including every new relative automorphism. This holds both for the measure-preserving and the nonsingular groups, and for every ground set I .*

Proof. Apply Lemma 3.5 to the product/Fubini factorization $\mathbb{B} = \mathbb{B}_S * Q$. Its measure-preserving clause applies because conditional probabilities integrate to the original product probability. $\square$

Lemma 9.12. *The atoms of C^Γ form a frame fixed pointwise by Γ and K_{cs} .*

Proof. By Lemma 9.9, C^Γ is atomic, with at most κ atoms. Every such atom U has scalar support 1 and a constant fiber cardinal by scalar Γ -ergodicity. We show that this cardinal is one.

For any countably infinite $S \subseteq \Omega$, first force with $\mathbb{B}_{\Omega \setminus S}$. This ccc forcing preserves κ and $\text{cf}(\kappa) = \omega$ and supplies κ distinct reals. Since the continuum has uncountable cofinality, the intermediate continuum is strictly larger than κ . Hence Lemmas 9.10 and 9.11 give a full invariant frame for the original relative cover Q^I , where Q is the recomputed standard algebra on S .

Suppose first that U has infinite degree. Its counting measure $\nu(U)$ is infinite, while its Γ -action is ergodic. Its invariant L^2 subspace is therefore zero. Choose a selector $u \leq U$. The closed convex hull of its orbit indicator contains zero; choose finite rational convex averages z_n with $\|z_n\|_2 < 2^{-n}$. The countably many selectors in these averages together have only countably many positive coordinates, each with countable scalar support. Choose S carrying all of them, u , and the finite translation supports used. After forcing the complementary factor, the norm inequalities are unchanged. Some selector v in the relative invariant frame has positive overlap with u , whereas invariance gives

$$\langle \mathbf{1}_v, z_n \rangle = \langle \mathbf{1}_v, \mathbf{1}_u \rangle > 0 \quad (n < \omega),$$

contradicting $\|z_n\|_2 \rightarrow 0$.

Suppose next that U has finite degree m . Since $\sum_i \mu(U_i) = m$, only countably many labels occur positively in U . Enumerating them, select their first m active labels in each fiber. This identifies $C|U$ over $\mathbb{B}$ with $\mathbb{B}^m$, with a countably supported choice of the m selectors. We construct a countable S on which this finite Γ -action is ergodic, with witnesses that remain valid after forcing the complementary factor.

Start with a countably infinite S_0 carrying that identification, adding a reserve if necessary. Given S_r , include in S_{r+1} the supports of all gauged matrix entries for translations in Γ_{S_r} . For every rational simple finite-cylinder function f on $2^{S_r} \times m$ and positive rational ε , ergodicity supplies a finite rational convex average of translates of f within ε of the constant $\bar{f} = m^{-1} \int f d\nu$. Include the supports of the translations and their matrix entries in S_{r+1} . There are only countably many requirements. Put $S = \bigcup_r S_r$.

Every finite translation on S appears at some stage, and its matrix is supported on S . Every rational finite-cylinder function has the recorded arbitrarily accurate averages. These functions remain dense in the recomputed $L^2(Q^m)$ after complementary forcing, and all recorded norm inequalities are unchanged. If v is any new invariant vector, taking inner products with these averages gives $\langle v, f \rangle = \langle v, \bar{f} \rangle$ on that dense set. Thus the relative Γ_S -action on U remains ergodic. Intersect U with the selectors of the full relative invariant frame. These intersections are invariant partial selectors and cover U . Ergodicity makes a nonzero one equal to U , so U is a selector and $m = 1$. In both cases we descend the original full cover. We do not assume that the relative full group preserves U .

The atoms of C^Γ therefore form a Γ -fixed frame, indexed by I . Finally fix any countably infinite S . In the relative nonsingular invariant frame on Q^I , a global Γ -fixed selector has scalar coefficients fixed by Γ_S . The latter acts ergodically on the recomputed standard Q , so those coefficients are zero or one. The selector is therefore a member of the relative invariant frame, so every relative nonsingular automorphism fixes it. This proves the assertion in the ground, including for automorphisms whose action depends on parameters from the entire complementary factor. $\square$

Lemma 9.13 (Initial frame). *Under the hypotheses of this subsection, a full local diagonal K -action on C has a frame fixed pointwise by Γ , Σ , and K_{cs} .*

Proof. Let $F = (f_i : i \in I)$ be the frame from Lemma 9.12. We first show that it is fixed by diagonal standard actions on disjoint countable coordinate blocks with a κ -sized complement.

Fix an infinite ground set $E \subseteq \Omega$ with $|\Omega \setminus E| = \kappa$, and force with $A = \mathbb{B}_{\Omega \setminus E}$. In $W' = W[G_A]$, let Q_E be the recomputed probability algebra on 2^E . By Lemma 9.11, the full action descends to Q_E^I with the same frame F . Let $L_E \leq \text{Aut}(Q_E, \mu)$ consist of all automorphisms fixing the coordinate algebra outside some countable subset of E pointwise. Every member of L_E fixes F .

Indeed, ccc covering puts its named countable support inside a ground countable $S \subseteq E$. Its ground lift fixes both A and $\mathbb{B}_{E \setminus S}$, hence belongs to K_{cs} . Names given only below a condition are mixed with the identity elsewhere. This includes every new relative operator.

The L_E -fixed algebra of the cover is precisely the copy of $\mathcal{P}(I)^{W'}$ formed by unions of frame selectors. To see this, write an element uniquely as

$$z = \bigvee_{i \in I} (b_i \wedge f_i), \quad b_i \in Q_E,$$

where scalar events act diagonally. Since L_E fixes each f_i , an invariant z has invariant coefficients b_i . Finite coordinate translations belong to L_E and act ergodically on Q_E , so all these coefficients are zero or one.

Partition E into countably infinite blocks $(D_t : t \in T)$, with ground bijections $\omega \rightarrow D_t$. In W' , put $P = \text{Aut}(Q_\omega, \mu)$ for the recomputed standard algebra. For $h \in P$, define $\Delta(h)$ by applying h independently on every block. Define this map first on finite tensor cylinders, then extend by measure completion. Its inverse is $\Delta(h^{-1})$, and the assignment is a homomorphism $P \rightarrow \text{Aut}(Q_E, \mu)$.

Each $\Delta(h)$ normalizes L_E . If $\ell \in L_E$ fixes the complement of a countable S , let

$$S^* = \bigcup \{D_t : D_t \cap S \neq \emptyset\}.$$

This is countable. Both $\Delta(h)$ and its inverse preserve each block algebra, so $\Delta(h)\ell\Delta(h)^{-1}$ fixes the coordinate algebra outside S^* pointwise. Applying the same argument to h^{-1} gives normalization. Thus the cover action of $\Delta(P)$ preserves the L_E -fixed algebra $\mathcal{P}(I)^{W'}$ and permutes its atoms f_i . We obtain an ordinary homomorphism $P \rightarrow \text{Sym}(I)$.

The forcing A preserves κ and its countable cofinality and supplies κ distinct reals. Hence $W' \models |I| \leq \kappa < 2^{\aleph_0}$. Malicki's theorem [24, Theorem 2.3] states that the standard probability-algebra group has no proper subgroup of index less than the continuum. Applied in W' to each point stabilizer, it makes this homomorphism trivial. Consequently every ground diagonal map fixes the original frame, since the factorization identification reflects equality of ground cover elements. The theorem is applied to the full recomputed group P and requires no regularity of its action.

Now let τ be a coordinate involution whose moved support E has κ -sized complement. If E is countable, then $\tau \in K_{\text{cs}}$. Otherwise group its transposed pairs into countably infinite collections. Their unions form blocks on which τ is the same standard automorphism, interchanging successive pairs of bits. The preceding argument therefore makes τ fix F .

Every coordinate involution is a product of at most two of this kind. If its moved support lacks a κ -sized complement, it has κ transposed pairs; divide these into two sets of size κ . Each restricted involution has κ fixed coordinates. Finally, every permutation is a product of two involutions, by factoring the cyclic shift on each finite or bilateral orbit into two reflections. Thus all of Σ fixes F . $\square$

9.3 Proper factors and relative frames

The proper-factor lemmas apply both to Cohen and to random algebras. Let κ be uncountable, $|\Omega| = \kappa$, and let $B = B_\Omega$ be either the finite-condition Cohen algebra or the fair product probability algebra. Let $H = \text{Aut}(B)$ in the Cohen case and $H = \text{Aut}(B, \mu)$ in the random case. Assume a full local diagonal H -action. Let N be generated by independently extended automorphisms in H on banks E satisfying $|E| = |\Omega \setminus E| = \kappa$.

In a displayed frame $(f_i)_{i \in I}$, use target comparison matrices P_h : their (i, j) -entry is the scalar event on which $\Theta_h f_j$ agrees with f_i . Rows and columns are Boolean partitions of 1, and diagonal equivariance gives

$$P_{hk} = P_h h(P_k). \quad (43)$$

Here matrix multiplication uses joins and meets, and h acts entrywise. These matrices are invertible. Locality computes the comparison of a scalar pasting on its target pieces. None of the following arguments requires a cardinality bound on I once the displayed frame is given.

Lemma 9.14. *If every row coordinate permutation fixes the frame, then every element of N fixes it. Whenever the frame is N -fixed, forcing a row factor B_X , $|X| < \kappa$, leaves its descended frame fixed by the entire recomputed complementary proper-factor group, including newly named banks and automorphisms.*

Proof. Let g act independently on a bank E with $|E| = |\Omega \setminus E| = \kappa$. Partition Ω into banks $(E_j)_{j \in \mathbb{Z}}$ of size κ , with $E_0 = E$, and fix coordinate identifications between them. Let R apply a copy of g on each bank $j \geq 0$ and the identity on each bank $j < 0$. This defines an automorphism in H . In the Cohen case, the compatible maps on finite products extend from their order-dense union; in the random case they preserve measure and extend from the metric-dense finite-product algebra. The copied inverses give the inverse.

The map R commutes with every permutation of the nonnegative banks and, independently, every permutation of the negative banks. These coordinate permutations fix the frame. Equation (43) therefore makes every coefficient of P_R invariant under both permutation groups. Their common scalar fixed algebra is $\{0, 1\}$. For random algebras, approximate a function by finite coordinate cylinders, move the finitely many banks involved to disjoint banks in their respective half-lines, and average; independence makes the centered averages tend to zero in L^2 . For Cohen algebras, finite conditions below an invariant event and its complement could be made compatible by moving their bank supports apart, giving the same zero-one conclusion. It follows that P_R is a constant ground permutation.

Let S shift E_j to E_{j+1} . It fixes the frame, so $P_S = 1$. On the subgroup generated by R, S , the constant comparison matrices multiply without scalar transport. But

$$RSR^{-1}S^{-1} = g :$$

the exponent of the copied g on bank j is $1_{j \geq 0} - 1_{j-1 \geq 0}$, which is 1 just at $j = 0$. The comparison of this commutator is the commutator of P_R with 1, hence is 1.

For the relative assertion, assume the frame is N -fixed and write $A = B_X$. Put $Y = \Omega \setminus X$. By Lemma 3.5, the full relative action includes every new quotient automorphism in either setting. For a prescribed ground bank $E \subseteq Y$ with $|E| = |Y \setminus E| = \kappa$, a named independent relative E -automorphism lifts to an independent automorphism in H of $B_{X \cup E}$. Its complementary bank has size κ , so the lift belongs to N .

We must also allow newly named banks. Let $\hat{\pi}$ be an A -name for a permutation of Y , and set $a_{yz} = \|\hat{\pi}(y) = z\|_A$. Each row and each column of (a_{yz}) has countably many positive entries, by ccc. The undirected graph joining y, z when $a_{yz} > 0$ or $a_{zy} > 0$ therefore has countable components. Each component is forced to be preserved by $\hat{\pi}$. Split these components into two collections of cardinality κ , giving ground unions Y_0, Y_1 of size κ .

The lift of $\hat{\pi}$ is the product of its restrictions to Y_0 and Y_1 . Each restriction fixes X and the other union, and is an independent automorphism in H on $X \cup Y_j$, hence belongs to N . In the random case, partition A by the images of each finite list of coordinates: these images are distinct

on every positive piece, so their conditional distribution is still the fair product. This verifies measure preservation. In both cases the named permutation and its inverse extend to inverse automorphisms of the completed product iteration.

Any named bank of size and complementary size κ is the image of a prescribed ground bank under such a named permutation. Conjugating its relative map to that ground bank gives a lift in N ; the conjugating lift is itself in N . Thus every new proper-factor generator fixes the descended frame. Only conjugation by members of N has been used. $\square$

Lemma 9.15. *Assume the frame is N -fixed. Suppose P_h preserves a ground set $J \subseteq I$ and restricts there to a constant ground permutation γ . For every raw factor $A = B_S$ with $|S| < \kappa$, some $n \in N$ makes nh fix A pointwise, with the same comparison γ on J .*

Proof. Close S under countable raw supports of the images of its literals by h and h^{-1} , repeating ω times. The result T has size at most $\max(|S|, \aleph_0) < \kappa$, and $h[B_T] = B_T$. Independently extend $(h|_{B_T})^{-1}$ to n . This map belongs to H and is supported inside a proper κ -bank, so $n \in N$. The comparison equation changes P_h only by scalar transport, leaving its constant restriction γ unchanged. $\square$

9.4 A common construction of commuting bit flips

Lemma 9.16 (Countable clocks). *In either of the preceding two settings, suppose $\text{cf}(\kappa) = \omega$ and the frame is N -fixed. If P_h preserves a ground block J and is constant there, there are commuting involutions and clock bits satisfying (44), with a complementary fixed algebra D and the product factorizations (45).*

Proof. Write γ for that restriction. We construct commuting involutions $\tau_i \in H$ and independent bit events b_i , $i < \omega$, such that each finite-stage map preserves J and

$$\tau_i(b_i) = \neg b_i, \quad \tau_i(b_j) = b_j \quad (i \neq j), \quad P_{\tau_i}|J = \begin{cases} \gamma & \text{on } b_i, \\ \gamma^{-1} & \text{on } \neg b_i. \end{cases} \quad (44)$$

The two comparison regions are target regions. We will arrange that later involutions fix successively larger short factors whose union is dense in B .

Suppose the first n generators have been constructed. Denote their group by $F = (\mathbb{Z}/2\mathbb{Z})^n$, its elements by σ_s , and the old bit phases by d_s , so $\sigma_s(d_0) = d_s$. Equation (43) gives

$$P_{\sigma_s}|J = \gamma^{\sum_{\{i:s_i=1\}} (2b_i-1)},$$

interpreted on the finite bit partition. These comparisons are powers of γ with coefficients in the old clock algebra.

Choose a short raw factor A containing the old clocks, invariant under F , and containing any prescribed short coordinate set. To obtain it, close the desired coordinate set under the countable supports of images of its literals by the finitely many $\sigma_s^{\pm 1}$, repeating ω times. This remains short. Choose a fresh raw bit x , and let $v \in N$ be its independent raw flip, fixing A . By Lemma 9.15, choose h_0 with comparison γ on J which fixes A and x .

On phase d_s , use respectively

$$h_s = \sigma_s h_0 \sigma_s^{-1}, \quad w_s = \sigma_s v \sigma_s^{-1}.$$

Both maps fix A pointwise, so they preserve d_s . Their comparisons on J are γ and 1. For example, put $Q_s = P_{\sigma_s}|J$ and compare $h_s\sigma_s = \sigma_s h_0$:

$$(P_{h_s}|J) h_s(Q_s) = Q_s \gamma.$$

The coefficients of Q_s lie in A , and its values commute with γ , proving the assertion. The calculation for w_s has 1 in place of γ .

Paste these maps on their preserved phases to obtain u, w . They fix A , commute with F , and satisfy $P_u|J = \gamma$, $P_w|J = 1$, and $w^2 = 1$. Commutation follows because conjugating the piece on d_s by σ_t gives exactly the prescribed piece on d_{s+t} . The event

$$b = \bigvee_{s \in F} \sigma_s(d_0 \wedge x)$$

is F -invariant and independent of A . For random algebras, its contribution on phase d_s has measure $\mu(a \wedge d_s)/2$ for $a \in A$. For Cohen algebras, both halves meet every positive $a \in A$, since the fresh bit does so before each conjugation. Moreover u fixes b and w flips it. Define the next generator on its source halves by

$$\tau = wu \text{ on } \neg b, \quad \tau = u^{-1}w \text{ on } b.$$

These restrictions are inverse maps in H . Thus τ is an involution, fixes A , commutes with F , and satisfies (44) for the new bit. This proves the induction step.

Now use $\text{cf}(\kappa) = \omega$. At stage n choose the factor A_n increasing, invariant under the old finite group, and containing a prescribed increasing exhaustion of Ω by sets of size less than κ . All τ_j with $j \geq n$ fix A_n pointwise, and the earlier generators preserve A_n . For $t \in T = (\mathbb{Z}/2\mathbb{Z})^\omega$, define its action on A_n by the finite product of $\tau_i^{t_i}$, $i < n$. These definitions and their inverses agree on overlaps and extend to B : the union of the A_n is order dense in the Cohen case and metric dense in the random case. They define a T -action on B . In the random case its scalar orbit maps are continuous by approximation in some A_n , where only finitely many group coordinates matter. This construction requires no regularity of the comparison maps.

Let $D = B^T$. The bits give the product factorizations

$$B \cong \begin{cases} D \hat{\otimes} B_\omega & (\text{random}), \\ \text{RO}(D^+ \times \text{Fn}(\omega, 2)) & (\text{Cohen}). \end{cases} \quad (45)$$

To verify these factorizations, let $a \in A_n$ and saturate each $a \wedge d_s$ under the first n generators. Its saturation belongs to D : the finite group permutes its terms, and the tail fixes them all. Its intersection with d_s is precisely $a \wedge d_s$. Thus each A_n lies in the algebra generated by D and the clock bits. In the random case, transitivity gives $\mu(d \wedge d_s) = 2^{-n}\mu(d)$ for $d \in D$; completion proves the product identity. In the Cohen case it makes every positive $d \in D$ meet every finite clock pattern. The displayed product map therefore preserves incompatibility, and the saturation calculation makes its image order dense, proving its completion is B . $\square$

We now specialize again to the random algebra, with $H = \text{Aut}(B, \mu)$ and $\text{cf}(\kappa) = \omega$.

Lemma 9.17. *In a full local H -action with an N -fixed frame, suppose one comparison P_h preserves a ground block $J \subseteq I$ and is constant there. Its restriction to J is the identity.*

Proof. Write γ for the restriction, and take the clocks of Lemma 9.16. Let C be the complete algebra generated by their bits, so $B = D \widehat{\otimes} C$. Each nonzero principal algebra of D has Maharam type κ . Its type is at most κ as a metric subspace of B ; if it were smaller, adjoining the countable clock in (45) could not give a principal algebra of the homogeneous B_κ . Maharam's classification [9, 331I] therefore gives an MP automorphism q carrying a raw countable clock and its raw complement onto C and D , with its i th raw bit sent to b_i . If $a_i \in N$ is the corresponding raw bit flip,

$$\tau_i q = q a_i. \quad (46)$$

Let ρ_{ij} exchange clock coordinates i, j and fix D . It is also the scalar pasting of $\tau_i \tau_j$ where those bits differ and the identity where they agree. Both maps fix D and agree on all clock generators, so (45) identifies them on B . The signed exponents in (44) cancel on the differing-bit region. Hence $P_{\rho_{ij}}$ preserves J and is identity there. This remains true for every finite clock permutation ρ_π .

Put $Q = P_q$. Since $\rho_\pi q = q a_\pi$ for a raw coordinate permutation $a_\pi \in N$, we have

$$P_{\rho_\pi} \rho_\pi(Q) = Q. \quad (47)$$

For $j \in J$, the j th row of P_{ρ_π} is the j th identity row. Therefore every Q_{jk} , $j \in J$, $k \in I$, is fixed by all finite clock permutations. Their common scalar fixed algebra is D . To see this without a separability assumption on D , approximate an invariant L^2 function using D and finitely many clock coordinates. Move the clock coordinates to many disjoint blocks and average. After subtracting conditional expectation onto D , these copies are conditionally orthogonal; their averages tend to zero in L^2 . The approximation error is unchanged by the permutations, so the invariant function belongs to $L^2(D)$.

Thus the rows of Q indexed by J are fixed by every τ_i . Applying (43) to (46), and restricting to the target half b_i , gives

$$Q_{\gamma^{-1}(j),k} = Q_{jk} \quad (j \in J, k \in I), \quad \text{below } b_i.$$

If γ moved j , these would be two distinct rows. Distinct rows are disjoint in each column, so every entry of the j th row would vanish below b_i , contradicting that the row has join 1 and $\mu(b_i) = 1/2$. Consequently $\gamma = 1$.

This argument requires only the constructed finite-stage maps and finite clock permutations to preserve J . The conjugator q need not preserve J . We therefore use the rows of the whole matrix Q . $\square$

9.5 Full rigidity and ordinal definitions

Theorem 9.18 (Full frame rigidity). *Let κ be any uncountable cardinal of countable cofinality in a ZFC ground. Every full local, diagonally equivariant $\text{Aut}(B_\kappa, \mu)$ -action with an N -fixed frame fixes that frame pointwise. The frame may have any ground set of labels.*

Proof. Fix $h \in H$ and a label i . Close $\{i\}$ under all possible images and inverse images under P_h . Each row and each column has only countably many positive entries, so this produces a countable ground block J containing i which P_h preserves. The restricted matrix has countable raw support. Add a countable reserve and close this support under the supports of images of its literals by h and h^{-1} , for ω steps. The resulting countable raw factor A satisfies $h[A] = A$ and carries every entry of $P_h|J$.

Let $k \in N$ independently extend $h|A$. The right residual $r = hk^{-1}$ fixes A pointwise and satisfies $P_r = P_h$. Force first with A . By Lemmas 9.11 and 9.14, the whole cover descends with its

full recomputed MP action and its proper-factor-fixed frame. The relative comparison of r on J is now a constant ground permutation, since its entries belonged to A . Apply Lemma 9.17 in this relative ZFC ground. Its cardinal κ is still uncountable of countable cofinality, as required.

The relative restricted comparison is the identity. It follows that $P_h|J$ was already the identity, so P_h fixes i . Since h and i were arbitrary, the whole group fixes the frame. $\square$

Proof of Theorem 9.1. We prove the result over any ground $W \models \text{ZFC} + \text{GA} + (V = \text{HOD})$. The empty family has the empty enumeration. Let κ be uncountable with $\text{cf}^W(\kappa) = \omega$, and let A be a nonempty OD family of size at most κ in the B_κ^W -extension. Scalar homogeneity makes its definition and cardinality hold with Boolean value 1. Choose an invariant name $\dot{A}$ and a name for a bijection from a ground cardinal $I \leq \kappa$ onto it. As in section 2, the algebra of named subfamilies is B_κ^I , with a full local diagonal action of $K = \text{Aut}(B_\kappa)$. Locality follows by induction on names; there is no rank bound on their values.

The hypotheses of Lemma 9.13 hold in W . It supplies a frame fixed by all coordinate permutations and by every relative countable-coordinate nonsingular map. By Lemma 9.14, this frame is N -fixed, and Theorem 9.18 makes it fixed by $H = \text{Aut}(B_\kappa, \mu)$.

Finally every $g \in K$ has a factorization $g = ch$, where $h \in H$ and c changes only a countable coordinate factor. Indeed the positive finite density $f = d(g_*\mu)/d\mu$ has countable raw support. On a countably infinite raw factor carrying it, the standard atomless isomorphism theorem gives c with $c_*\mu = f\mu$; extend c independently outside that factor. Then $c^{-1}g \in H$. Both factors fix the frame, so the frame is K -fixed.

Relabeling the original enumeration by this frame gives an invariant name for a bijection $I \rightarrow \dot{A}$, since every member-name and the ground ordinal I are fixed. By Lemma 2.2, the ground and the function-name are ordinal-definable in the extension. Thus Lemma 2.3 gives an OD bijective enumeration of A . In particular $A \subseteq \text{OD}$. $\square$

9.6 Small families at regular random width

Lemma 9.19 (The regular measure-preserving small-index theorem). *Let λ be regular uncountable with $\lambda^{<\lambda} = \lambda$. If C is a Boolean subalgebra of $\mathbb{B}_\lambda$ of cardinality less than λ and $H \leq \text{Aut}(\mathbb{B}_\lambda, \mu)_{(C)}$ has index at most λ , then H contains the pointwise stabilizer of some coordinate factor on fewer than λ coordinates containing C . In particular, CH gives countable supports at $\lambda = \omega_1$.*

Proof. We verify the hypotheses of the small-index theorem for homogeneous abstract elementary classes [12, Theorem 4.1]. Consider Boolean algebras equipped with a strictly positive finitely additive probability, in the countable language consisting of the Boolean operations and the predicates $\mu(x) < q$, $q \in \mathbb{Q}$. Strong substructure is Boolean subalgebra with the restricted probability. The rational cuts determine the measure, so embeddings are exactly measure-preserving embeddings. Increasing unions remain in the class: positivity and finite additivity are checked at one stage. The subalgebra generated by X has size at most $|X| + \aleph_0$. Thus this is an abstract elementary class of Löwenheim–Skolem number $\aleph_0$. Adding constants for C gives Löwenheim–Skolem number at most $\max(|C|, \aleph_0) < \lambda$.

The metric completion of a probability Boolean algebra for $d(x, y) = \mu(x \triangle y)$ is a complete probability algebra, and measure-preserving embeddings extend uniquely to completions. An isomorphism between two subalgebras of $\mathbb{B}_\lambda$ of cardinality less than λ consequently extends to their completions, of metric density less than λ . Every positive principal algebra of $\mathbb{B}_\lambda$ has relative Maharam type λ over either such completion: fewer than λ relative generators would, together with the base generators, make its absolute type less than λ . The relative extension theorem

[9, 333C(b)] therefore extends the isomorphism to an automorphism of $(\mathbb{B}_\lambda, \mu)$. This also proves the required homogeneity after naming C . Every probability Boolean algebra of cardinality less than λ embeds in $\mathbb{B}_\lambda$, by Maharam's theorem applied to its completion; homogeneity makes such embeddings extend a prescribed embedding of a subalgebra of cardinality less than λ .

We now verify the simultaneous automorphism amalgamation required by [12, Definition 3.5]. Let $A_0 \subseteq A_1, A_2$ be probability Boolean algebras of cardinality less than λ , and let fewer than λ paired automorphisms of A_1, A_2 agree on A_0 , each preserving A_0 setwise. Complete the three algebras and take the relatively independent product over the completion of A_0 . Its probability on rectangular generators is

$$\mu(x_1 \wedge x_2) = \int \mathbb{E}(\mathbf{1}_{x_1} \mid A_0) \mathbb{E}(\mathbf{1}_{x_2} \mid A_0) d\mu.$$

Each pair preserves this probability, since conditional expectation commutes with its common base automorphism, and hence induces an automorphism of the product. The Boolean algebra generated by the two images has cardinality less than λ and is invariant under all these maps and their inverses. In forming this algebra, take the quotient by the null ideal. Strict positivity of the marginals makes both original embeddings injective. These are the usual relative probability products [34, 458N–458P]. Embed the amalgam into $\mathbb{B}_\lambda$ over A_0 . Homogeneity extends its two embeddings to global automorphisms, giving precisely the simultaneous amalgamation demanded by the cited theorem. The construction retains any named constants.

Finally $|\mathbb{B}_\lambda| = \lambda^{\aleph_0} = \lambda$. The small-index theorem now supplies the stabilizer of a subalgebra of cardinality less than λ containing C . The union of the countable coordinate supports of its elements has size less than λ . Fixing that coordinate factor therefore fixes the subalgebra. $\square$

Corollary 9.20 (Regular random families). *Suppose $W \models \text{ZFC} + \text{GA} + (V = \text{HOD})$ and κ is regular uncountable with $\kappa^{<\kappa} = \kappa$ in W . In the extension by κ random reals, every $A \in \text{OD}_{<\kappa}$ with $|A| < \kappa$ has a bijective enumeration belonging to $\text{OD}_{<\kappa}$. There is no rank restriction.*

Proof. Capture the short parameter defining a nonempty A and a condition deciding its definition and cardinality $\mu < \kappa$ in a short coordinate extension U of W . The empty case is immediate. The residual family name is fully invariant; choose a forced bijection $(\tau_i : i < \mu)$ onto it. The arithmetic $\kappa^{<\kappa} = \kappa$ still holds in U : the original random algebra has size κ , and nice names for sequences in κ of any length $\xi < \kappa$ number at most $\kappa^{\xi \cdot \aleph_0} = \kappa$. Ccc preserves regularity. The set of all member-name classes has size at most $(\mu \cdot |\mathbb{B}_\kappa|)^{\aleph_0} = \kappa$, by countable Boolean mixing. Apply Lemma 9.19 to the stabilizer of each τ_i in the measure-preserving group. It contains the fixer of a short coordinate factor $\mathbb{B}_{S_i}$. Regularity makes $S = \bigcup_{i < \mu} S_i$ short.

Force $\mathbb{B}_S$ first. In the new ground U_1 , the original enumeration is fixed by the full recomputed measure-preserving tail group, by Lemma 3.5: every named operator lifts to a ground map fixing $\mathbb{B}_S$, hence all τ_i .

We show that the same original enumeration is fixed by nonsingular maps as well. First let g fix the complement of a countably infinite coordinate bank T pointwise, and temporarily force that complement. Its width is κ , so its extension has at least κ reals. The quotient on T is the recomputed standard random algebra Q , with the full relative action on Q^μ by Lemma 9.11. Since $\mu < \kappa \leq 2^{\aleph_0}$ there, Lemma 9.10 supplies a fully invariant frame. Every original selector is still fixed by every new relative measure-preserving map, by Lemma 3.5. Its coefficients in the invariant frame are consequently invariant scalars for the standard measure-preserving group, hence are zero or one. It is therefore one of that frame's selectors, and the induced g fixes it. Returning to

U_1, g therefore fixes every original entry. This temporary forcing verifies a Boolean equality. Its generic is not needed as a parameter in the final definition.

For arbitrary nonsingular g , its Radon–Nikodym density has countable support. On a countable factor containing that support choose a nonsingular c having the same pushforward probability; then $c^{-1}g$ is measure-preserving, as in the density correction in Lemma 11.3. Both maps fix the enumeration. Thus its single name is fully invariant, and Lemmas 2.2 and 2.3 give the required definition. The parameters are the generic from the initial short capture and the generic on S . They have one combined short code. $\square$

10 Singular Cohen extensions

At singular width, families of size strictly below the width admit uniform enumerations without a cardinal-arithmetic hypothesis. At countable cofinality, the conclusion extends to families of size equal to the width, and OD families have OD enumerations without an additional short parameter. We first prove the countable-cofinality theorem, then give the additional argument needed at other singular cardinals. Throughout this section $B = \mathbb{C}_\Omega = \text{RO}(\text{Fn}(\Omega, 2))$, where $|\Omega| = \kappa$. A *raw factor* is the complete algebra B_S generated by the coordinates in $S \subseteq \Omega$. We use the full local diagonal actions on B^I introduced in section 2. In a frame $(f_i)_{i \in I}$ their target comparison matrices satisfy

$$P_{gh} = P_g g(P_h). \quad (48)$$

As in (43), rows and columns are Boolean partitions of 1. A constant comparison is an ordinary permutation of the labels.

10.1 Initial frames at countable cofinality

In this subsection all Cohen conditions are finite. Work in an arbitrary ZFC ground W , let κ be uncountable with $\text{cf}(\kappa) = \omega$, and put

$$B = \text{RO}(2^\Omega), \quad |\Omega| = \kappa, \quad C = B^I, \quad 0 < |I| \leq \kappa.$$

Suppose $\Theta : \text{Aut}(B) \rightarrow \text{Aut}(C)$ satisfies diagonal equivariance and locality, as in (1)–(2). Write $\delta(b) = (b)_{i \in I}$, $s(z) = \bigvee_i z_i$, and let ϵ_i be the canonical coordinate selectors. Let Γ be the finite bit translations and $\Sigma = \text{Sym}(\Omega)$ the coordinate permutations. As before, a selector has coordinates partitioning 1_B , and a frame is a partition of 1_C into selectors.

Theorem 10.1 (Initial frame). *The atoms of C^Γ form a frame of cardinality $|I|$. Every member of this frame is fixed by all coordinate permutations and by every automorphism fixing $B_{\Omega \setminus S}$ pointwise for some countable $S \subseteq \Omega$. Such an automorphism may depend on all the complementary coordinates.*

We first prove atomicity. We then use a standard-factor result to show that the atoms are selectors. No cardinal arithmetic hypothesis is used.

Lemma 10.2. *Every nonzero atomless complete Knaster algebra has a positive condition forcing a nonground real.*

Proof. Suppose no new real is added. Every countable sequence of maximal antichains then has a common maximal refinement. Indeed, all antichains are countable, and the generic sequence of

their selected indices belongs to the ground. Equivalently, the nonzero meets of ground choices from these antichains cover 1.

Build refining maximal antichains through ω_1 , splitting every node into two positive pieces at successor stages and taking a common refinement at countable limits. Each level is countable. Choose one node at each level. Knaster gives an uncountable pairwise-compatible set of chosen nodes, hence an unbounded chain in this refinement tree. Along that chain, the unused children at uncountably many successive splits form an uncountable antichain, a contradiction. If b is the positive Boolean value that a witnessing real is nonground, its binary digit tree below b has zero meet along every ground branch. $\square$

Lemma 10.3. *The algebra C^Γ is atomic with at most κ atoms, and it is fixed pointwise by every countably supported coordinate permutation and every countably supported bit translation.*

Proof. Every complete subalgebra $R \subseteq C$ has order density at most κ . Indeed, the positive finite conditions in single coordinates form an order-dense family in C of that size, and their upper projections $\pi_R(p) = \bigwedge \{r \in R : p \leq r\}$ are order-dense in R . Finitary permutations fix C^Γ by locality: on each pattern of their finite support they agree on the whole principal algebra with a finite translation. Since Σ normalizes Γ , Lemma 4.4 gives

$$C^\Gamma \subseteq C^{\Sigma_c}, \quad \Sigma_c = \{\sigma : |\text{supp}(\sigma)| \leq \aleph_0\}.$$

A Σ_c -invariant name for a family of at most κ reals is forced to consist of ground members. Suppose otherwise, and choose a member name $\dot{r}$ which is nonground on a positive condition b . Weak homogeneity allows us to take the family to be nonempty everywhere, and we can extend the member name outside its original condition. The values $\|\dot{r} = \check{x}\|$ for ground reals x have only countably many nonzero terms. Thus $\dot{r}$ and the positive Boolean value

$$b = \neg \bigvee_{x \in (2^\omega)^W} \|\dot{r} = \check{x}\|$$

have a common countable coordinate support S . Copy S onto the nonempty initial segments of each branch $t \in \kappa^\omega$; each prescribed copy extends to a countably supported permutation. Denote the copies by $\dot{r}_t, b_t$. For distinct t, u ,

$$b_t \wedge b_u \wedge \|\dot{r}_t = \dot{r}_u\| = 0. \tag{49}$$

Indeed, decide the finitely many common input bits. In the two remaining independent factors, a product condition forcing equality must decide every digit of each real. Otherwise, two opposite decisions on one side could both be combined with the fixed condition on the other side. The product condition therefore forces a ground real, contradicting b_t . The branches for which b_t holds inject into the given family, so ccc covering puts them in a ground set of size at most κ . Every b_t is nonzero, whereas $\kappa^\omega > \kappa$, a contradiction.

Put $E = C^{\Sigma_c}$. For each coordinate homomorphism $E \rightarrow B$, let e_i be the complement of the join of its kernel. The e_i cover 1_E , and $E \restriction e_i$ embeds completely into $B \restriction (e_i)_i$. These intervals are Knaster: the Cohen algebra is Knaster by the finite-condition Δ -system argument, and the property passes to complete subalgebras. If E had an atomless part, one such interval would have a positive binary tree with zero meets along all ground branches, by Lemma 10.2. Each label in the tree's root selects a branch. The resulting family of at most $|I|$ reals is Σ_c -invariant, so consists of ground reals. But on a nonzero coordinate of the root its selected branch equals no ground branch, a contradiction. Thus E is atomic. Upper projections of the atoms of a complete atomic

algebra are atoms of any complete subalgebra and cover its unit. Hence C^Γ is atomic. Its number of atoms is at most κ , by ccc in each coordinate of C .

Finally, for a countably supported translation t_u , choose on a countable supporting set a vector v such that both v and $v + u$ have infinitely many zeros and ones. A permutation σ of that set takes v to $v + u$, and $t_u = \sigma t_v \sigma^{-1} t_v^{-1}$. The maps σ and t_v normalize Γ , and σ acts trivially on C^Γ . Their commutator therefore acts trivially there as well. $\square$

Lemma 10.4 (Standard Cohen factors below the continuum). *Let $Q = \text{RO}(2^\omega)$ and $\max(|J|, \aleph_0) < 2^{\aleph_0}$. Every full local diagonal-equivariant action of $\text{Aut}(Q)$ on Q^J has a frame fixed pointwise by $\text{Aut}(Q)$.*

Proof. Write $X = 2^\omega$, let Γ_0 be the finite translations, T all translations, and $\Sigma_0 = \text{Sym}(\omega)$. The proof of Lemma 10.3, using Lemma 4.4(ii), shows that both Σ_0 and T fix $(Q^J)^{\Gamma_0}$. To prove atomicity, we need a separate argument because a name may use every coordinate of X .

Suppose a T -invariant family of reals has size at most $\lambda < 2^{\aleph_0}$ and has a nonground member $\dot{r}$ on a positive b . Read $\dot{r}$ by a Borel function $f : X \rightarrow X$ and represent b by a nonempty regular open E . Each $E \cap f^{-1}(y)$ is meager, so $Z = \{(x, y) \in E^2 : f(x) = f(y)\}$ is meager by Kuratowski–Ulam. The set

$$F = \{(a, c, x) : x + a, x + c \in E, f(x + a) = f(x + c)\}$$

is meager as well, by the change of variables $(a, c, x) \mapsto (x + a, x + c, x)$. Cover F by closed nowhere dense F_n . For a nonempty basic clopen $U \subseteq X$, the set

$$R_{n,U} = \{(a, c) : \{(a, c)\} \times U \subseteq F_n\}$$

is closed nowhere dense. Off their meager union R , each section $F_{a,c}$ is meager. There is a perfect $P \subseteq X$ whose distinct pairs avoid R . To construct it, recursively split finitely many clopen sets and shrink the children so that every ordered product of distinct children avoids the first finitely many closed nowhere dense relations. Choose the children with diameters tending to zero. For distinct $a, c \in P$, therefore,

$$t_a(b) \wedge t_c(b) \wedge \|t_a(\dot{r}) = t_c(\dot{r})\| = 0.$$

The conditions $t_a(b)$ are all nonzero. As in (49), ccc covering contradicts $|P| = 2^{\aleph_0} > \lambda$. Thus all members of the family are ground. Apply Lemma 10.2 to the coordinate-kernel intervals of $(Q^J)^T$, exactly as above, to obtain atomicity of $(Q^J)^T$ and then of $(Q^J)^{\Gamma_0}$.

Fix an atom U of the latter algebra. Its scalar support is 1. For any j with $U_j > 0$, the join of the countable orbit of $U \wedge \epsilon_j$ is U . Each translate is a partial selector and has only countably many nonzero coordinates, so U has countable coordinate support. Its nonzero fiber cardinality is constant by scalar ergodicity. Choose a Boolean fiber enumeration, identifying $Q^J|U$ over its diagonal with Q^{J_0} for a nonempty finite or countable J_0 . The Γ_0 -action on this cover is ergodic, and Σ_0, T preserve it.

Apply Lemma 4.7 with $\kappa = \omega$ to this ergodic cover. Its set-action hypothesis follows from Lemma 4.4(ii), applied to $\mathcal{P}(D)$ for any countable set D , since this complete algebra has countable order density. Short-support permutations here are precisely the finitary ones. The lemma therefore makes the actual map $\sigma \mapsto \Theta_\sigma$ ordinary Borel in Boolean-matrix codes, without any assumed regularity of the original action.

Choose $v_0, v_1 \in X$ such that $v_0, v_1, v_0 + v_1$ each have infinitely many zeros and ones. The two countable sets $\{u : v_i + u \text{ is finite or cofinite}\}$ are disjoint. For each u , choose the first i for which

$v_i + u$ is neither finite nor cofinite, and match the zeros and ones of v_i increasingly with those of $v_i + u$. The resulting Borel $\sigma(u)$ satisfies

$$t_u = \sigma(u)t_{v_i}\sigma(u)^{-1}t_{v_i}^{-1}.$$

The two fixed lifts $\Theta_{t_{v_i}}$ therefore make the actual translation action Borel as well.

Choose jointly Borel representatives of its countable matrices, replacing the matrix by the identity where it is not a permutation. This gives $c : T \times X \rightarrow \text{Sym}(J_0)$ with $(\Theta_{t_u}z)(x) = c(u, x)z(x + u)$. For each fixed u, v the identity $c(u + v, x) = c(u, x)c(v, x + u)$ holds for comeager x . Kuratowski–Ulam and the change of variables $(u, v, x) \mapsto (x, x + u, x + u + v)$ show that

$$F(x, z) = F(x, y)F(y, z), \quad F(x, y) = c(x + y, x),$$

holds for comeager triples. Choose a such that the identity holds at $z = a$ for comeager pairs, and put $K(x) = F(x, a)$. Then $c(u, x) = K(x)K(x + u)^{-1}$ for comeager (u, x) . The selectors $q_j(x) = K(x)(j)$ are therefore fixed by a comeager set of translations. Their common stabilizer is a subgroup of T containing a comeager set. Since a comeager set meets each of its translates, this subgroup is all of T . Ergodicity of Γ_0 now forces this frame to have a single member. Thus U is a selector.

The atoms form a Γ_0 -fixed frame $\mathcal{F}$. Given $g \in \text{Aut}(Q)$, apply Lemma 3.4 to $\langle \Gamma_0, g \rangle$ and Γ_0 . It gives q such that every qhq^{-1} is piecewise given by Γ_0 . Locality makes $\Theta_{q^{-1}}[\mathcal{F}]$ an $\langle \Gamma_0, g \rangle$ -fixed frame. Every Γ_0 -fixed selector is a member of $\mathcal{F}$, since its coefficients relative to that frame are invariant scalars and therefore zeros or ones. The transported frame is consequently the same frame, and g fixes it pointwise. Finally its ground cardinality is $|J|$, since its matrix is a Boolean bijection and ccc preserves ground cardinals. $\square$

Lemma 10.5. *Every atom of C^Γ is a selector. Its resulting frame is fixed by every automorphism fixing the complement of a countable coordinate set pointwise.*

Proof. Let U be an atom. Its scalar support is 1. If it is not a selector, choose $i \neq j$ with $U_i \wedge U_j > 0$. Write $m_{jk}^\gamma = (\Theta_\gamma \epsilon_j)_k$. Each row and each column has countably many nonzero entries, and every entry has countable coordinate support. Build increasing countable $S_n \subseteq \Omega$ and $J_n \subseteq I$, starting with the overlap and its supports. For every $\gamma \in \Gamma_{S_n}$, include the labels occurring in nonzero entries of rows and columns indexed by J_n . Include the scalar supports of these entries and the supports of the newly included U_j . Also, for every pair of positive basic partial selectors below U with labels in J_n and conditions on S_n , choose a finite translation whose image of the first meets the second and add its support. Such a translation exists because the orbit join of any positive element below U is U .

Put $S = \bigcup_n S_n$, $J = \bigcup_n J_n$, and let V agree with U on J and be zero elsewhere. The recorded matrices define a Γ_S -action on $B_S^J|V$. Its basic partial selectors are order-dense, and the recorded intersection witnesses make the action ergodic. Its original two-coordinate overlap remains nonzero.

Force first with $A = B_{\Omega \setminus S}$. The quotient is the recomputed standard Cohen algebra Q on S . The recorded inequalities persist, and the basic partial selectors remain order dense. The intersection test therefore proves ergodicity for new elements of $Q^J|V$ as well. By Lemma 3.5, the whole original cover descends to Q^I with its full local relative action, including all new automorphisms. The complementary extension has at least κ distinct reals, preserves $\text{cf}(\kappa) = \omega$, and therefore has continuum strictly greater than κ . By Lemma 10.4, the full cover Q^I has an invariant frame. Intersect it with V . The intersections are invariant partial selectors covering V ;

ergodicity makes any nonzero one equal to V . This contradicts its overlapping coordinates. Hence U is a selector.

Let $\mathcal{F}$ be the frame of these atoms. For any countably infinite S , pass again to the complementary extension and choose a full invariant frame $\mathcal{E}$ there. Each $f \in \mathcal{F}$ is still a Γ_S -fixed selector. Its coefficients relative to $\mathcal{E}$ are Γ_S -invariant scalars, hence zeros or ones, so f is one row of $\mathcal{E}$. Every new relative automorphism fixes f . By the lifting in Lemma 3.5, the corresponding ground automorphisms fix f as well, including those whose action depends arbitrarily on the complement. The frame has cardinality $|I|$ by ccc and its Boolean bijection with I . $\square$

Lemma 10.6 (Small actions of the standard Cohen group). *Every homomorphism $\text{Aut}(Q) \rightarrow \text{Sym}(J)$, where $|J| < 2^{\aleph_0}$ and Q is the standard Cohen algebra, is trivial.*

Proof. We first prove that $G = \text{Aut}(Q)$ is simple. Let G_a consist of the automorphisms supported on a principal region a , and use $[x, y] = xyx^{-1}y^{-1}$. If $N \triangleleft G$ is nontrivial, choose $g \in N$ and $a > 0$ with $a \wedge g(a) = 0$. For $u, v \in G_a$,

$$[[u, g], v] = [u, v],$$

since $gu^{-1}g^{-1}$ is supported on the disjoint region $g(a)$. Thus $[G_a, G_a] \subseteq N$. For $0 < b < a$, every $h \in G_b$ is such a commutator. Partition a into nonzero b_n , $n \in \mathbb{Z}$, with $b_0 = b$, and choose $q \in G_a$ shifting them. Let R equal $q^n h q^{-n}$ on b_n for $n \geq 0$ and the identity on the negative banks. Then $[R, q] = h$. All proper nonzero principal regions are conjugate, since they and their complements are standard Cohen algebras. Hence N contains every G_c for $0 < c < 1$.

These subgroups generate G . Given $f \neq 1$, choose $a > 0$ with $a \wedge f(a) = 0$ and $a \vee f(a) < 1$. The involution t equal to f on a , f^{-1} on $f(a)$, and the identity elsewhere belongs to $G_{a \vee f(a)}$, while $tf \in G_{-a}$. Thus $f = t(tf) \in N$, proving simplicity.

Identify Q with $\text{RO}(\mathbb{R})$. Increasing homeomorphisms of $\mathbb{R}$ embed faithfully in G . Rosendal–Solecki’s small-index theorem [28, Theorem 19] says that $\text{Homeo}_+(\mathbb{R})$ has no nontrivial permutation action on fewer than continuum many points. The kernel of the given action of G therefore contains this nontrivial subgroup, and simplicity makes the entire action trivial. $\square$

Proof of Theorem 10.1. Only fixation by all coordinate permutations remains. Let $\mathcal{F}$ be the frame from Lemma 10.5. Fix an infinite $E \subseteq \Omega$ with $|\Omega \setminus E| = \kappa$, and force the complementary factor $A = B_{\Omega \setminus E}$. In that extension, let L_E be all automorphisms of the recomputed B_E fixing the complement of some countable subset of E . Every such automorphism fixes $\mathcal{F}$. Ccc covering puts its named countable support inside a ground countable set, and the lift fixes that set’s full complement in B . Thus Lemma 10.5 applies to its lift. Since L_E contains the finite translations, coefficientwise scalar ergodicity gives

$$((B_E)^I)^{L_E} = \mathcal{P}(\mathcal{F}).$$

Partition E into countably infinite blocks. The full recomputed $\text{Aut}(Q)$ acts diagonally by the same standard automorphism on each block. To define this Boolean action, choose homeomorphism representatives between comeager subspaces on each block. Their countable products give compatible automorphisms on every countable collection of blocks. Every event has countable support, so these actions and their inverses extend to B_E ; ccc reduces arbitrary joins to countable ones. This diagonal group normalizes L_E , since the countably many blocks meeting a countable support still have countable union. It therefore permutes the atoms $\mathcal{F}$ of the displayed fixed algebra. The complementary extension has continuum greater than $\kappa \geq |\mathcal{F}|$. By Lemma 10.6, this permutation action is trivial. In particular every specified old diagonal map fixes $\mathcal{F}$ in the ground.

Every coordinate permutation is a product of two involutions. An involution with countably many pairs is covered by Lemma 10.5. If it has uncountably many pairs and its support has a κ -sized complement, group its pairs into countably infinite blocks. It is the diagonal action of one fixed-point-free standard involution, so the preceding argument applies. If its complement is smaller than κ , split its κ pairs into two collections of size κ . Each restriction has a κ -sized complement and fixes the frame. This treats every involution and completes the proof. $\square$

10.2 Proper factors and countable cofinality

Let N be generated by independently extended automorphisms of B_E with $|E| = |\Omega \setminus E| = \kappa$. In particular, an independently extended automorphism of a short raw factor belongs to N .

The proper-factor and short-adjustment lemmas, Lemmas 9.14 and 9.15, apply to this group N . In particular, an N -fixed frame remains fixed by the full recomputed relative proper-factor group after any short raw coordinate extension.

Lemma 10.7. *Suppose $\text{cf}(\kappa) = \omega$, $|I| \leq \kappa$, and a full local action on B^I has an N -fixed frame. Any constant comparison on a preserved ground block is the identity.*

Proof. Write γ for the comparison on J . The Cohen case of Lemma 9.16 gives commuting involutions τ_n , bits b_n with target comparisons (44), and a fixed algebra D such that $B \cong \text{RO}(D^+ \times \text{Fn}(\omega, 2))$.

In a B -extension the continuum is greater than κ , since there are at least κ distinct new reals and the continuum cannot have countable cofinality. One standard Cohen real preserves its ground continuum, by counting nice names. Consequently the D -extension in (45) already has continuum greater than κ . Descend the whole cover to the standard Cohen quotient there. Every new quotient automorphism lifts to a D -fixing automorphism of B , so the relative action is full. By Lemma 10.4, it has a fully invariant frame.

The permutation exchanging clock bits k, l and fixing D is the pasting of $\tau_k \tau_l$ where the bits differ and the identity where they agree. The two exponents in (44) cancel, so it fixes each old f_j , $j \in J$. Express f_j in the new invariant frame in the D -extension. Every scalar coefficient is fixed by all finite permutations of the standard Cohen coordinates, hence is 0 or 1 by the finite-condition zero-one argument. Thus f_j is a member of that frame and is fixed by all τ_k . Equation (44) now gives $\gamma(j) = j$. $\square$

Theorem 10.8 (Cohen frames at countable cofinality). *Let κ be uncountable of countable cofinality in a ZFC ground. Every full local diagonally equivariant action of $\text{Aut}(\mathbb{C}_\kappa)$ on $\mathbb{C}_\kappa^I$, with $0 < |I| \leq \kappa$, has a frame fixed pointwise by the whole group.*

Proof. Use Theorem 10.1 and Lemma 9.14 to obtain an N -fixed frame. Given h and $i \in I$, close $\{i\}$ under the possible images and inverse images of P_h . The ccc makes this a countable ground block J preserved by P_h . Its matrix entries have countable raw support. Close that support under $h^{\pm 1}$ to obtain a countable invariant factor A . If $k \in N$ independently extends $h|A$, then $r = hk^{-1}$ fixes A and $P_r = P_h$. After forcing A , the relative action is full and its frame remains proper-factor fixed by Lemma 9.14. The comparison on J is now a constant permutation. Apply Lemma 10.7 in that relative ground. It follows that P_h fixes i . Varying h, i proves the theorem. $\square$

Corollary 10.9. *Let $W \models \text{ZFC} + \text{GA} + (V = \text{HOD})$, let $\text{cf}^W(\kappa) = \omega < \kappa$, and let G be $\text{Add}(\omega, \kappa)^W$ -generic. Every OD family of size at most κ in $W[G]$ has an OD bijective enumeration by a ground cardinal at most κ . Every $\text{OD}_{<\kappa}$ family of that size has an $\text{OD}_{<\kappa}$ bijective enumeration. No GCH hypothesis is required.*

Proof. For a nonempty OD family, choose an invariant name and a named bijection indexed by a ground cardinal. The theorem supplies a fixed frame, which relabels this bijection to an invariant function-name. Apply Lemmas 2.2 and 2.3. If the family is defined from a short parameter, choose a nice name for that parameter supported on a short raw coordinate factor. Code that restricted generic by a subset of a short ordinal, and apply the same argument to the residual κ -Cohen extension. Stable ground codes retain this one short parameter. The empty family has its empty enumeration. $\square$

10.3 Eliminating constant comparisons at higher cofinality

The next lemma applies to frames of any size. We will use it after one short capture has made the frame proper-factor fixed. The proof uses an ω_1 -long chain of proper raw factors. These factors may have size equal to the full width.

Lemma 10.10. *Let $\kappa > \omega_1$ in a ZFC ground. In a full local action on $\mathbb{C}_\kappa^I$ with an N -fixed frame, a constant comparison on any preserved ground block $J \subseteq I$ is the identity.*

Proof. Let u have constant comparison γ on J . First construct commuting automorphisms J_i , $i < \omega_1$, on one raw bank A of size ω_1 . Partition A into countable banks indexed by $\omega \leq \delta < \omega_1$. In bank δ , partition 1 into positive events indexed by the countable group $E_\delta = \bigoplus_{i < \delta} \mathbb{Z}$, choose coherent isomorphisms between their principal Cohen algebras, and let E_δ translate the indices. Let J_i perform its i th translation in every bank with $i < \delta$, and act trivially in the others. These products and their inverses are Cohen automorphisms, obtained on finite products and then extended by order completion. All J_i belong to N .

Each countable prefix E_θ has a fundamental event $p_\theta \in B_A$ satisfying

$$\{J_v(p_\theta) : v \in E_\theta\} \text{ partitions } 1, \quad J_\theta(p_\theta) = p_\theta. \quad (50)$$

For infinite θ , use the identity cell in bank θ ; for finite θ , use the union of cells in bank ω whose first θ entries vanish. Also, each $b \in B$ is fixed by all sufficiently late J_i , since the indices of the banks meeting its countable raw support are bounded in ω_1 .

We next prepare an increasing chain $D_i = B_{S_i}$, $i < \omega_1$, of u -invariant proper raw factors, all containing B_A , such that

$$|S_i| = |\Omega \setminus S_i| = \kappa, \quad B = \bigcup_{i < \omega_1} D_i \quad \text{as a set.} \quad (51)$$

For each raw coordinate x , let C_x be its countable closure under chosen supports for $u^{\pm 1}$ of raw literals. Thus $y \in C_x$ implies $C_y \subseteq C_x$. The set $K = \bigcup_{x \in A} C_x$ has size at most ω_1 . Hajnal's free-set theorem, in its arbitrary-cardinal form [7, p. 69], supplies $Z \subseteq \Omega$ of size κ with $C_z \cap Z = \{z\}$ for $z \in Z$. The theorem requires the fixed bound $|C_x| < \omega_1 < \kappa$; it does not require κ to be regular. Remove K from Z and color Z with ω_1 colors, each used κ times. Put

$$\rho(x) = \sup\{\text{col}(z) : z \in C_x \cap Z\}, \quad S_i = \{x : \rho(x) \leq i\},$$

where the empty supremum is 0. Each $\rho(x) < \omega_1$, and $y \in C_x$ implies $\rho(y) \leq \rho(x)$. Hence each S_i is closed under the dependencies from both u and its inverse, and contains K . Since $\rho(z) = \text{col}(z)$ on Z , both S_i and its complement have size κ . Every countable support is contained in some S_i , proving (51).

Call a map chain preserving if it preserves every D_i setwise. The maps u, J_i have this property. So do their compositions and inverses, independent extensions of their restrictions to any D_i , and principal pastings on countable partitions in B_A . For independent extensions, this follows by restricting to earlier factors and using the product decomposition on later factors. For pastings, compute images and inverse images as joins in each complete D_i .

We construct permanent commuting maps h_i and full maps F_θ , all chain preserving, with

$$\begin{aligned} h_i|_{B_A} &= J_i|_{B_A}, & P_{h_i}|_J &= \gamma, \\ F_\theta h_i F_\theta^{-1} &= J_i \quad (i < \theta), & F_\theta|_{B_A} &= \text{id}, \\ F_\eta|_{D_\theta} &= F_\theta|_{D_\theta} \quad (\theta < \eta). \end{aligned} \tag{52}$$

Start with $F_0 = \text{id}$. The source prefix generated by the h_i , $i < \theta$, is a copy of E_θ , since it agrees with the target action on A . Its comparisons on J are $\gamma^{\sum_i v_i}$ for $v \in E_\theta$.

We use two pasting constructions. First, if a countable action h_v has fundamental event p , conjugate a map supported on p by h_v to define it on each $h_v(p)$. The pasted map centralizes the action. If the action has comparisons which are powers of γ , this construction preserves a constant comparison γ or 1. Second, suppose source and target actions agree on A , share $p \in B_A$, and an equivariant automorphism T of an invariant raw factor $D \supseteq B_A$ fixes B_A . Independently extend T to Q . On the common partition $h_v(p) = J_v(p)$ put

$$F = J_v Q h_v^{-1}. \tag{53}$$

The inverses of these restrictions show that F is a full automorphism. Equivariance shows that it extends T , fixes B_A , and conjugates the source action to the target action. These pastings preserve the chain whenever their defining restrictions do.

At successor stage θ , put $D = D_\theta$ and $T = F_\theta|_D$. The automorphism $\sigma = T^{-1}(J_\theta|_D)T$ has an independent extension in N . Localize that extension to p_θ , and mirror it under the old source prefix. The resulting map j centralizes the prefix, restricts to σ on D , and has identity comparison on J . This restriction holds because σ commutes on D with the old source action. It preserves the chain, as does each restriction used in its construction. The action generated by the old maps and j has fundamental event $p_{\theta+1}$ in B_A .

Independently extend $(u|D)^{-1}$ to $q \in N$. Then $w = qu$ fixes D and retains comparison γ . Localize w to $p_{\theta+1}$ and apply the first pasting construction to the enlarged action. This gives a chain-preserving map r centralizing that action, fixing D , and having comparison γ on J . Set $h_\theta = rj$. It is a permanent extension of the required source restriction, commutes with the old maps, and satisfies (52). Apply (53) with $p_{\theta+1}$ and the independent extension of T to obtain $F_{\theta+1}$. This preserves all the inductive requirements, including preservation of every D_i .

At a countable limit θ , coherent restrictions and inverse restrictions give an automorphism T of $D^- = B_{\bigcup_{i < \theta} S_i}$: the union of the earlier raw factors is order dense in D^- . It is equivariant with the permanent source prefix, fixes B_A , and preserves each earlier D_i . Its independent extension preserves the whole chain, since the raw coordinates of D^- are contained in every later S_i . Use (53) with p_θ to obtain the full F_θ . No continuity of the chain at θ is needed.

Finally, by (51), the coherent restrictions define a full automorphism F with $Fh_i = J_i F$ for every $i < \omega_1$. Put $P = P_F$. Since $P_{J_i} = 1$, (48) gives, on columns in J ,

$$P\gamma = J_i(P) \quad (i < \omega_1).$$

If $\gamma(a) \neq a$, choose b with $P(b, a) > 0$. A sufficiently late J_i fixes this event exactly, so $P(b, \gamma(a)) = P(b, a) > 0$, contradicting disjointness within row b . Thus $\gamma = 1$. At limits we took unions only of

restrictions of the conjugators. Every source map used in the final identity remained unchanged after its construction. $\square$

Theorem 10.11 (Uniform strict-small Cohen families). *Let $W \models \text{ZFC} + \text{GA} + (V = \text{HOD})$ and let κ be any singular cardinal of W . In a $\text{Add}(\omega, \kappa)^W$ -extension, every $A \in \text{OD}_{<\kappa}$ of size less than κ has an $\text{OD}_{<\kappa}$ bijective enumeration. No GCH or strong-limit assumption is required.*

Proof. Countable cofinality is Corollary 10.9. Suppose $\text{cf}(\kappa) > \omega$. Capture the original short parameter and represent the family by a full local cover with $|I| < \kappa$ labels. The uniform normalization lemma proved in the next section, Lemma 11.4, supplies, after one further short capture, a frame fixed by all raw permutations and all relative short-factor maps. For Cohen algebras, that lemma uses only Theorem 10.8, so the present argument is not circular. By Lemma 9.14 the frame is proper-factor fixed, also in every subsequent short relative ground.

For any scalar automorphism g , the entries of P_g have a joint raw support of size at most $\max(|I|, \aleph_0) < \kappa$. Close it under $g^{\pm 1}$ to obtain a short invariant factor B_T . Independently extend $g|_{B_T}$ to h , which fixes the frame. Then $r = h^{-1}g$ fixes B_T and $P_r = h^{-1}(P_g)$ has all entries in B_T . After forcing B_T , it is a constant permutation of the whole frame. The residual width is still κ and $\kappa > \omega_1$, so Lemma 10.10 makes it the identity. This proves $P_g = 1$ for every g .

We used the factor on T separately for each g to prove invariance. Its generic is not a parameter of the final definition. The invariant enumeration exists by Lemmas 2.2 and 2.3 using just the original parameter capture and the single normalization capture. Together these are one short code. $\square$

Remark 10.12. The strict inequality is necessary for uniform enumeration. Under GCH and uncountable cofinality, the Cohen extension has continuum κ . Its family of all reals cannot have an $\text{OD}_{<\kappa}$ enumeration: after capturing a proposed short parameter, homogeneity would place every real enumerated from it in that short coordinate extension, contrary to the existence of new reals in the remaining Cohen extension. The next section proves the pointwise endpoint for singular strong-limit κ .

11 Small index and small families at singular strong limits

We prove the Cohen and random small-index theorems together. All intermediate domains are coordinate factors, so we can amalgamate maps by taking products over a common coordinate base. The two constructions use different completions at limit stages.

Throughout the first subsection, Ω has singular strong-limit cardinality κ . Write B_E for the complete coordinate factor on $E \subseteq \Omega$, and let $B = B_\Omega$ be either the ordinary Cohen algebra or the fair product probability algebra. In the former case put $G = \text{Aut}(B)$, and in the latter put $G = \text{Aut}(B, \mu)$. A coordinate set is *short* if it has cardinality less than κ . Subscripts in parentheses denote pointwise stabilizers.

11.1 A common small-index theorem

Theorem 11.1. *Let D be a complete subalgebra of B contained in a short coordinate factor, and put $P = G_{(D)}$. If $H \leq P$ and $[P : H] \leq \kappa$, then there is a short $S \subseteq \Omega$ such that*

$$D \subseteq B_S \quad \text{and} \quad G_{(B_S)} \subseteq H.$$

This holds also when $\text{cf}(\kappa) = \omega$.

The proof uses the tree method of Melles and Shelah [27]. We give the tree construction in detail and prove its extension step using coordinate products.

Every element of either algebra has countable coordinate support. Hence, given a short set T and fewer than κ specified full maps, closing T under supports of the images and inverse images of its coordinate bits, in countably many rounds, gives a common invariant coordinate factor. Its coordinate set has cardinality at most

$$\max(\aleph_0, |T|, \text{number of specified maps}) < \kappa. \quad (54)$$

For an increasing chain of coordinate sets, the union of their complete factors is order dense in the factor on their union in the Cohen case, and metrically dense in the random case. Indeed, it contains every finite-coordinate cylinder. Compatible maps and compatible inverses therefore extend uniquely to the appropriate completion. In the random case these extensions remain measure preserving.

We also use the following elementary product amalgamation. Suppose S, Y, Z are disjoint, and automorphisms u of $B_{S \cup Y}$ and v of $B_{S \cup Z}$ preserve B_S and have the same restriction β there. Extend β independently, remove this extension on the left of both maps, and extend the two residual maps by the identity on the other bank. These residual extensions commute, since each fixes the entire factor containing the images of the other's additional coordinate bits. Their product, followed by the extension of β , amalgamates u, v . All maps are complete automorphisms, with the corresponding product inverses. If u, v preserve measure, every extension and product used here preserves measure. Dependence on the common base B_S is allowed.

Lemma 11.2 (One row of equations). *Let $N_0 = B_S \subseteq N_1 = B_T$ be short coordinate factors containing D , and let V be a set of fewer than κ indices with a distinguished index η . Suppose full maps $h_\nu \in P$ preserve N_0, N_1 , with $h_\eta = \text{id}$. Suppose automorphisms g_0, m_ν, l_ν of N_0 , fixing D , satisfy*

$$g_0 m_\eta m_\nu^{-1} g_0^{-1} = l_\eta l_\nu^{-1} (h_\nu \upharpoonright N_0). \quad (55)$$

Prescribe extensions m_ν^+, l_ν^+ to automorphisms of N_1 fixing D . There are full maps $g, m'_\nu, l'_\nu \in P$ retaining these restrictions, with $g \upharpoonright N_0 = g_0$, such that

$$g m'_\eta (m'_\nu)^{-1} g^{-1} = l'_\eta (l'_\nu)^{-1} h_\nu \quad (\nu \in V). \quad (56)$$

In the random case all the prescribed and constructed maps are measure preserving.

Proof. Put $Y = T \setminus S$, and choose a fresh coordinate copy Y' of Y outside T . Let g act as g_0 on B_S , swap Y with Y' , and fix all remaining coordinates. Then

$$C' = g^{-1}[N_1] = B_{S \cup Y'}.$$

Independently extend m_η^+, l_η^+ to full maps a, b . Both preserve B_S and fix Y' , hence preserve C' . For $\nu \neq \eta$ define an automorphism v_ν of C' by

$$v_\nu(a^{-1} g^{-1} b(x)) = g^{-1} h_\nu^{-1} l_\nu^+(x) \quad (x \in N_1). \quad (57)$$

Both parameterizations are isomorphisms of N_1 onto C' , and are measure preserving in the random case. Rearranging (55) gives

$$m_\nu m_\eta^{-1} g_0^{-1} l_\eta = g_0^{-1} (h_\nu \upharpoonright N_0)^{-1} l_\nu.$$

Thus v_ν and m_ν^+ agree on B_S . The product amalgamation above, followed by independent extension to B , gives m'_ν extending both. Put $m'_\eta = a$, and define on all of B

$$l'_\nu = h_\nu g m'_\nu a^{-1} g^{-1} b. \quad (58)$$

For $\nu = \eta$ this is b ; otherwise (57) says exactly that it extends l_ν^+ . Equation (56) is a rearrangement of (58). Every map fixes D . $\square$

Proof of Theorem 11.1. Put $\theta = \text{cf}(\kappa)$. Choose increasing infinite cardinals μ_i , $i < \theta$, cofinal in κ , with $\mu_i^+ < \kappa$, and consider the tree

$$T_\alpha = \prod_{i < \alpha} \mu_i^+ \quad (\alpha \leq \theta).$$

Every level below θ has size less than κ . Indeed, for some infinite $\xi < \kappa$, its size is at most $\xi^\xi = 2^\xi < \kappa$. König's theorem gives $|T_\theta| > \kappa$. Fix continuous increasing short coordinate sets X_α , $\alpha < \theta$, with union Ω .

Suppose the conclusion fails. We construct increasing short coordinate factors N_α containing D and B_{X_α} . At every node $\eta \in T_\alpha$ we construct three automorphisms g_η, m_η, l_η of N_α fixing D , coherent along branches. Once N_α is chosen, choose permanently

$$h_\alpha \in G_{(N_\alpha)} \setminus H.$$

All later coordinate closures include $h_\alpha^{\pm 1}$, so it preserves every later domain and fixes every earlier one pointwise. For nodes η, ν at the same level, let $h_{\eta\nu} = h_\gamma$ if their first difference is at γ and $\eta(\gamma) < \nu(\gamma)$; in the opposite order, and for equal nodes, let $h_{\eta\nu} = \text{id}$. Maintain

$$g_\eta m_\eta m_\nu^{-1} g_\eta^{-1} = l_\eta l_\nu^{-1} (h_{\eta\nu} \upharpoonright N_\alpha). \quad (59)$$

The initial maps can all be the identity.

At a successor stage, give each child its parent's maps on N_α . The equations hold there also for new siblings, since h_α fixes N_α . Independently extend all inherited m, l maps, and use (54) to obtain a common short invariant coordinate factor containing N_α , $B_{X_{\alpha+1}}$, and invariant under all previously chosen h 's. Initially retain each g only on N_α .

Choose an infinite $\chi < \kappa$ bounding the size of this coordinate set, the new level, and the previously chosen maps. In an iteration of length χ , visit every row cofinally often. Such a schedule exists even when χ is singular. Partition χ into as many sets of cardinality χ as there are rows, and visit each row at the stages in its assigned set. When row η is visited, let N_0 be its last completed domain and let N_1 be the current common domain. Its old equation holds on N_0 , all current m, l maps extend their restrictions there, and all $h_{\eta\nu}$ preserve both domains. Apply Lemma 11.2. Capture the resulting full maps, the previously chosen maps, and the current domain in a common invariant coordinate factor of size at most χ . This extends all m, l restrictions and the selected g restriction. Every unvisited row keeps its old g and its old equation on its own domain; no equation is imposed for that row on the larger domain until it is visited.

At an inner limit complete the increasing union of common coordinate factors. For a particular g , complete the union of its visit domains, which is an earlier stage, possibly a limit stage, of the same chain. The coordinate bound remains χ . At the end all rows have been visited cofinally, so all three maps act on the same factor $N_{\alpha+1}$ and satisfy (59) there. Their inverses are coherent as well.

At an outer limit $\alpha < \theta$, the union of the coordinate sets is still short. Complete the coherent maps along each branch. At height θ , every branch similarly gives full maps $g_\rho, m_\rho, l_\rho \in P$: the union contains every finite-coordinate cylinder, so order completion or metric completion gives all of B . This last assertion applies also when $\theta = \omega$.

For two distinct branches, ordered so that $\rho(\gamma) < \nu(\gamma)$ at their first difference, the two instances of (59) imply

$$aLa^{-1} = Lh_\gamma, \quad a = g_\rho g_\nu^{-1}, \quad L = l_\rho l_\nu^{-1}. \quad (60)$$

There are more than κ branches and at most κ pairs of right cosets (Hg_ρ, Hl_ρ) . Two branches therefore give $a, L \in H$, and (60) gives $h_\gamma \in H$, a contradiction. The argument does not assume that membership in H is preserved under limits. $\square$

11.2 A uniform normalization for fewer than κ labels

We now prove the frame normalization used in both kinds of extension. This argument does not require κ to be strong limit. A full local diagonal action on B^I and its selector frames are understood as in (1)–(2) and section 9.2. In the random case “full” refers to all Boolean automorphisms, including the nonsingular ones. Relative actions are always recomputed in the intermediate extension.

Lemma 11.3. *In any ZFC ground, if ν is uncountable of countable cofinality and $0 < |I| \leq \nu$, every full local diagonal action on the random B_ν^I has a frame fixed by the full Boolean automorphism group.*

Proof. By Lemma 9.13, there is a frame fixed by raw permutations and all maps fixing the complement of a countable coordinate set. Lemma 9.14 and Theorem 9.18 then give fixation by every measure-preserving automorphism. For a nonsingular g , the density $f = d(g_*\mu)/d\mu$ has countable coordinate support. On a countably infinite factor containing that support, the isomorphism theorem for atomless standard probability spaces supplies a nonsingular c with $c_*\mu = f\mu$. Extend it independently. Then $c^{-1}g$ is measure preserving, while c acts only on a countable coordinate factor. Both therefore fix the frame. $\square$

Lemma 11.4 (Uniform normalization). *Work in any ZFC ground. Let κ be singular with $\text{cf}(\kappa) > \omega$, let B be Cohen or random of width κ , and let a full local diagonal action on B^I be given, where $0 < |I| < \kappa$. Fix any original frame $(e_i : i \in I)$, and put $\lambda = \max(\aleph_0, |I|)$. There is a coordinate set S of size at most λ such that, after forcing B_S , the descended original frame is fixed by every raw permutation and by every full automorphism fixing the complementary factor of some short coordinate set pointwise. The latter maps may depend on all complementary parameters.*

Proof. Use comparison matrices in the original frame, with convention

$$P_{gh} = P_g g(P_h). \quad (61)$$

Choose $\lambda < \nu < \kappa$ of countable cofinality, and a coordinate bank T of size ν . Force the complementary factor first. The relative cover is the original recomputed B_T^I , with its full action. This is Lemma 9.11 for random forcing; the same argument for Cohen forcing uses names for operators and their inverses in the product iteration, and descends their equality tests and the locality identity. In particular, the relative action includes every operator in the intermediate extension.

By Theorem 10.8 and Lemma 11.3, this relative cover has a fully invariant frame. Lift it to a frame F over B . It is fixed by every automorphism fixing the complement of T pointwise. The change-of-frame matrix from F to (e_i) has at most λ entries, supported together by a set U of size at most λ . Choose $Q \subseteq T \setminus U$ of size ν . Every automorphism fixing the complement of Q fixes both F and the change-of-frame matrix, and hence fixes the original frame.

Let t_α flip coordinate α , and put

$$S = \{\alpha \in \Omega : P_{t_\alpha} \neq \text{id}\}.$$

For a raw permutation π , choose a support U_π of its comparison matrix of size at most λ . If $\alpha \in Q$ and $\beta = \pi(\alpha)$, then $t_\beta \pi = \pi t_\alpha$, so

$$P_{t_\beta} t_\beta(P_\pi) = P_\pi.$$

For $\beta \notin U_\pi$ this gives $P_{t_\beta} = \text{id}$. Thus $S \cap \pi[Q] \subseteq U_\pi$. If $|S| > \lambda$, choose a permutation sending Q to a set of size ν containing λ^+ elements of S . This contradicts the bound on U_π . Consequently $|S| \leq \lambda$.

Force B_S . The original frame in the recomputed tail is fixed by every finite translation. Let E now be any short coordinate set in this intermediate ground. Choose a countable-cofinality cardinal ν strictly between $\max(|E|, \lambda)$ and κ , and a bank $T \supseteq E$ of size ν . After forcing its complement, choose a full invariant frame F as above. The coefficients of every e_i in the F coordinates are fixed by finite translations of T . The scalar fixed algebra of those translations is $\{0, 1\}$, by the finite-condition argument for Cohen forcing and product-measure ergodicity for random forcing. Hence each selector e_i is a member of F and is fixed by the full relative group. Every automorphism fixing the complement of E belongs to this relative T group. This proves the claimed fixation for all such new E and all new relative maps.

It remains to consider raw permutations. The finite-translation fixed cover algebra consists of unions of the original frame members. Permutations normalize these translations, and therefore act ordinarily on I . Every action of $\text{Sym}(\kappa)$ on fewer than κ points is trivial. For completeness, the small-index support theorem of Dixon–Neumann–Thomas [5, Theorem 2^b, p. 582] gives, for any point stabilizer L of index less than κ , a short A with $\text{Sym}(\kappa)_{(A)} \subseteq L$. Among κ permutations sending A to pairwise disjoint sets, two have the same left L coset. Thus some $h \in L$ sends A disjointly from itself. The pointwise stabilizers of two disjoint short sets A, D generate $\text{Sym}(\kappa)$: given f , choose v fixing A with $vf[A] \cap D = \emptyset$, extend $vf|A$ to w fixing D , and note that $w^{-1}vf$ fixes A . Hence $L = \text{Sym}(\kappa)$, as required. $\square$

11.3 Arbitrary-rank definable families

Theorem 11.5 (The Cohen endpoint). *Suppose $W \models \text{ZFC} + \text{GA} + (V = \text{HOD})$ and κ is a singular strong-limit cardinal of uncountable cofinality in W . In an extension $M = W[G]$ by κ ordinary Cohen reals, every $A \in \text{OD}_{<\kappa}$ of cardinality at most κ satisfies $A \subseteq \text{OD}_{<\kappa}$. The members may have arbitrary rank.*

Proof. Capture the short parameter defining A , and a condition deciding the definition, nonemptiness, and cardinal bound, in a short coordinate extension V_0 of W . The empty case is immediate. This gives a fully invariant residual name $\dot{A}$, with a forced bijection $(\tau_i : i < \mu)$ onto it, where $\mu \leq \kappa$.

The required arithmetic holds also in V_0 . In W , strong limitness and uncountable cofinality give $\kappa^{\aleph_0} = \kappa$; a short Cohen or random factor has size less than κ . Forcing of size $\delta < \kappa$ has at most

$2^{\delta \cdot \xi} < \kappa$ names for subsets of each infinite $\xi < \kappa$. Since our short factors are ccc, κ remains strong limit singular of the same cofinality. Thus in V_0 the residual Cohen algebra B has cardinality κ .

Let Z be the set of names forced to belong to $\dot{A}$, modulo forced equality. Each is a countable Boolean mixture of the τ_i , so

$$|Z| \leq (\mu \cdot |B|)^{\aleph_0} \leq \kappa.$$

The full automorphism group of B acts on the set Z . For each i , Theorem 11.1 gives a short S_i such that the full pointwise fixer of B_{S_i} fixes τ_i . Force this factor first. By Lemma 2.4, the quotient name is fixed by every automorphism recomputed in $V_0[G_{S_i}]$: names for an operator and its inverse lift through the iteration to maps fixing B_{S_i} . The stable-code and invariant-name lemmas 2.2 and 2.3 give an $\text{OD}_{<\kappa}$ definition of its value. The original parameter capture and this further generic have one combined short code. Only the original W is required to satisfy the stable-ground hypotheses. Applying the argument separately to every i proves the claim; no union of the S_i is taken. $\square$

Theorem 11.6 (Uniform enumeration for random families). *Under the same ground and cardinal assumptions, let $M = W[G]$ be the extension by κ random reals. Every $A \in \text{OD}_{<\kappa}$ with $|A| < \kappa$ has a bijective enumeration belonging to $\text{OD}_{<\kappa}$. There is no restriction on the ranks of its members.*

Proof. Capture the defining parameter and a condition deciding nonemptiness and $|A| = \mu < \kappa$ in a short coordinate extension of W . A name for a bijection of μ onto the fully invariant family gives the original frame $(e_i : i < \mu)$ of its cover. Apply Lemma 11.4 and make its one common short capture. In the resulting ground V_* this original frame is fixed by every short relative full automorphism. The preceding arithmetic argument shows that the residual probability algebra still has strong-limit singular type κ , uncountable cofinality, and cardinality κ .

Its selectors form an ordinary set of size at most $(\mu \cdot |B|)^{\aleph_0} = \kappa$. For each i , apply Theorem 11.1 to the stabilizer of e_i in $\text{Aut}(B, \mu)$, obtaining a short S_i whose full measure-preserving pointwise fixer fixes e_i . Given an arbitrary $g \in \text{Aut}(B, \mu)$, close S_i under coordinate supports of $g^{\pm 1}$ to a short invariant set E . Let n independently extend $g|_{B_E}$. The map n is short and measure preserving, so fixes the entire original frame. The map $n^{-1}g$ fixes B_E pointwise, so fixes e_i . Therefore g fixes e_i . This holds for all g and i .

Every nonsingular automorphism factors into a nonsingular map acting on a countable coordinate factor and a measure-preserving map, by the density correction in Lemma 11.3. Both fix the original frame. Thus its single enumeration name is fully invariant. Apply Lemmas 2.2 and 2.3. The parameters are the generics from the original short capture and the common normalization capture. They have one short code for the entire enumeration. The memberwise sets S_i and the auxiliary sets E were used only to prove invariance and are not added to this parameter. $\square$

Remark 11.7. The Cohen conclusion at size κ is pointwise; it does not assert a common short-parameter enumeration. The random strict inequality is sharp: at the present cardinals $\kappa^{\aleph_0} = \kappa$, and Propositions 9.5 and 9.6 give a definable family of size κ with no $\text{OD}_{<\kappa}$ member.

12 Definable models of arithmetic

We first show that a definable structure whose domain is pointwise definable has an isomorphic copy in the corresponding hereditary inner model. We use this to exclude definable models with full binary standard system at $\aleph_1$ in ordinary Cohen and random extensions. We then construct definable saturated models at successors of singular strong limits.

12.1 Definition codes and hereditary classes

Write $\text{OD}_{\mathbb{R}}$ for definability from ordinals and one real, and $\text{HOD}_{\mathbb{R}}$ for its hereditary part. We use the same construction with parameters in H_λ , and occasionally with one additional fixed set a . In this subsection let

$$Z = \mathbb{R} \text{ or } H_\lambda, \quad O = \text{OD}_{a,Z}, \quad S = \{x : \text{tc}(\{x\}) \subseteq O\},$$

where λ is regular uncountable. The real-parameter assertions hold in ZF; for H_λ we work in ZFC. The fixed parameter a may be omitted and need not belong to S . Finite tuples of allowed parameters can be coded by one allowed parameter. The hereditary-model and copy lemmas also allow any nonempty $Z \subseteq \mathbb{R}$ definable from a and ordinals and closed under coding finite tuples of its members.

There is a uniform definable partial evaluation map

$$\text{Val}_a : Z \times \text{Ord} \rightarrow V \quad \text{with range } O. \tag{62}$$

The ordinal input codes a formula, ordinal parameters, and a rank segment containing a and the other parameters in which uniqueness is witnessed. Reflection supplies such a code for every definition; conversely, evaluating a code gives an allowed definition. Only satisfaction for set-sized ranks is used.

Lemma 12.1 (Hereditary definability). *S is a transitive inner model of ZF containing every member of Z . For $Z = H_\lambda$ and every infinite $\kappa < \lambda$, it contains $L(\mathcal{P}(\kappa)^V)$ and computes $\mathcal{P}(\kappa)$ correctly.*

Proof. Substitution of definitions makes O closed under definitions using finitely many of its members. The class S is definable from a and ordinals, transitive, contains all ordinals and all members of Z , and is closed under pairing and union. For Separation and Replacement, relativize the formula to S and form the required set in V . It belongs to O by substitution and has all its elements in S , so belongs to S . The same argument applies to $\mathcal{P}^V(x) \cap S$, giving the internal power set. The other ZF axioms are inherited, with $L \subseteq S$ supplying Infinity. In the H_λ case, every subset of κ belongs to S , as does their ordinal-definable collection $\mathcal{P}(\kappa)^V$. Minimality gives the stated constructible inner model. $\square$

Lemma 12.2 (The all-codes copy lemma). *Every structure in O , in a language indexed by ω , whose domain is pointwise in O has an externally isomorphic copy in S . No bound on the domain's cardinality is required.*

Proof. For each domain point d , take the least ξ such that $\text{Val}_a(z, \xi) = d$ for some $z \in Z$. Replacement bounds these ordinals by one δ . The set

$$C = \{(z, \xi) \in Z \times \delta : \text{Val}_a(z, \xi) \in \text{dom}(\mathcal{M})\}$$

maps onto the domain. Pull back equality, the relations, and the function graphs through Val_a . These sets, and their language-indexed collection, are in O and have hereditary entries in S , so belong to S . Their quotient by the pulled-back equality exists in S by ZF; evaluation induces the external isomorphism. The construction retains all codes below the ordinal bound and requires no choice of representatives. $\square$

Lemma 12.3 (Closure from ordinal covers). *Work in ZFC. Let either $Z = \mathbb{R}$ and $\mu = \omega$, or $Z = H_{\mu^+}$ for an infinite cardinal μ . Suppose that the range of every ordinal sequence of length at most μ is contained in the range of an ordinal-definable map $e : \mu \rightarrow \text{Ord}$. Then S is closed under ambient sequences of length at most μ of its members, and consequently satisfies DC_μ .*

Proof. Choose codes $x_i = \text{Val}_a(z_i, \xi_i)$ for such a sequence, and choose e covering the ξ_i . Put $j_i = \min\{j : e(j) = \xi_i\}$. The sequence of (z_i, j_i) is coded by one real in the first case and belongs to H_{μ^+} in the second. Together with an ordinal definition of e , it defines the whole sequence in O . All other sets in its transitive closure belong to S , so the sequence belongs to S . Sequences witnessing ambient dependent choice therefore also witness dependent choice in S . $\square$

The cover hypothesis holds in a cardinal-preserving μ^+ -cc extension of a ground with the stable ordinal codes of Lemma 2.2: antichains deciding the ordinal values give a ground cover of size at most μ , and a ground enumeration of that cover is ordinal-definable in the extension.

Corollary 12.4 (A countable ground-model copy). *Let a be a set of ordinals and let c be Cohen-generic over $L[a]$. In $V = L[a][c]$, every countable $\text{OD}(a)$ structure in a fixed countable language coded in $L[a]$ has an $\text{OD}(a)$ isomorphic copy on a finite ordinal or on ω whose code belongs to $L[a]$. In particular, for a countable $\text{OD}(a)$ model $\mathcal{M}$ of $I\Delta_0$,*

$$\text{SSy}(\mathcal{M}) \subseteq \mathcal{P}(\omega)^{L[a]}.$$

Proof. The proof applies to any original ground W_0 with the stable ordinal codes of Lemma 2.2, with $a \in W_0$ as a fixed parameter; $L[a]$ supplies the same codes from a . Write W for the ground. The empty-domain case is immediate. Otherwise Cohen homogeneity gives an invariant name for the domain. Apply Theorem 3.1 to obtain an invariant bijective enumeration. Applying Lemma 2.3 to the name for the entire enumeration gives an $\text{OD}(a)$ bijection from a finite ordinal or ω onto the domain. Pull back the structure along it. The atomic diagram is an $\text{OD}(a)$ real. Every Boolean value deciding a bit is fixed by the Cohen automorphism group, hence is zero or one, so the diagram belongs to W . Satisfaction in a set-sized first-order structure is absolute. Every real coded by an element of the pulled-back arithmetic model therefore belongs to W . $\square$

12.2 Canonical exponential cuts in bounded arithmetic

We use $I\Delta_0$ in the ordinary language of arithmetic. For each standard n , let $\beta_n(a)$ assert that the n th binary digit of a is one, using the numeral for 2^n :

$$\beta_n(a) \longleftrightarrow \exists u \leq a \exists v < \overline{2^n} (a = (2u + 1)\overline{2^n} + v).$$

For a set-sized model $M \models I\Delta_0$, put

$$\text{SSy}_{\text{bin}}(M) = \{\{n < \omega : M \models \beta_n(a)\} : a \in M\}.$$

This convention requires no total exponentiation in M . For models with exponentiation it agrees with the binary standard system used below. The metatheory of this subsection is ZF; standard natural numbers and satisfaction refer to that ambient universe.

Fix the Paris Δ_0 graph $E(x, y)$ for partial exponentiation $y = 2^x$. In $I\Delta_0$ it is functional, its domain is initial, and $E(x, y)$ implies $E(x + 1, 2y)$. The usual algebraic and order laws hold wherever the values exist [6, Section 2, Lemma 2.1]. For standard n define ordinary arithmetic formulas

$$T_0(x) := (x = x), \quad T_{n+1}(x) := \exists y [E(x, y) \wedge T_n(y)],$$

and define, externally,

$$C(M) = \{a \in M : M \models T_n(a) \text{ for every } n < \omega\}. \quad (63)$$

The restricted structure is uniformly definable from M alone, by the satisfaction relation for set-sized structures. Here an initial cut is allowed to be all of M .

Lemma 12.5. *For every $M \models I\Delta_0$, the structure $C(M)$ is an initial arithmetic substructure satisfying $I\Delta_0 + \text{Exp}$, or EFA . Its operations and partial-exponential graph are inherited from M .*

Proof. Every standard numeral lies in $C(M)$. Downward closure and monotonicity of partial exponentiation, iterated through each standard finite number of steps, show that $C(M)$ is initial. If $a \in C(M)$ and $E(a, b)$, functionality and $T_{n+1}(a)$ give $T_n(b)$ for every standard n . Thus exponentiation is total on $C(M)$ and has values there.

The elementary bound

$$E(m, e) \wedge m \geq 4 \implies m^2 \leq e$$

is provable in $I\Delta_0$. Indeed, with m, e fixed, all exponential values through m exist and are bounded by e . Bounded induction therefore applies to the assertion that $x^2 \leq y$ whenever $4 \leq x \leq m$, $y \leq e$, and $E(x, y)$; the successor step uses $(x+1)^2 \leq 2x^2$ for $x \geq 3$. For $a, b \in C(M)$, take $m = \max(a, b, 4)$ and $e = 2^m \in C(M)$. Both $a + b$ and ab are at most e , so initiality gives arithmetic closure, including successor.

An initial arithmetic substructure is Δ_0 -elementary. Bounded induction holds in $C(M)$: a counterexample there would have a least counterexample below it in M , and this element and its predecessor would still lie in the cut. The Δ_0 graph E is absolute as well. Hence $C(M) \models I\Delta_0 + \text{Exp}$. $\square$

Lemma 12.6. *If D is a nonstandard initial arithmetic substructure of $M \models I\Delta_0$ on which exponentiation is total with values in D , then $\text{SSy}_{\text{bin}}(D) = \text{SSy}_{\text{bin}}(M)$.*

Proof. Fix a nonstandard $c \in D$ and put $d = 2^c \in D$. For any $a \in M$, division with remainder gives $a = qd + r$ with $r < d$, so $r \in D$. For every standard n , the initial-domain property and $n + 1$ applications of the exponential recurrence show that 2^{n+1} divides d . Thus a and r have the same standard binary digits. Bounded-formula absoluteness gives both inclusions of standard systems. $\square$

Theorem 12.7. *There is a fixed recursive real G_{tow} such that, for every $M \models I\Delta_0$, if $G_{\text{tow}} \in \text{SSy}_{\text{bin}}(M)$, then $C(M)$ is nonstandard and*

$$\text{SSy}_{\text{bin}}(C(M)) = \text{SSy}_{\text{bin}}(M).$$

In particular this holds whenever M has the full ambient binary standard system. No saturation assumption is required.

Proof. In the standard natural numbers let $e_0 = 1$, $e_{i+1} = 2^{e_i}$, and use the injective polynomial pairing $\pi(i, y) = (i + y)^2 + i$. Set

$$G_{\text{tow}} = \{\pi(i, e_i) : i < \omega\}.$$

This set is recursive: for any input, test its finitely many possible paired coordinates and compute the required finite tower. Suppose that $a \in M$ codes this real. Its infinite trace makes a nonstandard. Introduce the fixed bounded formulas

$$\begin{aligned} B(k, a) &:\iff \exists p \leq a [E(k, p) \wedge \exists u \leq a \exists v < p (a = (2u + 1)p + v)], \\ F(i, y, a) &:\iff y \leq a \wedge B(\pi(i, y), a) \wedge \forall z < y \neg B(\pi(i, z), a). \end{aligned}$$

For standard k , functionality of E shows that $B(k, a)$ is equivalent to $\beta_k(a)$; if $a < 2^k$, both are false. For fixed i , at most one y satisfies $F(i, y, a)$. For standard i it is e_i : the marked position is standard, and every smaller candidate is standard and unmarked.

Let $\text{Good}(n, a)$ be the following single Δ_0 formula:

$$\begin{aligned} n \leq a \wedge F(0, 1, a) \\ \wedge \forall i \leq n \exists y \leq a [F(i, y, a) \wedge y \geq i + 1] \\ \wedge \forall i < n \forall y \leq a \forall z \leq a [(F(i, y, a) \wedge F(i + 1, z, a)) \longrightarrow E(y, z)]. \end{aligned}$$

Every standard n satisfies it. Bounded maximum in M gives a greatest $N \leq a$ satisfying it, and N is nonstandard. Put $j = \lfloor N/2 \rfloor$ and let b be the unique row value with $F(j, b, a)$. The explicit growth clause gives $b \geq j + 1$, so b is nonstandard. For every standard k , the rows $j, j + 1, \dots, j + k$ exist below N and satisfy successive E -relations. This standard finite list of relations implies $T_k(b)$. Therefore $b \in C(M)$. Apply Lemma 12.6. $\square$

The code for G_{tow} is used only to prove nonstandardness; it is not a parameter in the definition of $C(M)$. Thus this construction preserves definability from any specified parameters and does not increase the size of the domain. It also takes a definable family of such structures to a definable family of EFA structures without selecting a member of the original family. No saturation of $C(M)$ is asserted.

12.3 A common category environment for Cohen extensions

The same category argument works for ordinary and generalized Cohen forcing. Fix a ground $W_0 \models \text{ZFC} + \text{GA} + (V = \text{HOD})$, an infinite regular κ with $\kappa^{<\kappa} = \kappa$ in W_0 , and a ground cardinal $\Lambda > \kappa$. Let G be generic and put

$$V = W_0[G], \quad G \subseteq \text{Add}(\kappa, \Lambda)^{W_0}, \quad \lambda = \kappa^+, \quad \mathcal{S} = \text{HOD}_{H_\lambda^V}^V. \quad (64)$$

Write $\mathcal{D} = \text{OD}_{H_\lambda^V}^V$ for the corresponding nonhereditary class. Write $P_I^W = \text{Add}(\kappa, I)^W$, and P_1^W for one column. The forcing is $< \kappa$ -closed and κ^+ -cc, the latter by $\kappa^{<\kappa} = \kappa$ and the delta-system argument. Thus it preserves cardinals and adds no ordinal sequences of length below κ . These facts also hold over a one-column extension and require no regularity of Λ . The support calculation itself needs no stable-ground assumption.

Lemma 12.8. *Let $W_0 \models \text{ZFC}$, let κ be infinite regular with $\kappa^{<\kappa} = \kappa$ in W_0 , and let $V = W_0[G]$ for $\text{Add}(\kappa, \Lambda)^{W_0}$, where $\Lambda > \kappa$ is any ground cardinal. Put $\lambda = \kappa^+$.*

- (i) *If $\delta \leq \kappa$ and $f : \delta \rightarrow \text{Ord}$ belongs to V , its range is contained in a set $B \in W_0$ with $|B|^{W_0} \leq \kappa$.*
- (ii) *For every $z \in H_\lambda^V$ there is a ground set $E \subseteq \Lambda$ of ground cardinality at most κ such that $z \in W_0[G \restriction E]$.*

Proof. For (i), maximal antichains deciding each $f(i)$ have size at most κ . Their at most κ sets of possible values give the required ground cover.

For (ii), in V code the transitive closure of $\{z\}$ by a well-founded extensional relation on an ordinal at most κ , with a distinguished point. Ground pairing functions code this relation and the distinguished point by a subset of κ . A nice name for this subset uses at most κ antichains of size at most κ . Each condition mentions fewer than κ columns. Consequently the name is supported on a ground set E of at most κ columns. The code belongs to $W_0[G \restriction E]$. Well-foundedness is downward absolute, and the transitive collapse in this intermediate model is the same as in V . Thus z belongs to it as well. $\square$

Lemma 12.9. *In (64), $\mathcal{S}$ is a transitive inner model of $\text{ZF} + \text{DC}_\kappa$, closed under ambient sequences of length at most κ of its members, and*

$$L(\mathcal{P}(\kappa)^V) \subseteq \mathcal{S}, \quad \mathcal{P}(\kappa)^\mathcal{S} = \mathcal{P}(\kappa)^V.$$

Every hereditary-small parameter belongs to the extension generated by at most κ original columns. For $\kappa = \omega$, $\mathcal{S} = \text{HOD}_{\mathbb{R}}^V$.

Proof. Apply Lemmas 12.1 and 12.3 with $Z = H_{\kappa^+}$. The ordinal covers in Lemma 12.8(i) have ground enumerations with stable ordinal codes, so the closure hypothesis holds. Part (ii) gives the assertion about parameters. Finally, a pointed well-founded relation on ω codes any hereditarily countable set by a real; conversely every real is hereditarily countable. Hence the two parameter conventions define the same class at $\kappa = \omega$. $\square$

12.3.1 Good bases and common extensions

Definition 12.10. With the ground and extension fixed as above, a *good base* is a subset h of κ which is one-Cohen generic over W_0 and satisfies

$$V = W_0[h][H] \quad \text{for some } P_\Lambda^{W_0[h]} \text{-generic } H. \quad (65)$$

Let $D \subseteq \mathcal{P}(\kappa)^V$ be the set of good bases. Equality of universes means that every set is the evaluation of a $W_0[h]$ -name under H . Define this relation in V using its mantle predicate for W_0 . It is OD, and its hereditary-small entries put D in $\mathcal{S}$. In inner-model arguments, D always denotes this actual set. An original block on κ columns recodes in W_0 as a good base, since removing those columns leaves Λ columns. Consequently any at most κ members of H_λ^V can be captured in a good base by joining their codes.

Lemma 12.11 (The actual quotient over a real). *Let W_0 be any ground model of ZFC and let $V = W_0[G]$ for $\text{Add}(\omega, \Lambda)^{W_0}$, where Λ is uncountable. For every $z \in \mathbb{R}^V$, the actual extension $V/W_0[z]$ is an extension by a forcing equivalent to $\text{Add}(\omega, \Lambda)^{W_0[z]}$.*

Proof. A nice name for z uses a countable set $T \subseteq \Lambda$ of the original coordinates. Enlarge T to be countably infinite and let $\mathbb{A}$ be the complete Cohen algebra on T . Let $\mathbb{D} \subseteq \mathbb{A}$ be the complete subalgebra generated by the Boolean values of the bits of this name. The intermediate-model theorem identifies its generic extension with $W_0[z]$; see [16, 17]. Factor first through $\mathbb{D}$ and then through the quotient, using the quotient completion in the intermediate extension. The standard quotient factorization is also described in [10, 556F–556G].

We verify the needed density bound. Fix a countable dense subset $D \subseteq \mathbb{A}^+$ in W_0 , and let

$$\pi(b) = \bigwedge \{d \in \mathbb{D} : b \leq d\} \quad (b \in \mathbb{A})$$

be the projection onto $\mathbb{D}$. If H is the induced $\mathbb{D}$ -generic filter, the quotient image of b is positive exactly when $\pi(b) \in H$. Projection preserves arbitrary joins, so

$$\pi(b) = \bigvee \{\pi(d) : d \in D, d \leq b\}.$$

When $\pi(b) \in H$, a ground maximal antichain refining the displayed cover ensures that $\pi(d) \in H$ for some $d \in D$ below b . Thus the positive images of D are dense in the quotient and in its completion, taken in the intermediate extension.

A complete Boolean algebra with a countable dense subset has at most countably many atoms. The complement of their join, if nonzero, is atomless and has countable density; a countable dense Boolean subalgebra there is the countable atomless Boolean algebra, so its completion is the Cohen algebra. The quotient is consequently a disjoint sum of countably many atomic components and at most one Cohen component. The untouched Λ Cohen coordinates absorb each of these components: one additional Cohen real can be reindexed into the tail. A nonempty finite or countable disjoint sum of full Λ -Cohen algebras is again that algebra, by partitioning one coordinate into a maximal antichain of the required size and using homogeneity on its cones. This proves the assertion for the actual factorization of V . All the forcings involved are ccc and preserve cardinals. $\square$

Lemma 12.12. *Let $(h_i : i < \delta)$ be an ambient family of good bases, with $\delta \leq \kappa$, and let $a \subseteq \kappa$. There is an actual original coordinate model $N = W_0[G \restriction I]$, where $I \in W_0$ and $|I|^{W_0} = \kappa$, which contains a and all h_i , and satisfies*

$$N = W_0[h_i][k_i] \quad (i < \delta)$$

for a one-Cohen generic k_i over $W_0[h_i]$. A ground recoding h of $G \restriction I$ is itself good and has $N = W_0[h]$. The conclusion can capture any additional family of at most κ members of H_λ^V .

Proof. First let $\kappa = \omega$. Capture the sequence of h_i and all additional parameters in an original countable block I_0 , and adjoin one fresh original coordinate c . Put $N = W_0[G \restriction I_0][c]$. For each i , the proof of Lemma 12.11 gives countable density for the quotient of $W_0[G \restriction I_0]$ over $W_0[h_i]$. Its product with the fresh Cohen coordinate has countable density and is atomless, so its completion is a Cohen algebra. Thus N is an actual one-Cohen extension of every $W_0[h_i]$. The enlarged original block is countable and leaves Λ columns, so its recoding is good. This proves the countable case.

Now let $\kappa > \omega$. Write $W_i = W_0[h_i]$. In the ambient ZFC universe choose actual presentations $V = W_i[K_i]$ by $P_\Lambda^{W_i}$. The sequence (h_i) and a is coded by a subset of κ . The support bound gives a ground set I_0 of size κ capturing this code. The case $\delta = 0$ ends here.

Recursively choose increasing $I_n \in W_0$ and, for each $i < \delta$, increasing $J_{i,n} \in W_i$, all of size κ in their grounds. Put

$$N_n = W_0[G \restriction I_n], \quad N_{i,n} = W_i[K_i \restriction J_{i,n}].$$

Given I_n , capture a subset-of- κ code for $G \restriction I_n$ in the relative presentation over each W_i . Include the previous $J_{i,n-1}$ and pad to size κ . This gives $N_n \subseteq N_{i,n}$. Recode each relative block as a subset $k_{i,n}$ of κ . These δ subsets have one code, which can be captured in an original block I_{n+1} containing I_n . Because $h_i \in N_0$, we obtain

$$N_n \subseteq N_{i,n} \subseteq N_{n+1} \quad (i < \delta, n < \omega). \quad (66)$$

For the limit step, if transitive ZFC models $W \subseteq V$ have the same ordinals and no new countable ordinal sequences, every countable ambient sequence of members of W belongs to W : bound their ranks, well-order that rank segment in W , and code the entries by a countable ordinal

sequence. Consequently $(I_n)_{n<\omega} \in W_0$ and $(J_{i,n})_{n<\omega} \in W_i$. Put $I = \bigcup_n I_n$ and $J_i = \bigcup_n J_{i,n}$, and form

$$N = W_0[G \restriction I], \quad N_i = W_i[K_i \restriction J_i].$$

Their complementary coordinate sets have size Λ . Their remaining forcing is $< \kappa$ -closed, so both N and N_i are closed under countable ambient sequences of their own elements.

By (66), N_i contains each partial generic function $\bigcup(G \restriction I_n)$. It contains their countable sequence and its union, which reconstructs the full generic on $I \times \kappa$. Hence $N \subseteq N_i$. Conversely, N contains each relative function $\bigcup(K_i \restriction J_{i,n})$, its sequence, and its union; it also contains W_i . Thus $N_i \subseteq N$. We have $N = N_i$ for every i .

Recode the J_i -generic in W_i as one Cohen generic k_i , and recode the original I -generic in W_0 as h . The remaining Λ columns show that h is good. To include a family of hereditary-small parameters, first combine their relation codes into one subset of κ . $\square$

12.3.2 Good points on a ground Cohen tree

For $\kappa = \omega$ take $\mathcal{T} = 2^{<\omega}$ and $X = 2^\omega$. For $\kappa > \omega$, fix a ground tree isomorphism $\kappa^{<\kappa} \cong \mathcal{T}$ and put $X = [\mathcal{T}]^V$. In either case X is the full ambient branch space. Its basic cylinders are $[s]$, for $s \in \mathcal{T}$. A fixed ground indexing codes its nodes and branches by subsets of κ ; if the original nodes have higher rank, retain the ground decoding map as an ordinal-definable parameter. In particular $X \in \mathcal{S}$ and every ambient branch is present there.

The tree forcing is equivalent to P_1 . For $\kappa = \omega$ this is immediate. For uncountable κ , concatenate the words $1^\alpha 0$, $\alpha < \kappa$, to identify the successor-length nodes of $\kappa^{<\kappa}$ with a dense set of binary strings. Regularity keeps each short concatenation below length κ . This identifies the forcing presentations and their generic extensions.

For a good $W = W_0[h]$ and positive finite n , let

$$\begin{aligned} \Gamma_n(W) = \{ \vec{x} \in X^n : \vec{x} \text{ is } \mathcal{T}^n\text{-generic over } W, \text{ and} \\ V = W[\vec{x}][H] \text{ for a } P_\Lambda^{W[\vec{x}]}\text{-generic } H \}. \end{aligned} \tag{67}$$

All short nodes and partial-function conditions are unchanged in the relevant extensions. The relation $(h, \vec{x})$ in (67) is an ambient OD set with hereditary entries in $\mathcal{S}$, so it too belongs to $\mathcal{S}$. Inside $\mathcal{S}$ we use the relation formed in V , so no computation of the mantle inside $\mathcal{S}$ is required. Whenever forcing relations are taken, expand the original ambient definitions; Lemma 2.2 keeps their ground predicate equal to W_0 in each relevant ZFC extension.

Lemma 12.13. *The good points have the following properties.*

- (i) *If $W' = W[k]$ is a one-Cohen extension and both are good bases, then $\Gamma_n(W') \subseteq \Gamma_n(W)$.*
- (ii) *For positive finite r, s ,*

$$(\vec{x}, \vec{y}) \in \Gamma_{r+s}(W) \iff \vec{x} \in \Gamma_r(W) \text{ and } \vec{y} \in \Gamma_s(W[\vec{x}]).$$

The model $W[\vec{x}]$ on the right is a good base after a ground recoding as one Cohen extension of W_0 .

- (iii) *Every nonempty basic open subset of X^n meets $\Gamma_n(W)$.*

Proof. For (i), the pair $(k, \vec{x})$ is product generic over W . Its ground forcing factors are unchanged, so we may reverse their order. Over $W[\vec{x}]$, adding k and then forcing with the full tail is equivalent,

by reindexing, to forcing with a full Λ -column tail. This gives (i). For (ii), use the product forcing theorem. In the forward direction the finite $\vec{y}$ -factor can be absorbed into the remaining full tail. In the reverse direction the asserted joint genericity and actual tail are precisely (67). The forcing adding h and $\vec{x}$ over W_0 is equivalent to one Cohen column.

For (iii), factor an actual full-tail presentation over W into $\mathcal{T}^n$ and a full tail. A tree cone is isomorphic in the ground to the entire tree, so the distinguished generic can be placed in any prescribed product of cylinders while preserving the intermediate extension and its remaining tail. $\square$

Definition 12.14. For each positive finite n , define in $\mathcal{S}$

$$\mathcal{I}_n = \{E \in \mathcal{P}(X^n)^{\mathcal{S}} : E \cap \Gamma_n(W_0[h]) = \emptyset \text{ for some } h \in D\}.$$

We abbreviate $\mathcal{I}_1$ to $\mathcal{I}$.

Proposition 12.15. *The $\mathcal{I}_n$ are proper ideals in $\mathcal{S}$, closed under unions of at most κ members, even when the family is initially given in V . They contain every $\mathcal{S}$ -set covered by at most κ nowhere dense sets and contain no nonempty open set. They are invariant under permutations of the finite coordinates and under ground tree automorphisms.*

Proof. For a family $(E_i : i < \delta)$, $\delta \leq \kappa$, choose witnessing good bases in V . A common base from Lemma 12.12, together with Lemma 12.13(i), has good points avoiding every E_i . The union belongs to $\mathcal{S}$ by Lemma 12.9. This proves closure under these unions. Downward closure is immediate. Part (iii) of Lemma 12.13 proves positivity of nonempty basic open sets, hence properness and positivity of every nonempty open set.

A closed set is coded by a subset of the ground basis, which has size κ . Capture the codes of κ closed nowhere dense covers in one good base. Their nowhere denseness is absolute: every basic condition has a stronger one avoiding the coded set, and all these conditions are unchanged. Generic tuples over that base avoid the covers, proving the assertion for nowhere dense sets. Ground tree automorphisms and coordinate permutations preserve genericity and the generated intermediate extension. They therefore preserve the good points over each ground containing their codes, and hence preserve the ideals. $\square$

The same argument applies to any tree automorphism coded by a subset of κ : first capture its code in a common good base. In particular it applies to the ground XOR translations used below. Also $X^n \setminus \Gamma_n(W) \in \mathcal{I}_n$. Consequently, for every nonempty basic open U ,

$$U \cap \Gamma_n(W) \notin \mathcal{I}_n. \tag{68}$$

Otherwise, taking the union with that complement would put U in the ideal.

12.3.3 Regular-open traces and the product theorem

Theorem 12.16. *For every $E \subseteq X^n$ in $\mathcal{S}$ there are a good base W and a regular open set O , coded in W , such that*

$$E \cap \Gamma_n(W) = O \cap \Gamma_n(W).$$

In particular $E \triangle O \in \mathcal{I}_n$.

Proof. Capture the hereditary-small parameter defining E , and retain its ordinal parameters, in a good W . Expand every ambient class definition before taking forcing relations. The actual full tail over W is homogeneous. Since the definition is unique in V , its uniqueness and the relevant type of its value are forced by the greatest condition.

Factor the tail as $\mathcal{T}^n$ followed by a full tail. In the first forcing, take the Boolean value of the assertion that the top of the remaining tail forces the distinguished tuple to belong to the defined set. Represent it by the regular open set O in the completion of $\mathcal{T}^n$. Its code is a subset of a basis of size κ . Its interpretation in V is still regular open, because closure and interior of this coded open set are computed by compatibility and refinement among the unchanged basic conditions.

For each $\vec{x} \in \Gamma_n(W)$, the ambient universe V is a full tail extension of $W[\vec{x}]$. Homogeneity and the two forcing theorems therefore identify membership in the actual E with membership of $\vec{x}$ in O . The set where the two memberships differ misses $\Gamma_n(W)$ and therefore belongs to the ideal. $\square$

Corollary 12.17. *If $E \subseteq X^n$ belongs to $\mathcal{S}$ and $E \cap U$ is positive on a nonempty basic open set U , there is a nonempty basic $U' \subseteq U$ such that $U' \setminus E \in \mathcal{I}_n$. Every $\mathcal{S}$ -function with range of ambient size at most κ is constant modulo $\mathcal{I}_n$ on a smaller basic open subset of any given nonempty basic open domain.*

Proof. A regular-open trace of E must meet U , since otherwise $E \cap U$ would be small. Choose U' inside their intersection. For a function, its at most κ fibers cannot all be small on U , by Proposition 12.15; apply the first assertion to a positive fiber. An ambient enumeration of its small range belongs to $\mathcal{S}$ by Lemma 12.9. $\square$

Theorem 12.18 (Finite-product category theorem). *For $E \subseteq X^{r+s}$ in $\mathcal{S}$, where r, s are positive finite,*

$$E \in \mathcal{I}_{r+s} \iff \{\vec{x} \in X^r : E_{\vec{x}} \notin \mathcal{I}_s\} \in \mathcal{I}_r.$$

Proof. If E misses $\Gamma_{r+s}(W)$, then for every $\vec{x} \in \Gamma_r(W)$ its section misses $\Gamma_s(W[\vec{x}])$, by the product law. This is a good base, so the section is small. The set of positive sections consequently misses $\Gamma_r(W)$.

Conversely, a positive E has a nonempty regular-open trace. Choose a nonempty basic rectangle $U \times U'$ inside this trace. For $\vec{x} \in U \cap \Gamma_r(W)$, the product law gives

$$U' \cap \Gamma_s(W[\vec{x}]) \subseteq E_{\vec{x}}.$$

The left side is positive by (68). Thus the set of positive sections contains $U \cap \Gamma_r(W)$, itself positive. This proves the converse by contraposition. All sets of sections used here belong to $\mathcal{S}$ by separation from the ideals already constructed there. $\square$

12.3.4 Uniformization with arbitrary codomain

Theorem 12.19. *Suppose $Y, R \in \mathcal{S}$, $R \subseteq X \times Y$, and every section of R is nonempty. There is a partial function $F \in \mathcal{S}$ uniformizing R such that $X \setminus \text{dom}(F) \in \mathcal{I}$. More precisely, its domain contains $\Gamma_1(W')$ for some good base W' .*

Proof. Capture the definition parameters of Y, R in a good $W = W_0[h]$, retaining their ordinal parameters. Expand ambient class definitions before forcing. By homogeneity, the greatest condition forces their membership in $\mathcal{S}$ and the nonemptiness of all sections.

Present the tail as $\mathcal{T} \times P_\Lambda^W$. The evaluation (62), after coding its hereditary-small parameter by a subset of κ , has the form $\text{Val} : \mathcal{P}(\kappa) \times \text{Ord} \rightarrow \mathcal{D}$. The maximum principle gives names $\dot{r} \subseteq \kappa$ and $\dot{\xi} \in \text{Ord}$ such that the top forces

$$\text{Val}(\dot{r}, \dot{\xi}) \in Y, \quad R(\dot{x}, \text{Val}(\dot{r}, \dot{\xi})). \quad (69)$$

The κ^+ -cc bounds the possible values of $\dot{\xi}$ by a ground set of size at most κ . Enumerate it by $b : \kappa \rightarrow B$ and replace $\dot{\xi}$ by $b(\dot{j})$, taking the least suitable index. Nice names for $\dot{r}, \dot{j}$ use at most κ tail columns. Pass to the extension by their actual generic a , keeping the distinguished tree factor in the remaining forcing. The new base $W' = W[a]$ is good, since a nonempty captured block recodes as one Cohen column and Λ columns remain. This also holds for singular Λ .

Over W' , the retained names are $\mathcal{T}$ -names. Indexing the unchanged tree conditions by κ makes their nice-name reading tables hereditary of size at most κ , hence members of H_Λ^V . For trees with higher-rank nodes use the fixed ground decoding map. Evaluate the tables at the initial-segment filter of x , leaving invalid readings undefined, to obtain $r(x), j(x)$. These readings are correct at every W' -generic branch. The ordinal table $b \in W_0[h]$ is definable from h and ordinals by Lemma 2.2. Consequently

$$F(x) = \text{Val}^V(r(x), b(j(x)))$$

defines an $\text{OD}_{H_\Lambda}^V$ partial function when its value is retained precisely if it exists, belongs to the actual Y , and witnesses the actual R . Its arguments and values are hereditary members of that class, so $F \in \mathcal{S}$.

The Boolean value of (69) remains one after passing to the new base. For $x \in \Gamma_1(W')$, the first forcing theorem says that the residual full tail forces the retained reading to witness R . The ambient universe V is such a tail extension of $W'[x]$. Thus $x \in \text{dom}(F)$, proving the claim. $\square$

12.4 The ordinary Cohen case

Specialize the preceding construction to $V = W_0[G]$, $G \subseteq \text{Add}(\omega, \omega_1)^{W_0}$, and write $S = \text{HOD}_{\mathbb{R}}^V$ and $C = 2^\omega$. Taking $W_0 = L$ gives the stated version below. Addition on C is coordinatewise modulo two. For $M = W_0[z]$, let $\text{NC}_n(M)$ be the tuples in C^n that are not jointly Cohen generic over M ; equivalently, they lie in a meager Borel set coded in M . Define in S

$$\mathcal{I}_n = \{E \in \mathcal{P}(C^n)^S : E \subseteq \text{NC}_n(W_0[z]) \text{ for some real } z\}, \quad \mathcal{I} = \mathcal{I}_1. \quad (70)$$

The actual non-genericity relation is formed in V using the stable ground predicate and belongs to S ; no internal computation of the mantle is intended.

Theorem 12.20 (The Cohen category environment). *In the stated L -extension, the following hold, with (ii)–(v) interpreted in S .*

- (i) S is a transitive model of $\text{ZF} + \text{DC}$ and contains all the reals of V . In fact, it is closed under countable V -sequences of its elements.
- (ii) For each positive integer k , $\mathcal{I}_k$ is a proper σ -ideal containing every ordinary meager set in S and containing no nonmeager Borel set. It is invariant under translations of C^k and permutations of the coordinates.
- (iii) Every $X \subseteq C^k$ in S differs from a Borel set by a member of $\mathcal{I}_k$.

(iv) For positive integers k, l and $E \subseteq C^{k+l}$ in S ,

$$E \in \mathcal{I}_{k+l} \iff \{x \in C^k : E_x \notin \mathcal{I}_l\} \in \mathcal{I}_k. \quad (71)$$

In particular, if every section is in $\mathcal{I}_l$, then $E \in \mathcal{I}_{k+l}$.

(v) If $Y, R \in S$, $R \subseteq C \times Y$, and every section of R is nonempty, there is a function $F \in S$ uniformizing R on a domain $D \subseteq C$ with $C \setminus D \in \mathcal{I}$.

Proof. For a good base $W_0[h]$, its good points are exactly its actual Cohen-generic tuples in V : apply Lemma 12.11 to the full tail over that base and the real coding the tuple. Moreover, any real z is captured by an original countable block, whose recoding h is good. Thus $\text{NC}_n(W_0[z]) \subseteq \text{NC}_n(W_0[h])$. Conversely, every good h is a real. Consequently (70) is precisely the ideal of Definition 12.14.

Now (i) is Lemma 12.9; (ii)–(v) follow from Proposition 12.15 and Theorems 12.16, 12.18 and 12.19. A nonmeager Borel set cannot lie in $\mathcal{I}_n$: modulo a meager Borel set it contains a nonempty open set, and the ideals contain all meager sets and no nonempty open set. Translation by any real tuple is a coded homeomorphism preserving genericity over a base capturing that tuple, so the symmetry assertion applies. $\square$

We will also use the following trace description. If $E \subseteq C^m$ is defined from z and ordinals, then some regular open B coded in $W_0[z]$ satisfies

$$\vec{x} \text{ Cohen generic over } W_0[z] \implies (\vec{x} \in E \iff \vec{x} \in B). \quad (72)$$

Indeed, the proof of Theorem 12.16 only needs a full homogeneous tail over the base, which Lemma 12.11 supplies over $W_0[z]$ and over $W_0[z, \vec{x}]$. In particular $E \triangle B \subseteq \text{NC}_n(W_0[z])$.

Corollary 12.21 (Local decisions). *Let $A \subseteq C$ belong to S , and let U be a nonempty basic cylinder. If $A \cap U \notin \mathcal{I}$, there is a nonempty basic cylinder $W \subseteq U$ such that $W \setminus A \in \mathcal{I}$. Consequently every function in S with standard finite range is constant modulo $\mathcal{I}$ on some subcylinder of any prescribed nonempty basic cylinder.*

Proof. Apply Corollary 12.17 at $\kappa = \omega$. $\square$

12.5 The category Scott obstruction

Work in S with the notation of Theorem 12.20. A statement holds *on U modulo $\mathcal{I}$* when its exceptions in U belong to $\mathcal{I}$; a set is *conull* when its complement belongs to $\mathcal{I}$.

We use $\text{EFA} = Q + I\Delta_0 + \text{Exp}$ in its conservative definitional expansion by $x \mapsto 2^x$, with induction for bounded formulas in the exponential language; see [1, Section 1.1]. Binary digits are boundedly definable, for example by

$$\text{Bit}(a, m) \iff \exists u \leq a \exists v < 2^m \ a = (2u + 1)2^m + v.$$

For EFA, $\text{SSy}(\mathcal{M})$ means the family of binary-coded traces $\{n < \omega : \mathcal{M} \models \text{Bit}(a, \bar{n})\}$, as a ranges over M . No equivalence with traces of arbitrary unbounded formulas is required.

Theorem 12.22 (The Cohen Scott obstruction). *There is no model $\mathcal{M} \in S$ of $I\Delta_0$ such that*

$$\text{SSy}(\mathcal{M}) = \mathcal{P}(\omega)^S.$$

By Theorem 12.7, any such model has a canonically definable EFA initial substructure with the same binary standard system. It suffices to exclude that substructure. We therefore work with EFA models throughout the proof below; the metatheory is ZF. The proof uses only the preceding category environment, so we may work over its fixed stable ground W_0 . The arithmetic strategy follows [14]. Local uniformization and Kuratowski–Ulam turn pointwise arithmetic overspill into a bound on a nonempty open set. Periodic translations then give a countable coinital sequence in the nonstandard part. A repeated schedule of basic open sets produces infinitely many arithmetic blocks, from which one formula defines the standard cut.

12.5.1 Local overspill

A basic cylinder always means a nonempty set of the form $[s] = \{x \in 2^\omega : s \subseteq x\}$, where $s \in 2^{<\omega}$. We use Corollary 12.21: a positive subset of a cylinder is conull on a smaller cylinder, and a function with standard finite range is constant modulo $\mathcal{I}$ on some smaller cylinder.

Lemma 12.23 (Local overspill). *Let $\mathcal{M} \in S$ be a nonstandard model of EFA, and identify ω with its standard copy in $\mathcal{M}$. Suppose $P \subseteq C \times M$ belongs to S and, outside a set in $\mathcal{I}$, every x admits a nonstandard $h \in M$ such that*

$$\forall N < h \ P(x, N).$$

For every basic cylinder U there are a nonstandard $H \in M$ and a basic cylinder $W \subseteq U$ such that

$$\forall N < H \ P(x, N) \quad \text{holds on } W \text{ modulo } \mathcal{I}. \quad (73)$$

The same assertion holds for a space X with a basis of positive open sets and ideals $\mathcal{I}, \mathcal{I}_2$, provided conull uniformization, local decision, Kuratowski–Ulam, and invariance under transposition hold and nonempty basic rectangles are positive. One may replace $M \setminus \omega$ by any final segment A of positive arithmetic elements closed under $h \mapsto \lfloor h/2 \rfloor$.

Proof. Conull uniformization gives a function $h : C \rightarrow M \setminus \omega$ whose bound works outside a set in $\mathcal{I}$. To apply uniformization to a relation with all sections nonempty, use a fixed nonstandard default value where no bound exists; use the same default off the resulting conull domain.

For nonstandard $a \in M$, put $E_a = \{x : h(x) > a\}$. Suppose that $E_a \cap U \in \mathcal{I}$ for every such a , and consider

$$Q = \{(x, y) \in U^2 : h(y) > \lfloor h(x)/2 \rfloor\}.$$

For each $x \in U$, the element $\lfloor h(x)/2 \rfloor$ is nonstandard. Thus every vertical section of Q belongs to $\mathcal{I}$, and Kuratowski–Ulam gives $Q \in \mathcal{I}_2$. Coordinate-permutation invariance gives $Q^\top \in \mathcal{I}_2$ as well. But $Q \cup Q^\top = U^2$: of two positive integers in $\mathcal{M}$, at least one exceeds half the other. This contradicts the fact that the nonempty open rectangle U^2 does not belong to $\mathcal{I}_2$.

Consequently $E_a \cap U$ is positive for some nonstandard a . Local decision gives a basic $W \subseteq U$ on which $h(x) > a$ modulo $\mathcal{I}$. Discard also the exception to the chosen bound property, and take $H = a$. The proof uses only the stated properties and therefore proves the general form as well. If a suitable function h is already given, no uniformization assumption is needed. $\square$

To prove Theorem 12.22, suppose that $\mathcal{M} \in S$ is an EFA model with full standard system. It is necessarily nonstandard. Uniformize the coding relation to obtain a conull $X \subseteq C$ and a function $F : X \rightarrow M$ in S such that $F(x)$ codes x . Extend F to C by a fixed default value. Write $\text{Bit}(a, m)$ for the assertion that the m th binary digit of a is one, and put

$$\text{Ext}(x)(m) = 1 \iff \mathcal{M} \models \text{Bit}(F(x), m).$$

For every $x \in X$, all its standard bits are correct simultaneously:

$$\text{Ext}(x)(n) = x(n) \quad (n < \omega). \quad (74)$$

Here and below arithmetic positions such as n denote their images in $\mathcal{M}$.

For finite binary words s, t , with t nonempty, $s \smallfrown t^\omega$ is the ordinary eventually periodic real. The notation $s \smallfrown t^\infty$ denotes the corresponding arithmetically defined pattern on all positions of $\mathcal{M}$; it is not an assertion that an integer codes an infinite periodic string.

Lemma 12.24 (Local periodic overspill). *For every such s, t and every basic cylinder U , there are a nonstandard $H \in M$ and a basic $W \subseteq U$ such that*

$$\text{Ext}(x + s \smallfrown t^\omega)(m) = \text{Ext}(x)(m) \oplus (s \smallfrown t^\infty)(m) \quad (m < H) \quad (75)$$

holds on W modulo $\mathcal{I}$. Addition of reals is Cantor-group addition, and $\oplus$ on the right is addition of bits modulo two.

Proof. Discard the two coding exceptions $C \setminus X$ and its translate by $s \smallfrown t^\omega$. They belong to $\mathcal{I}$ by translation invariance. For each remaining x , the displayed identity holds at every standard position by equation (74). For this fixed x , the identity is a bounded formula of the exponential language, using only the two integer codes and the finite periodic pattern as parameters. Division by the positive standard length of t defines the periodic part by a bounded formula. Choose any nonstandard cutoff. Either the identity holds below it, or bounded minimization gives its least failure below that cutoff. This least failure is nonstandard, and the identity holds below it. Thus bounded induction alone gives the required pointwise nonstandard bound. Apply Lemma 12.23 to make one bound work on a subcylinder modulo $\mathcal{I}$. $\square$

12.5.2 A countable coinital sequence

Let

$$a_n = 0 \smallfrown 1^n \smallfrown 0^\omega, \quad a_\infty = 0 \smallfrown 1^\omega.$$

For $n < \omega$, define $u_n(x)$ to be the least $m > n + 1$ such that $\text{Ext}(x)(m) \neq \text{Ext}(x + a_n)(m)$, if such an m exists, and put $u_n(x) = \infty$ otherwise. The least witness is obtained by bounded minimization: both bits vanish above $F(x) + F(x + a_n)$, so it suffices to search below $F(x) + F(x + a_n) + 1$. The symbol ∞ is a formal value larger than every element of M . For each basic cylinder U , set

$$I_{n,U} = \{b \in M \setminus \omega : \{x \in U : u_n(x) \geq b\} \in \mathcal{I}\}. \quad (76)$$

These are final segments of the nonstandard part.

Lemma 12.25. *Every $I_{n,U}$ is proper, and*

$$\bigcup_{n < \omega, U \text{ basic}} I_{n,U} = M \setminus \omega. \quad (77)$$

Consequently S contains a strictly decreasing coinital sequence $N_0 > N_1 > \dots$ in $M \setminus \omega$.

Proof. Apply Lemma 12.24 for a_n inside U . On a basic $W \subseteq U$, modulo $\mathcal{I}$, the two strings agree throughout $(n + 1, H)$ for some nonstandard H . Thus $u_n \geq H$ there. Since W is positive, $\{x \in U : u_n(x) \geq H\} \notin \mathcal{I}$. This proves $H \notin I_{n,U}$ and hence properness.

Suppose equation (77) fails, and take a nonstandard d outside the union. Apply Lemma 12.24 for a_∞ to obtain a basic U and nonstandard H on which

$$\text{Ext}(x + a_\infty)(m) \neq \text{Ext}(x)(m) \quad (0 < m < H)$$

modulo $\mathcal{I}$. Choose a nonstandard m with $m + 1 < \min(d, H)$. By local decision, refine U to a basic cylinder W on which $f(x) = \text{Ext}(x + a_\infty)(m)$ is constant modulo $\mathcal{I}$.

The translations $t_n = a_n + a_\infty$ tend to zero. For every sufficiently large standard n , t_n vanishes on the prefix defining W , so $W + t_n = W$. Outside the constancy exception and its translate, both of which belong to $\mathcal{I}$, we have

$$\text{Ext}(x + a_n)(m) = f(x + t_n) = f(x) = \text{Ext}(x + a_\infty)(m) \quad (x \in W).$$

This bit differs from $\text{Ext}(x)(m)$ modulo $\mathcal{I}$ on W . Since m is nonstandard, $m > n + 1$, and therefore $u_n(x) \leq m$ on W modulo $\mathcal{I}$. It follows that $m + 1 \in I_{n,W}$, whence $d \in I_{n,W}$ by finality, a contradiction.

There are only countably many pairs (n, U) . Enumerate their proper final segments as $\langle J_i : i < \omega \rangle$, and use countable choice in S to choose $c_i \in (M \setminus \omega) \setminus J_i$. The sequence $\langle c_i : i < \omega \rangle$ is coinital: if $d \in J_i$, finality and $c_i \notin J_i$ give $c_i < d$. Define

$$N_0 = \lfloor c_0/2 \rfloor, \quad N_{i+1} = \left\lfloor \frac{\min(N_i, c_{i+1})}{2} \right\rfloor. \quad (78)$$

All these elements are nonstandard and the sequence strictly decreases. It is coinital because $N_i \leq c_i$. The use of countable choice is justified by DC in S . $\square$

12.5.3 Infinitely many successful blocks

The proof below requires only a real-coded nonincreasing coinital sequence. We use a strictly decreasing one for convenience. Fix the sequence from Lemma 12.25 and a sequence $\langle U_n : n < \omega \rangle$ of basic cylinders in which every basic cylinder occurs infinitely often. Put

$$w_n = 0 \frown 1^n, \quad p_n = n + 1, \quad L_n = 2p_n. \quad (79)$$

The internal periodic pattern w_n^∞ has exactly two zeros in each interval of length L_n , separated by p_n .

Lemma 12.26. *There are a real $s_\omega \in S$, strictly increasing indices $r_n \geq n$, and $x \in C$ with $x, x + s_\omega \in X$ such that, writing $M_n = N_{r_n}$, the following hold. The successive gaps in the ordinary zero set of s_ω are nondecreasing and unbounded. For infinitely many standard n , the strings $\text{Ext}(x)$ and $\text{Ext}(x + s_\omega)$ agree at exactly two positions in $[M_n, M_n + L_n)$, separated by $n + 1$.*

Proof. Construct finite words s_n , beginning with $s_0 = \emptyset$. Given s_n , let $z_n = s_n \frown w_n^\omega$. Local periodic overspill inside U_n gives a basic $W \subseteq U_n$ and a nonstandard H_n on which the comparison equation (75) for z_n holds below H_n modulo $\mathcal{I}$. By coinitality choose $r_n \geq n$, larger than all previous indices, such that

$$M_n = N_{r_n}, \quad M_n + L_n < H_n. \quad (80)$$

This is possible because $H_n - L_n$ is nonstandard.

The map

$$G_n(y) = \langle \text{Ext}(y + z_n)(M_n + j) : j < L_n \rangle$$

has standard finite range. Refine W to a basic cylinder V_n on which G_n has one constant value modulo $\mathcal{I}$. Let ℓ_n be the length of the prefix defining V_n . Choose an integer $q_n \geq 1$ with $|s_n| + q_n p_n \geq \ell_n$, and set

$$s_{n+1} = s_n \frown w_n^{q_n}, \quad s_\omega = \bigcup_{n < \omega} s_n. \quad (81)$$

Dependent choice carries out this recursion in S . Since $q_n \geq 1$, the union is a real, and

$$s_\omega \upharpoonright \ell_n = z_n \upharpoonright \ell_n. \quad (82)$$

Put $t_n = s_\omega + z_n$. By equation (82), t_n fixes V_n under translation. Let $E_n \subseteq V_n$ be the exception to the chosen constant value of G_n . Then $E_n \in \mathcal{I}$, and for $y \in V_n \setminus (E_n \cup (E_n + t_n))$,

$$\langle \text{Ext}(y + s_\omega)(M_n + j) : j < L_n \rangle = G_n(y + t_n) = G_n(y). \quad (83)$$

Combining this equality with the periodic-overspill comparison on W , we obtain, on V_n modulo $\mathcal{I}$,

$$\text{Ext}(y + s_\omega)(M_n + j) = \text{Ext}(y)(M_n + j) \oplus (s_n \frown w_n^\infty)(M_n + j) \quad (j < L_n). \quad (84)$$

The position M_n is nonstandard and hence lies beyond the standard finite prefix s_n . Thus the agreement positions in this interval are exactly the two zeros of the periodic pattern, at distance $p_n = n + 1$.

For every standard k , the set

$$D_k = \bigcup_{n \geq k} V_n$$

is dense open. Indeed, every basic cylinder occurs as U_n for some $n \geq k$, and contains that stage's nonempty V_n . Consequently $\bigcap_k D_k$ is ordinary comeager. For each n , the failure set of equation (84) inside V_n belongs to $\mathcal{I}$: it is contained in the two translated constancy exceptions and the periodic comparison exception. Remove their countable union, together with the two coding exceptions for y and $y + s_\omega$. The complement of $\bigcap_k D_k$ is meager and also belongs to $\mathcal{I}$. Since $\mathcal{I}$ is proper and closed under countable unions, there is some x in $\bigcap_k D_k$ outside all these exceptional sets.

This x belongs to infinitely many V_n , so infinitely many blocks satisfy the required comparison. Finally, equation (81) shows that the gaps between successive zeros of s_ω are $1, 2, 3, \dots$, each repeated a positive finite number of times. Within stage n the gaps are $n + 1$, and the gap to the first zero of the next stage is also $n + 1$. $\square$

12.5.4 Defining the standard cut

Proof of Theorem 12.22. Assume that $\mathcal{M} \in S$ has full standard system, and carry out the preceding constructions. Take s_ω, x from Lemma 12.26, and let $p = F(x)$ and $q = F(x + s_\omega)$. Define the arithmetic set

$$B = \{b \in M : \mathcal{M} \models \text{Bit}(p, b) \leftrightarrow \text{Bit}(q, b)\}. \quad (85)$$

Define the bounded formula

$$\text{Next}_B(u, v) \iff u < v \wedge B(u) \wedge B(v) \wedge \forall z < v (u < z \rightarrow \neg B(z)).$$

It says that u, v are consecutive agreement positions. Consider

$$\begin{aligned} C(d) \iff \forall u, v, r, s < d [& \text{Next}_B(u, v) \wedge \text{Next}_B(r, s) \\ & \wedge v \leq r \rightarrow v + r \leq s + u]. \end{aligned} \quad (86)$$

This compares every earlier consecutive gap with every later one, using addition instead of subtraction. After expanding B and Next_B , we obtain a bounded formula with parameters p, q . At standard positions, equation (74) identifies B with the ordinary zero set of s_ω . Its gaps are nondecreasing, so $C(d)$ holds for every standard d .

Let d be nonstandard. Since $M_n = N_{r_n}$ is decreasing and cointial, eventually $M_n < \lfloor d/2 \rfloor$. Every standard L_n is also less than $\lfloor d/2 \rfloor$. Choose a sufficiently late successful block; then $M_n + L_n < d$. Its two agreement positions $r < s$ are nonstandard, consecutive in B , and satisfy $s - r = n + 1$. The ordinary zero gaps of s_ω are unbounded, so choose standard consecutive agreement positions $u < v$ with $v - u > n + 1$. All four positions lie below d , and $v < r$. They violate the inequality in equation (86). Hence $C(d)$ fails for every nonstandard d .

We have proved externally that $C^{\mathcal{M}} = \omega$. In particular, $\mathcal{M} \models C(0)$ and $\mathcal{M} \models \forall d (C(d) \rightarrow C(d+1))$. Bounded induction for C , with parameters p, q , yields $\mathcal{M} \models \forall d C(d)$, contradicting nonstandardness. $\square$

12.6 A Borel obstruction for a finite fragment

A *coded Borel quotient presentation* consists of a Borel domain $P \subseteq \omega^\omega$, a Borel equivalence relation $\approx$ on P , and Borel operations and relations on representatives which respect $\approx$ and induce a first-order structure on $P/\approx$. Constants are given by distinguished representatives. Ordinary Borel presentations are the special case in which $\approx$ is equality. All Borel sets and maps in this subsection are given by real codes.

The proof uses the following facts about real codes. We state their metatheoretic strength because it is part of the theorem.

Lemma 12.27 (Coded category toolkit). *The following uniform statements are provable in $\Pi_1^1\text{-CA}_0$.*

- (i) *If a coded Borel relation $R \subseteq 2^\omega \times \omega^\omega$ has nonempty sections, there are codes for a comeager Borel set $X \subseteq 2^\omega$ and a Borel map on X which uniformizes R .*
- (ii) *From a sequence of coanalytic codes one can uniformly obtain Baire-property data. In particular, if their union contains a comeager set, then in every nonempty basic cylinder one member of the sequence is comeager on a smaller basic cylinder.*
- (iii) *For every uniformly nonempty sequence of Σ_1^1 relations, one real codes a sequence of witnesses. Ordinary countable recursions using uniform code operations and least natural-number choices likewise have single real codes.*

Proof. Borel evaluation and the usual operations on Borel codes are available already in ATR_0 . For (i), use the coded Jankov–von Neumann construction. For each basic open set in the range, replace the corresponding selector preimage by an open set modulo a coded meager set. Deleting the union of these countably many errors leaves a coded comeager set on which the selector is Borel. It may in fact be taken continuous relative to its domain. The same uniform Baire-property construction for coanalytic codes proves (ii). If every member were meager in a given cylinder, the coded union of their meager hulls would make that cylinder meager. The first assertion of (iii) is Σ_1^1 choice, which is provable already in ATR_0 [29, Theorem V.8.3]; the recursion assertion is the usual arithmetical recursion on the joined codes. These are the coded forms of separation, uniformization, and the Baire-property theorems developed in [29, Sections V.3 and VI.2]. $\square$

Theorem 12.28 (Finite-fragment Borel obstruction). *There are a fixed bounded formula $\text{Bin}(a, i)$ and a fixed finite subtheory F_{Bor} of the usual axiomatization of $I\Delta_0$ such that $\Pi_1^1\text{-CA}_0$ proves that no Borel quotient presentation of an F_{Bor} -model has every real as a Bin-trace. In every $I\Delta_0$ -model the Bin-traces are exactly the binary standard system. Consequently the same theory proves that no Borel quotient presentation of an $I\Delta_0$ -model has full binary standard system.*

Proof. We first prove the obstruction, identifying each arithmetic axiom as it is used. We then collect these axioms into a finite subtheory. Fix the partial-exponentiation graph E from section 12.2. For definiteness, put

$$\pi(u, v) = (u + v)^2 + u, \quad \rho(i, c, d) = 1 + \pi(i, \pi(c, d)).$$

The discretely ordered semiring axioms prove that π , and hence ρ , is injective and that $\rho(i, c, d) > i, c, d$. Put

$$\text{Rem}(c, d, i, r) \iff r < 1 + (i + 1)d \wedge \exists q \leq c [c = q(1 + (i + 1)d) + r], \quad (\text{Rem})$$

$$\text{BBit}(c, d, i) \iff d > 0 \wedge \text{Rem}(c, d, i, 1). \quad (\text{BBit})$$

Division with unique remainder makes this a two-valued decoder. The bounded formula

$$\delta(a, m) \iff m = 0 \vee \exists q \leq a \exists r < m [a = qm + r] \quad (87)$$

has an induction instance which proves existence of quotient and remainder; uniqueness follows from the discretely ordered semiring axioms. We first convert the binary standard-system hypothesis to this decoder without assuming total exponentiation.

For a finite binary word σ of length n , put $d_\sigma = n!$ (with $0! = 1$). The moduli $1 + (i + 1)d_\sigma$, $i < n$, are pairwise coprime. Indeed, a common divisor for the i th and j th moduli divides $(j - i)d_\sigma$, is coprime to d_σ , and therefore divides $j - i$; since $j - i$ divides d_σ , the i th modulus is congruent to 1 modulo $j - i$. Thus the common divisor is 1. The Chinese remainder theorem supplies c_σ whose remainder at the i th modulus is $\sigma(i)$. Choose the least such c_σ below the product of the moduli. This is a bounded search, so $\sigma \mapsto (c_\sigma, d_\sigma)$ is primitive recursive.

Define the fixed bounded formula for binary coding by

$$\text{Bin}(a, i) \iff i < a \wedge \exists p \leq a [E(i, p) \wedge \exists u \leq a \exists v < p a = (2u + 1)p + v]. \quad (88)$$

Functionality, the base value, and the successor law for E imply in $I\Delta_0$ that, for every standard i , this is equivalent to $\beta_i(a)$. The weaker fragment below requires no property of E , since its fullness hypothesis is stated directly in terms of Bin-traces. For $x \in 2^\omega$ let

$$Y_x = \{\rho(n, c_{x \upharpoonright n}, d_{x \upharpoonright n}) : n < \omega\}.$$

Fullness for Bin gives y whose standard Bin-trace is Y_x . Since the displayed support is unbounded and $\text{Bin}(y, i)$ includes $i < y$, the element y is nonstandard.

Write

$$\text{Mark}_y(i, c, d) \iff d > 0 \wedge \rho(i, c, d) < y \wedge \text{Bin}(y, \rho(i, c, d)),$$

$$\text{Row}_y(i, c, d) \iff \text{Mark}_y(i, c, d) \wedge \forall c', d' < y [\rho(i, c', d') < \rho(i, c, d) \longrightarrow \neg \text{Mark}_y(i, c', d')].$$

Thus Row chooses the least marked record in a row. Let $\text{GoodRow}(n, y)$ be the bounded formula asserting that $n < y$, that every row $i \leq n$ has a row pair $c, d < y$, and that whenever $i < j \leq n$, their selected pairs have the same BBit values at every $k < i$. Explicitly, its last clause is

$$\forall i < j \leq n \forall c, d, c', d' < y [\text{Row}_y(i, c, d) \wedge \text{Row}_y(j, c', d') \longrightarrow \forall k < i (\text{BBit}(c, d, k) \leftrightarrow \text{BBit}(c', d', k))].$$

This predicate is downward closed and holds for every standard n . For a standard row the intended record is standard, and every smaller record is standard, so the Bin-trace of y identifies the intended pair. Put

$$\Theta_{\text{row}}(z, y) \iff \forall n < z \text{ GoodRow}(n, y).$$

But $\text{GoodRow}(y, y)$ fails. The induction instance for Θ_{row} gives a least $b \leq y$ at which GoodRow fails. The element b is nonstandard, and its predecessor N is a nonstandard good row. If (c, d) is the pair in row N , comparison with the standard row $k + 1$ gives

$$\text{BBit}(c, d, \bar{k}) \iff x(k) = 1 \quad (k < \omega). \quad (89)$$

Thus fullness for Bin implies that every real has a remainder code (c, d) . In particular the model is nonstandard. Fix one nonstandard element for the least-failure arguments below.

Now let $P/\approx$ be the presented model. On representatives, BBit is analytic, since the quotient witness q in (Rem) is existentially quantified. Its complement is analytic as well. The part with $d = 0$ is Borel, and on $d > 0$ division supplies q, r with

$$r < 1 + (i + 1)d, \quad c \approx q(1 + (i + 1)d) + r, \quad r \not\approx \bar{1},$$

and uniqueness makes this equivalent to failure of BBit . Analytic separation therefore gives a Borel code for BBit , uniformly from the presentation. Consequently

$$R(x, c, d) \iff d > 0 \wedge \forall n < \omega [\text{BBit}(c, d, \bar{n}) \leftrightarrow x(n) = 1]$$

is Borel and has nonempty sections by (89). Jankov–von Neumann uniformization and Baire regularization give a comeager Borel set $X \subseteq 2^\omega$ and a Borel map $F : X \rightarrow P^2$ uniformizing R . Extend F by a fixed value off X , and set

$$\text{Ext}(x)(m) = 1 \iff \text{BBit}(F(x), m).$$

This relation is Borel in (x, m) , and $\text{Ext}(x)(n) = x(n)$ for $x \in X$ and standard n .

The set of representatives of nonstandard elements,

$$P_{\text{ns}} = \bigcap_{n < \omega} \{a \in P : \bar{n} < a\},$$

is Borel, and the reverse of the arithmetic order is a Borel linear quasiorder on it. We use the following coded consequence of the Harrington–Marker–Shelah theorem. In $\Pi_1^1\text{-CA}_0$, every real-coded Borel linear quasiorder has a real-coded countable cofinal sequence. Indeed, Marcone’s formalization of the representation theorem gives, already in ATR_0 , a coded countable wellorder W and a Borel strong order-preserving map

$$f : P_{\text{ns}} \longrightarrow (2^W, \leq_{\text{lex}})$$

[25, Theorem 4.2 and Corollary 4.9]; see also the original theorem [13, Theorem 3.1]. The coded lexicographic argument is as follows. Follow the lexicographically highest extendible branch b . If $b = f(x)$ for some x , then x is cofinal. Otherwise let λ be the first limit initial segment at which extendibility fails, and fix a coded increasing sequence (λ_n) cofinal in λ . For every n , the Borel set

$$E_n = \{x \in P_{\text{ns}} : f(x) \upharpoonright \lambda_n = b \upharpoonright \lambda_n\}$$

is nonempty. By Σ_1^1 choice [29, Theorem V.8.3], one real codes a sequence $x_n \in E_n$. The first-difference argument shows that $(f(x_n))$ is cofinal in the image, and strong order preservation shows that (x_n) is cofinal in P_{ns} . Applying this lemma to the reverse order gives a real-coded coinital family $(A_i)_{i < \omega}$ in the nonstandard part. Put

$$Q_n = \min(A_0, \dots, A_n),$$

retaining the first representative which attains the minimum. Then (Q_n) is a nonincreasing coinital sequence, obtained by finite comparisons and without choosing new representatives.

We next prove a Borel version of Lemma 12.23. If $a = s \smallfrown t^\omega$ is eventually periodic, with $t \neq \emptyset$, choose standard remainder codes (c_s, d_s) and (c_t, d_t) for the prefix and period, and put $\alpha = (c_s, d_s, |s|, c_t, d_t, |t|)$. The fixed bounded formula

$$\begin{aligned} \text{Per}(m; \alpha) \iff & [m < |s| \wedge \text{BBit}(c_s, d_s, m)] \\ & \vee [|s| \leq m \wedge \exists q \leq m \exists r < |t| (m = |s| + q|t| + r \wedge \text{BBit}(c_t, d_t, r))] \end{aligned}$$

defines the corresponding internal periodic predicate $a^*(m)$. Division supplies a unique residue modulo the fixed positive standard number $|t|$. Thus Per is Borel on representatives, since this predicate and its complement are analytic by the argument used for BBit . If P, Q are two remainder-code pairs, let

$$\Psi(m; P, Q, \alpha) \iff [\text{BBit}(P, m) \leftrightarrow \text{BBit}(Q, m)] \leftrightarrow \neg \text{Per}(m; \alpha),$$

where $\text{BBit}((c, d), m)$ abbreviates $\text{BBit}(c, d, m)$, and set

$$\Theta_{\text{per}}(z; P, Q, \alpha) \iff \forall m < z \Psi(m; P, Q, \alpha).$$

For $x \in X \cap (X + a)$, standard correctness gives

$$\text{Ext}(x + a)(m) = \text{Ext}(x)(m) \oplus a^*(m) \quad (m < \omega). \quad (90)$$

The induction instance for Θ_{per} gives, for each x , a nonstandard bound below which the identity holds. Indeed, below a fixed nonstandard cutoff, either there is no failure or there is a least one, which must be nonstandard. Hence the coanalytic sets

$$D_i^a = \{x \in X \cap (X + a) : (90) \text{ holds for all } m < Q_i\}$$

cover the comeager set $X \cap (X + a)$. In every nonempty basic cylinder U , some $D_i^a \cap U$ is nonmeager. The Baire property for coanalytic sets therefore gives a nonempty basic $W \subseteq U$ on which the identity below Q_i holds comeagerly. This gives the required local common bound without selecting overspill bounds.

We can now repeat the block construction in the proof of Lemma 12.26, with the meager ideal in place of $\mathcal{I}$ and (Q_i) in place of (N_i) . At stage n take

$$w_n = 0 \smallfrown 1^n, \quad p_n = n + 1, \quad L_n = 2p_n,$$

and apply the preceding paragraph to $z_n = s_n \smallfrown w_n^\omega$ inside the n th cylinder of a schedule which repeats every basic cylinder infinitely often. It gives a subcylinder W_n and a nonstandard $H_n = Q_i$ on which the periodic comparison holds comeagerly. Choose an increasing $r_n \geq n$ such that, for $M_n = Q_{r_n}$,

$$M_n + L_n < H_n.$$

The finite-valued Borel map

$$x \mapsto \langle \text{Ext}(x + z_n)(M_n + j) : j < L_n \rangle$$

is constant on a comeager part of a smaller basic cylinder V_n . Extend s_n by a positive finite number of copies of w_n far enough to cover the prefix defining V_n . Exactly as in (81)–(84), the union s_ω of these prefixes and the Baire category theorem supply an x such that $x, x + s_\omega \in X$ and unboundedly many stages are successful. At every successful stage the two internal strings agree at exactly two positions in $[M_n, M_n + L_n)$, and these positions are consecutive and separated by $n + 1$. The successful block positions are coinital. Once $Q_k < \lfloor e/2 \rfloor$, every sufficiently late M_n is below Q_k , while the standard L_n is below the nonstandard element $\lfloor e/2 \rfloor$.

Write $F(x) = (c, d)$ and $F(x + s_\omega) = (c', d')$, and define the fixed bounded predicate

$$B(m) \iff (\text{BBit}(c, d, m) \leftrightarrow \text{BBit}(c', d', m)).$$

Use the formula Next_B and the four-endpoint gap formula $C(e)$ from (86). At standard positions B is the zero set of s_ω , whose successive gaps are nondecreasing and unbounded, so $C(e)$ holds for every standard e . If e is nonstandard, choose a sufficiently late successful block below e . Its consecutive agreement points have gap $n + 1$. A standard construction stage $k > n$ supplies an earlier consecutive standard gap $k + 1 > n + 1$, and all four endpoints are below e . Thus $C(e)$ fails. The extension of C is exactly the standard cut, contradicting the one induction instance for this fixed bounded formula.

The proof uses only finitely many first-order axioms. In the standard presentation

$$I\Delta_0 = \text{PA}^- + \{\text{Ind}(\varphi) : \varphi \text{ is bounded}\},$$

where PA^- is the fixed finite theory of discretely ordered commutative semirings, take F_{Bor} to be PA^- together with exactly four universally closed induction instances: induction in a for $\delta(a, m)$ from (87), induction in z for $\Theta_{\text{row}}(z, y)$, induction in z for $\Theta_{\text{per}}(z; P, Q, \alpha)$, and induction in e for $C(e; P, Q)$. Division uniqueness, the polynomial-record facts, and the elementary arithmetic of standard finite intervals are already provable in PA^- . Thus F_{Bor} is a finite subset of the axioms of $I\Delta_0$. For any other conventional finite base, take the union of the finitely many $I\Delta_0$ axioms occurring in fixed proofs of the required PA^- facts together with the four displayed induction instances. Hence the conclusion that a finite fragment suffices is independent of the choice of finite base.

All the descriptive-set-theoretic objects in the proof have uniform real codes. Analytic separation gives the Borel code for BBit ; parts (i) and (ii) of Lemma 12.27 yield the codes for X, F and the local common bounds; the Harrington–Marker–Shelah representation followed by the coded lexicographic lemma yields the coinital sequence; and part (iii) codes the countable block recursion and all of its choices and error sets. The representation theorem is formalizable in ATR_0 by Marcone [25]. Since satisfaction of the fixed finite theory F_{Bor} expands as a finite formula about the presentation code, the whole assertion is one sentence of second-order arithmetic, with no third-order satisfaction predicate. $\square$

Remark 12.29. The forcing-theoretic transfer arguments above use the canonical exponential cut. In Theorem 12.28, we instead obtain the coinital sequence from the Borel order itself. Partial exponentiation occurs only in the fixed bounded trace formula used to produce remainder codes.

12.7 The obstruction at $\aleph_1$

Continue with $V = L[G]$, where G adds ω_1 Cohen reals, and $S = \text{HOD}_{\mathbb{R}}^V$.

Theorem 12.30 (No small definable model with full standard system). *In V there is no $\text{OD}_{\mathbb{R}}$ model $\mathcal{M}$ of $I\Delta_0$ with $|\mathcal{M}| \leq \aleph_1$ and $\text{SSy}(\mathcal{M}) = \mathcal{P}(\omega)^V$.*

Proof. The domain is an $\text{OD}_{\mathbb{R}}$ set of size at most $\aleph_1$. A sequence of fewer than $\kappa = \omega_1$ reals is coded by one real, so Theorem 8.14 implies that every domain element is $\text{OD}_{\mathbb{R}}$. By Lemma 12.2, an isomorphic structure $\widehat{\mathcal{M}}$ belongs to S . Absoluteness of satisfaction makes it a model of $I\Delta_0$ there.

The models S and V have the same reals, because each real is hereditarily definable from itself. Every such real is coded in $\mathcal{M}$, hence by the corresponding element of $\widehat{\mathcal{M}}$. This element belongs to S , and the bit coding relation is absolute. Thus S regards $\widehat{\mathcal{M}}$ as having full standard system, contrary to Theorem 12.22. $\square$

Corollary 12.31 (Absence of definable saturated copies). *For every complete consistent $T \supseteq I\Delta_0$, its saturated model of cardinality $\aleph_1$ exists in V and has no $\text{OD}_{\mathbb{R}}$ copy there. In particular it has no OD copy.*

Proof. The extension satisfies CH, so $(\aleph_1)^{\aleph_0} = \aleph_1$. Standard saturation theory gives a saturated model of T of cardinality $\aleph_1$, unique up to isomorphism. For every real $x \subseteq \omega$, the type

$$\{\beta_n(v) : n \in x\} \cup \{\neg\beta_n(v) : n \notin x\},$$

where $\beta_n(v)$ tests the residue modulo the standard integer 2^{n+1} , is finitely satisfiable by standard integers. Saturation realizes it, so the model has full standard system. Apply Theorem 12.30. $\square$

12.8 The arithmetic obstruction after ω_1 random reals

For the random analogue we use an additional parameter. The members of a small definable family are definable from reals and one fixed orbit of generic filters, although they need not be individually real-ordinal-definable. The hereditary class defined using this orbit has the probability and uniformization properties needed below.

Theorem 12.32. *Let $V = L[F]$, where F adds ω_1 random reals. There is no $\text{OD}_{\mathbb{R}}$ model $\mathcal{M}$ of $I\Delta_0$, on any domain, such that*

$$|\mathcal{M}| \leq \aleph_1, \quad \text{SSy}(\mathcal{M}) = \mathcal{P}(\omega)^V.$$

Here the standard system is defined by binary coding. The same conclusion holds with L replaced by any $W \models \text{ZFC} + \text{CH} + \text{GA} + (V = \text{HOD})$.

We prove the stated generalization. Write $\mathbb{B} = \mathbb{B}_{\omega_1}^W$ for the probability algebra, μ for its probability, and $P = \text{Aut}^W(\mathbb{B}, \mu)$. Fix a ground ordinal coding of $\mathbb{B}$ and, in $V = W[F]$, put

$$a = \{\text{code}(h[F]) : h \in P\}, \quad O_a = \text{OD}_{a, \mathbb{R}}^V, \quad S_a = \{x : \text{tc}(\{x\}) \subseteq O_a\}. \quad (91)$$

The parameter a is fixed throughout; it need not belong to S_a . Its canonical name is fixed by every member of P .

12.8.1 Small-index supports and descent to the orbit

Lemma 12.33 (Local patching). *Let $D \subseteq \mathbb{B}$ be a separable complete probability subalgebra. If $h \in P$ and $d > 0$ satisfy*

$$h(d \wedge c) = h(d) \wedge c \quad (c \in D), \quad (92)$$

there is $k \in P_{(D)}$ agreeing with h on the entire principal algebra $\mathbb{B} \restriction d$.

Proof. For every $c \in D$, the two sides of equation (92) have equal measure. Consequently

$$c \wedge \neg d \mapsto c \wedge \neg h(d)$$

is a well-defined measure-preserving isomorphism of the restricted D -algebras. Unless $d = 1$, their ambient complementary principal algebras have the same positive total measure and relative type ω_1 over these separable bases. After normalizing the measures, [9, 333C(b)] extends the displayed map to the whole complements. Paste this extension with $h \restriction (\mathbb{B} \restriction d)$. The result fixes D . If $d = 1$, the hypothesis already says that h fixes D . $\square$

Lemma 12.34 (Descent to one orbit). *Every member of every $\text{OD}_{\mathbb{R}}$ family of size at most ω_1 in V belongs to O_a .*

Proof. First suppose that a ground name τ is fixed modulo forced equality by every member of P fixing a countable coordinate factor $\mathbb{B}_T$. We recover τ^F from a , the real coding $F \cap \mathbb{B}_T$, and ground ordinal codes. Evaluate τ at all filters coded by a whose restriction to $\mathbb{B}_T$ is this same filter. The value at F occurs. For another such filter $h[F]$, equality of the two restricted filters holds below some $d \in F$; thus

$$d \wedge h^{-1}(c) = d \wedge c \quad (c \in \mathbb{B}_T).$$

After applying h , this is equation (92). Choose $k \in P_{(\mathbb{B}_T)}$ agreeing with h on $\mathbb{B} \restriction d$. Then $k[F] = h[F]$, whereas invariance of τ gives $\tau^{k[F]} = \tau^F$. The proposed evaluation therefore has one value. Stable ground codes, supplied by Lemma 2.2, make it a definition in O_a . Local patching is needed because agreement of two actual generic traces does not imply that h fixes $\mathbb{B}_T$ globally.

Now let A be the given family. Capture its real definition parameter and a condition forcing the definition, nonemptiness, and the size bound on a countable ground coordinate set T_0 . Below this condition use the named family, and elsewhere use a fixed singleton. The resulting everywhere nonempty family name is invariant under $P_{(\mathbb{B}_{T_0})}$. The set of classes of its member names modulo equality forced by 1 has ground cardinality at most ω_1 . To see this, choose a forced enumeration by ω_1 . Every member name is a countable Boolean mixture of entries of this enumeration, and there are at most

$$(\omega_1 \cdot |\mathbb{B}|)^{\aleph_0} = \omega_1$$

such mixtures. The indicated group acts on this ordinary ground set. Each point stabilizer has index at most ω_1 , so Lemma 9.19, with constants for the finite-cylinder algebra on T_0 , gives a countable factor $\mathbb{B}_T$ whose fixer fixes that member name. Apply the first paragraph to each member. $\square$

12.8.2 The hereditary model and its probability

Put $C = 2^\omega$, and use addition modulo two on C . All definitions in equation (91) are made in the ambient model V .

Lemma 12.35. *S_a is a transitive inner model of $\text{ZF} + \text{DC}$, contains every real of V , and is closed under countable V -sequences of its elements. Also $W[z] \subseteq S_a$ for every real $z \in V$. An O_a structure with pointwise O_a domain has an isomorphic copy belonging to S_a .*

Proof. Apply Lemmas 12.1 and 12.3 with $Z = \mathbb{R}$ and the fixed anchor a . The closure hypothesis follows from ccc and stable ordinal codes for ground enumerations. For every real z , the intermediate model $W[z]$ is a set-forcing extension of W ; its objects have definitions from z and ground ordinal codes, by Lemma 2.2. Its transitivity therefore gives $W[z] \subseteq \text{HOD}_{\mathbb{R}}^V \subseteq S_a$. The copy assertion is Lemma 12.2, which does not require $a \in S_a$. $\square$

The next lemma allows us to obtain traces at every random real from the traces at coordinate generics.

Lemma 12.36 (Realizing a random real locally). *In a homogeneous random extension of uncountable type over any ZFC ground U , let r_β be a fixed coordinate real. Every real x in the extension that is random over U is the actual value of $h r_\beta$ for some measure-preserving ground automorphism h supported on countably many coordinates. The same holds for finite tuples of jointly random reals.*

Proof. Read x as a ground measurable function f of countably many coordinates, and let ρ be its distribution. Write the absolutely continuous part as $g dm$, and choose a ground m -null Borel set Z carrying the singular part. Since x is random over U , it avoids Z and the null exception to finiteness of g . For some positive integer N the event $c_0 = \{f \notin Z, g(f) \leq N\}$ belongs to the actual generic. The marginal of f restricted to c_0 is bounded by Nm . Split c_0 into N equal slices using a fresh independent uniform variable, and select the slice c in the generic. Increasing N first if necessary ensures $\mu(c) < 1$. Then

$$\eta = f_*(\mu|_c) \leq m.$$

On $\neg c$, use a second independent uniform variable to sample $(m - \eta)/(1 - \mu(c))$. Pasting this sample with f on c gives a ground random variable y of distribution exactly m whose actual value is x .

Take a countable coordinate factor containing all coordinates used, β , and a further independent atomless variable. Conditional on y , this factor is atomless because the last variable remains independent. The standard relative product theorem identifies the factor with the product of the y -factor and a standard atomless factor, and also with the product of the r_β -factor and its complement; see Lemma 5.4 and [4]. The resulting isomorphism sends r_β to y . Extend it by the identity on the other coordinates. The slice containing the actual generic was selected externally, but every candidate map belongs to the ground. For finite tuples, use the standard finite product probability as the target marginal in the same construction. $\square$

Lemma 12.37 (Universal Borel traces). *For each $X \subseteq C^n$ belonging to O_a , there are a countable coordinate intermediate ground $U = W[F_T]$ and a Borel set B coded in U such that*

$$\vec{x} \text{ jointly random over } U \implies (\vec{x} \in X \iff \vec{x} \in B).$$

The ground U can be enlarged to include any prescribed real parameter.

Proof. Capture the real definition parameters and a suitable forcing condition on a countable coordinate set T . Every new measure-preserving tail automorphism over U fixes the interpreted

anchor. To prove this, take a name for the map, extend it by the identity off a condition, and lift it to the ground iteration while fixing $\mathbb{B}_T$. Preservation of conditional probability, followed by integration over $\mathbb{B}_T$, makes the lift measure-preserving. It therefore fixes the canonical name of a . This is also the scalar case of Lemma 9.11.

Choose n fresh coordinate reals $\vec{r}$. The Boolean value of $\vec{r} \in X$ is fixed by every measure-preserving map on the remaining coordinates, so belongs to the factor generated by $\vec{r}$. Indeed, conditional on that factor, the remaining product has no nonconstant event invariant under all its finite flips. Represent this Boolean value by a Borel set B over U . For any actual jointly random tuple $\vec{x}$ over U , Lemma 12.36 supplies a relative measure-preserving map taking $\vec{r}$ to $\vec{x}$ at the generic. This map fixes a and the Borel code. Applying it to the Boolean identity proves the asserted equivalence. The argument works after any further countable capture. $\square$

Proposition 12.38 (The probability environment). *Inside S_a , every subset of C has a countably additive, translation-invariant probability extending the usual Borel probability. There are compatible finite-product probabilities satisfying Fubini, and every set differs from a Borel set by a null set. Every relation $R \subseteq C \times Y$ in S_a with nonempty sections has, in S_a , a uniformizing function on a conull subset of C .*

Proof. Let $\text{NR}_n(W[z])$ denote the tuples of reals in V that are not jointly random over $W[z]$. Define in V

$$\mathcal{I}_n = \{X \in \mathcal{P}(C^n)^{S_a} : \exists z \in C \ X \subseteq \text{NR}_n(W[z])\}.$$

Joining a countable sequence of parameters into one real, using Lemma 12.35, shows that this is a σ -ideal. It contains the usual null Borel sets and contains no positive Borel set. For the latter assertion, capture a Borel code and z in a countable coordinate model. Among countably many fresh independent tuples, some belongs to the positive Borel set, and every such tuple is jointly random over that model. By Lemma 12.37, every $X \in \mathcal{P}(C^n)^{S_a}$ differs from a Borel set B by a member of $\mathcal{I}_n$. Thus

$$\nu_n(X) = m^n(B) \quad \text{whenever } X \triangle B \in \mathcal{I}_n \tag{93}$$

is well defined. Its null ideal is exactly $\mathcal{I}_n$, since a null Borel trace can be included in the exceptional ideal by adjoining its code. Simultaneous traces over a joined real parameter give countable additivity. Including a translation parameter in the trace ground gives translation invariance.

We must check that the ideals and measure graphs belong to S_a . Interpret $W[z]$ in V using stable ground codes. Membership in $W[z]$ is uniform in z . Range over ground names for reals and the complete subalgebras generated by their bit values. Ground Borel operations recover from z the generic trace that gives it as a value. Each resulting intermediate model is $W[z]$ by the intermediate model theorem. The ground names are quantified over, so no further parameter depending on z is introduced. Consequently $\mathcal{I}_n$ and the graph in equation (93) belong to O_a . Their entries and all their transitive constituents belong to O_a , since $X \in S_a$ and the values of ν_n are reals. Hence both graphs belong to S_a .

For Fubini, choose a joint trace B of $X \subseteq C^{k+l}$ over a countable coordinate ground U . If x is random over U and y is random over $U[x]$, the pair is jointly random over U . The section X_x therefore differs from B_x only on $\text{NR}_l(U[x])$. Its section measure agrees with the Borel section measure for ν_k -almost every x . Ordinary Borel Fubini gives Fubini for the probabilities ν_n . Here $U[x] = W[z, x]$ for a real code z of the captured generic. All these assertions hold internally in S_a , since its countable sequences are actual sequences and it has all the ambient reals. The same trace

argument shows that finite-valued functions are approximable in measure by functions of finitely many bits, and that

$$t_j \longrightarrow 0 \implies \nu_1(X \triangle (X + t_j)) \longrightarrow 0.$$

For uniformization use the witness-code construction in Theorem 12.19, now with $\text{Val}_a(z, \xi)$ from (62). Capture the definitions and a condition forcing nonempty sections on countably many coordinates, and choose a fresh coordinate β . The maximum principle selects a real name and an ordinal name coding a witness for r_β . The real name has countable support; ccc replaces the ordinal name by an integer index in a countable ground ordinal table. Pass to the intermediate model containing their coordinates other than β . Over this countable coordinate ground U_1 , the remaining names have Borel readings. Evaluate those readings and keep a value precisely when it belongs to the actual Y and witnesses the actual R . The reading codes and the stable ordinal-table code make this an O_a partial function, and its graph is hereditary, hence in S_a . Its failure set has zero Boolean value at r_β . Apply the proof of Lemma 12.37 over this same U_1 , with β as the distinguished coordinate. Local realization transports the zero failure identity to every real random over U_1 , since the relative measure-preserving maps fix the anchor and the reading codes. The failure set is therefore contained in $\text{NR}_1(U_1)$, hence belongs to $\mathcal{I}_1$. Thus the domain is conull. $\square$

The measures just constructed are extensions on all the subsets belonging to S_a . Their null sets need not be null for ordinary ambient outer measure.

12.8.3 The probability Scott argument

We now prove the probability counterpart of Theorem 12.22. The probability argument supplies arithmetic blocks to which the same final bounded formula applies.

Lemma 12.39 (The probability Scott obstruction). *Suppose a transitive model of $\text{ZF} + \text{DC}$ has the probability and uniformization properties of Proposition 12.38. It has no $I\Delta_0$ model with full binary standard system.*

Proof. Work in this model and write $\nu = \nu_1$. If a counterexample exists, Theorem 12.7 gives a canonical nonstandard EFA cut with the same binary standard system. Write M for this cut. Uniformization gives a function $F : C \rightarrow M$ whose value codes x on a conull set X . Extend it by a fixed value off its original domain. Put

$$\text{Ext}(x)(m) = 1 \iff M \models \text{Bit}(F(x), m).$$

For $x \in X$, all standard bits are correct simultaneously. The arithmetic operations and bounded overspill used below are exactly those justified in the proof of Lemma 12.24.

We first need a measure form of overspill. Suppose that for almost every x there is a nonstandard h with $P(x, N)$ for all $N < h$. Uniformize these bounds, taking nonstandard defaults on the exceptional set, to obtain $h(x)$. Choose k with $1/(k+1) < \varepsilon$. Among $k+1$ independent samples, the events that a specified sample has the unique strict minimum h -value are disjoint and equimeasurable. Hence

$$\nu_{k+1}\{h(y) < \min_{i < k} h(x_i)\} \leq \frac{1}{k+1}.$$

Fubini supplies a fixed tuple for which the corresponding section has measure less than ε . For $H = \lfloor \min_{i < k} h(x_i)/2 \rfloor$, which is nonstandard, the bound $\forall N < H \ P(y, N)$ therefore holds with probability greater than $1 - \varepsilon$.

Apply this observation to bounded overspill of the correct standard-bit identities. For finite words s, t with $t \neq \emptyset$, there is a nonstandard H such that, with probability greater than $1 - \varepsilon$,

$$\text{Ext}(x + s \frown t^\omega)(m) = \text{Ext}(x)(m) \oplus (s \frown t^\omega)(m) \quad (m < H). \quad (94)$$

The internal periodic pattern on the right is boundedly defined; no integer coding an infinite internal string is asserted.

We next obtain a countable coinital sequence in $M \setminus \omega$. Put $a_n = 0 \frown 1^n \frown 0^\omega$ and $a_\infty = 0 \frown 1^\omega$. Let $u_n(x)$ be the least $m > n + 1$ at which the codes of x and $x + a_n$ disagree, or a formal ∞ if no such position exists. A least witness, when present, is found by bounded minimization below $F(x) + F(x + a_n) + 1$. For fixed $0 < p < 1$, put

$$I_{n,p} = \{d \in M \setminus \omega : \nu\{x : u_n(x) < d\} > p\}.$$

These are final segments. Each is proper. Apply equation (94) to a_n with error less than p to find a nonstandard H with $\nu(u_n < H) < p$. They cover the nonstandard part as n varies. Otherwise take nonstandard d outside their union. Apply equation (94) to a_∞ with $\delta < (1 - p)/3$, and choose nonstandard $m + 1 < \min(d, H)$. With probability greater than $1 - \delta$, the codes of x and $x + a_\infty$ disagree at m . Translation continuity, applied to the set where the latter bit is one, gives, for large standard n ,

$$\nu\{x : \text{Ext}(x + a_n)(m) = \text{Ext}(x + a_\infty)(m)\} > 1 - \delta.$$

Thus $\nu(u_n < m + 1) > 1 - 2\delta > p$, contradicting the choice of d .

Take $p = 1/2$. By countable choice choose a nonstandard $c_n \notin I_{n,1/2}$. The sequence is coinital, since $d \in I_{n,1/2}$ implies $c_n < d$ by finality. Successively halving finite minima, as in equation (78), gives a strictly decreasing coinital sequence $N_0 > N_1 > \dots$ of nonstandard elements.

We now construct successful blocks with summable error. Put $w_n = 0 \frown 1^n$, $p_n = n + 1$, $L_n = 2p_n$, and $\varepsilon_n = 2^{-n-5}$. Starting with the empty word s_0 , apply equation (94) to $z_n = s_n \frown w_n^\omega$ with error ε_n . Choose increasing indices $r_n \geq n$ so that

$$M_n = N_{r_n}, \quad M_n + L_n < H_n.$$

Finite-coordinate approximation gives $\ell_n < \omega$ and a function $f_n : 2^{\ell_n} \rightarrow 2^{L_n}$ such that

$$\nu\{x : \langle \text{Ext}(x)(M_n + j) : j < L_n \rangle = f_n(x \upharpoonright \ell_n)\} > 1 - \varepsilon_n.$$

Choose $q_n \geq 1$ with $|s_n| + q_n p_n \geq \ell_n$, and let $s_{n+1} = s_n \frown w_n^{q_n}$. Dependent choice carries out the recursion; its union s_ω is a real, with $s_\omega \upharpoonright \ell_n = z_n \upharpoonright \ell_n$. Translation invariance and the two approximation errors imply, outside a set of measure at most $3\varepsilon_n$,

$$\text{Ext}(x + s_\omega)(M_n + j) = \text{Ext}(x)(M_n + j) \oplus (s_n \frown w_n^\omega)(M_n + j) \quad (j < L_n). \quad (95)$$

Indeed the block read from $x + s_\omega$ equals the block read from $x + z_n$ whenever both finite-coordinate approximations succeed, and equation (94) compares the latter block with that of x . Since $\sum_n 3\varepsilon_n < \infty$, countable additivity makes equation (95) hold eventually for almost every x . Choose such an x with $x, x + s_\omega \in X$.

The standard agreement positions of $F(x)$ and $F(x + s_\omega)$ are the zeros of s_ω . Their consecutive gaps are nondecreasing and unbounded. The gap $n + 1$ occurs a positive finite number of times at stage n . For all sufficiently large n , equation (95) gives exactly two agreement positions in $[M_n, M_n + L_n)$, separated by $n + 1$. The bounded-cut argument at the end of the proof of Theorem 12.22 now applies. Formula equation (86), with parameters $F(x), F(x + s_\omega)$, holds exactly at standard d , contradicting bounded induction. $\square$

Proof of Theorem 12.32. Suppose $\mathcal{M}$ is a counterexample. By Lemma 12.34, every element of its domain belongs to O_a . The structure itself is O_a , so Lemma 12.35 gives an isomorphic copy $\widehat{\mathcal{M}} \in S_a$. Satisfaction for this structure and its standard-index binary coding are absolute. Since S_a and V have the same reals, S_a regards the copy as an $I\Delta_0$ model with full binary standard system. This contradicts Proposition 12.38 and Lemma 12.39. $\square$

Corollary 12.40. *In the extension of Theorem 12.32, every complete consistent extension of $I\Delta_0$ has a saturated model of cardinality $\aleph_1$, and none of these models has an $\text{OD}_{\mathbb{R}}$ copy. In particular the saturated elementary extension of $\mathbb{N}$ of cardinality $\aleph_1$ has no $\text{OD}_{\mathbb{R}}$ presentation on any domain.*

Proof. CH is preserved, so the usual construction gives a saturated model of each such theory of cardinality $\aleph_1$. Saturation realizes the countable type prescribing the standard binary digits of any real, since each finite part is realized by a standard integer. The binary standard system is therefore full, and Theorem 12.32 applies. $\square$

12.9 Definable saturation at successors of singular strong limits

At a singular strong limit κ , a regular ultrafilter can be defined from data of hereditary size κ . Apply the finite-support ultrapower construction of Kanovei and Shelah [20] to all codes for these data. This gives the following theorem in ZFC.

Theorem 12.41. *For every singular strong limit cardinal κ , there is an OD κ^+ -saturated elementary extension of $\mathbb{N}$ of cardinality exactly 2^κ . The construction is uniform from κ .*

Proof. Put $\mu = \text{cf}(\kappa)$ and $J = [\kappa]^{<\omega}$. Choose an increasing cofinal sequence of infinite cardinals $(\kappa_i : i < \mu)$ below κ , a uniform ultrafilter D_0 on μ , and, for each i , an ultrafilter U_i on $J_i = [\kappa_i]^{<\omega}$ containing every cone $\{s \in J_i : \xi \in s\}$, $\xi < \kappa_i$. These ultrafilters exist by the finite-intersection property. Call a tuple q of this form *admissible*, and define

$$X \in U_q \iff \{i < \mu : X \cap J_i \in U_i\} \in D_0 \quad (X \subseteq J).$$

Restriction to the J_i 's respects intersections and complements, so U_q is an ultrafilter on the full power set of J . For every $\xi < \kappa$, the cone $R_\xi = \{s \in J : \xi \in s\}$ belongs to U_q , since $\xi < \kappa_i$ eventually and D_0 contains all final segments. Each $s \in J$ belongs to only finitely many R_ξ 's, so these cones regularize every U_q .

Since κ is strong limit, each admissible tuple has hereditary size at most κ :

$$|\text{tc}(\{q\})| \leq \kappa + \sum_{i < \mu} 2^{\kappa_i} + 2^\mu = \kappa.$$

Fix an ordinal coding of pointed well-founded extensional relations on ordinals at most κ by subsets of κ . Let $C \subseteq \mathcal{P}(\kappa)$ be the set of codes whose distinguished points collapse to admissible tuples. Admissibility is tested in the ambient universe, including the ultrafilter assertions on the full power sets of the J_i 's. Thus C is nonempty and OD from κ , and $|C| \leq 2^\kappa$. Decoding assigns a regular ultrafilter U_c to each $c \in C$. The lexicographic order of characteristic functions linearly orders C definably; no choice of a particular tuple is made.

For finite $u = \{c_0 < \dots < c_{n-1}\} \subseteq C$, take the Fubini ultrafilter D_u on J^u in this order:

$$X \in D_u \iff U_{c_{n-1}} s_{n-1} \cdots U_{c_0} s_0 ((s_0, \dots, s_{n-1}) \in X).$$

Here $U_c s \varphi(s)$ means $\{s : \varphi(s)\} \in U_c$. These products cohere when unused coordinates are added, and therefore define an ultrafilter D on the algebra of finitely supported subsets of J^C . For a structure N , let F_N be the set of all finitely supported functions $J^C \rightarrow N$ and put $T(N) = F_N/D$.

The truth set of a formula on finitely supported functions has finite support. For the existential step of Łoś's theorem, choose witnesses as a function of the coordinates in the union of the parameter supports. The witness function still has finite support. Thus constant functions give an elementary embedding $N \rightarrow T(N)$.

Moreover $T(N)$ realizes every finitely satisfiable type over the diagonal N with at most κ formulas. Write such a type as

$$p(y) = \{\varphi_\xi(y, \bar{m}_\xi) : \xi < \nu\}, \quad \nu \leq \kappa.$$

For each $s \in J$, choose $n_s \in N$ realizing the finite part indexed by $s \cap \nu$. Fix a coordinate $c \in C$ and put $f(z) = n_{z(c)}$. Its class realizes p , because the truth set for the ξ -th formula contains the D -large cylinder $z(c) \in R_\xi$. The choice of this coordinate and these witnesses is needed only to verify realization. The operator T uses all finitely supported functions.

Starting with $N_0 = \mathbb{N}$, iterate this fixed operation for κ^+ stages, taking canonical direct limits at limit stages. Quotients, embeddings, and direct limits are uniformly definable, so the final model N_{κ^+} is OD from κ . Identify stages with their images. Every parameter set of size below κ^+ is contained in some N_α , $\alpha < \kappa^+$, by regularity. A type over it has at most κ formulas. By elementarity each finite part is realized in N_α , and the one-step property realizes the whole type in $N_{\alpha+1}$. Thus N_{κ^+} is κ^+ -saturated.

Put $\Lambda = 2^\kappa$. If $|N| \leq \Lambda$, then

$$|F_N| \leq |[C]^{<\omega}| \cdot |N|^\kappa \leq \Lambda \cdot \Lambda^\kappa = \Lambda.$$

The direct limits through $\kappa^+ \leq \Lambda$ stages satisfy the same bound. For the reverse inequality, fix a coordinate c . For $X \subseteq \kappa$ and $s = \{\xi_0 < \dots < \xi_{n-1}\} \in J$, put

$$f_X(s) = \sum_{\substack{j < n \\ \xi_j \in X}} 2^j.$$

If $X \neq Y$ and $\xi \in X \triangle Y$, then $f_X(s) \neq f_Y(s)$ for every $s \in R_\xi \in U_c$. The functions $z \mapsto f_X(z(c))$, each supported on one coordinate, give 2^κ distinct elements already in $T(\mathbb{N})$. Elementary embeddings preserve their distinctness, proving $|N_{\kappa^+}| = 2^\kappa$. Definably relabel the diagonal images of standard integers by those integers and tag all other elements. The result is an elementary extension that contains $\mathbb{N}$ itself. $\square$

Corollary 12.42. *If κ is singular strong limit and $2^\kappa = \kappa^+$, there is an OD saturated elementary extension of $\mathbb{N}$ of cardinality κ^+ . In particular, GCH gives such a model at the successor of every singular cardinal, including $\aleph_{\omega+1}$.*

Proof. Apply Theorem 12.41. Under GCH every singular cardinal is strong limit and satisfies $2^\kappa = \kappa^+$. $\square$

The model in Corollary 12.42 is unique up to isomorphism among saturated models of $\text{Th}(\mathbb{N})$ of that cardinality. The structure is ordinal-definable, but its domain need not be pointwise OD.

13 The relative Feferman–Levy model

We use the finite-stage filtration from [15] and include the original ground in the relative construction. We identify the resulting model with the class defined by ambient hereditary definability. Finite collapse descent then proves countable choice for finite sets and shows that every infinite set is Dedekind-infinite.

13.1 The model and its finite stages

Begin with $W_0 = L$, and put

$$\kappa_i = \aleph_i^{W_0}, \quad P_i = \text{Fn}(\omega, \kappa_i, <\omega), \quad P = \prod_{i < \omega}^{\text{fin}} P_i.$$

Let G be P -generic and work in the ambient ZFC universe $V = W_0[G]$. Define

$$N_k = W_0[G \restriction k], \quad R_k = \mathcal{P}(\omega)^{N_k}, \quad \mathcal{A} = \langle R_k : k < \omega \rangle, \quad \mathcal{R} = \bigcup_{k < \omega} R_k.$$

Write

$$M_0 = W_0(\mathcal{R}, \mathcal{A}) \tag{96}$$

for the least transitive inner model of ZF containing W_0 and the sets $\mathcal{R}, \mathcal{A}$. We will identify it with

$$\begin{aligned} \mathcal{D}_{\text{FL}} &= \{x : \exists r \in \mathcal{R} \ x \in \text{OD}_{\mathcal{A}, r}^V\}, \\ S_{\text{FL}} &= \{x : \text{tc}(\{x\}) \subseteq \mathcal{D}_{\text{FL}}\}. \end{aligned} \tag{97}$$

The term *relative Feferman–Levy model* refers throughout to M_0 . All definability in (97) is computed in the ambient V , with $\mathcal{A}$ as an allowed parameter. All conclusions below also hold when the original ground W_0 satisfies $\text{ZFC} + \text{GCH} + \text{GA} + (V = \text{HOD})$. We give the proof in this form, using Lemma 2.2. Intermediate stages are not assumed to satisfy GA or $V = \text{HOD}$.

Use the ambient evaluation map of equation (62), with anchor $\mathcal{A}$ and allowed reals $\mathcal{R}$:

$$\text{Eval}_{\mathcal{A}} : \text{Ord} \times \mathcal{R} \rightarrow V, \quad \text{Eval}_{\mathcal{A}}(\alpha, r) = \text{Val}_{\mathcal{A}}(r, \alpha),$$

whose values at r are exactly $\text{OD}_{\mathcal{A}, r}^V$.

Lemma 13.1 (Stage facts). *(i) S_{FL} is a transitive inner model of ZF containing $\mathcal{A}, \mathcal{R}$, and every member of $\mathcal{R}$.*

(ii) $\mathcal{P}(\omega)^{S_{\text{FL}}} = \mathcal{R}$, and each R_k is countable in S_{FL} .

(iii) Every $N_k \subseteq S_{\text{FL}}$ has a real presentation $N_k = W_0[u]$, with $u \in R_k$. In V the class N_k is uniformly definable from R_k and the ambiently definable original ground W_0 .

(iv) For $k \geq 1$,

$$\omega_1^{N_k} = \kappa_k, \quad N_{k+1} \text{ is a } \text{Coll}(\omega, \omega_1^{N_k})\text{-extension of } N_k.$$

Proof. Finitely many reals in $\mathcal{R}$ lie in one finite stage, which contains their tuple code. Since $\mathcal{R}$ is definable from $\mathcal{A}$, Lemma 12.1 gives ZF and all members of $\mathcal{R}$. The sets $\mathcal{A}, \mathcal{R}, R_k$ are definable from $\mathcal{A}$; their other transitive constituents are built from allowed reals and natural numbers. Hence they also belong to S_{FL} , proving (i).

Every ground set is ambient OD by Lemma 2.2. A finite tuple of collapse generics has a real code. For a generic surjection $f : \omega \rightarrow \kappa_i$, code the relation $n \preceq m$ iff $f(n) \leq f(m)$. The ordinal ranks of its quotient well-order recover f . Joining finitely many such codes gives $N_k = W_0[u]$ with $u \in R_k$; for $k = 0$ use the zero real. The stable ground codes make every object of N_k ambiently definable from u and ordinals. Since N_k is transitive, $N_k \subseteq S_{\text{FL}}$.

More precisely, relative constructibility over the definable class W_0 , including the transitive closure of the added parameter, gives

$$N_k = W_0(R_k) = \bigcup_{g \in R_k} W_0[g]. \quad (98)$$

Each class on the right is contained in N_k , and one of them is $W_0[u] = N_k$. This also supplies a uniform ambient definition of N_k from R_k and W_0 , proving (iii).

If a real $x \in S_{\text{FL}}$ is ambiently defined from $\mathcal{A}, r$, take $r \in N_k$. The rowwise automorphisms of the remaining product preserve the canonical filtration name and r . Tail weak homogeneity decides every bit of x over N_k , so $x \in R_k$. Conversely every real in $\mathcal{R}$ belongs to the hereditary model. For $k \geq 1$, the first k rows have ground size κ_{k-1} , and their real names number at most $2^{\kappa_{k-1}} = \kappa_k$ in W_0 . Thus R_k has a well-order of size at most κ_k in N_k . Row k makes this well-order countable, giving an enumeration in $N_{k+1} \subseteq S_{\text{FL}}$. Similarly R_0 is countable in N_2 , since its ground cardinality is κ_1 . This proves (ii).

Finally, the first k rows preserve κ_k by their κ_k chain condition, and their last row makes κ_{k-1} countable. Hence $\omega_1^{N_k} = \kappa_k$. The next row is precisely $\text{Fn}(\omega, \kappa_k, <\omega)$, proving (iv). $\square$

13.2 Recovering the relative construction

We reconstruct ambiently defined relations from sets of forcing data in the original ground. The argument requires no definition of the ground or of ambient truth inside an intermediate ZF model.

Lemma 13.2 (Relations on definition codes). *Let T be any transitive ZF inner model of V containing W_0 and $\mathcal{A}$. Let δ be an ordinal and d a positive integer. If $B \subseteq (\delta \times \mathcal{R})^d$ is definable in V from $\mathcal{A}$, ordinals, and one real $z \in \mathcal{R}$, then $B \in T$.*

Proof. First $\mathcal{R} \in T$ and every $N_k \subseteq T$. Indeed a real in R_k codes the actual finite generic $G \restriction k$, and evaluation of ground names from this generic takes place in any such T . Choose m with $z \in N_m$, retain the actual filter $g_m = G \restriction m$ as a parameter in T , and fix a ground $P_{<m}$ -name $\dot{z}$ for z . Here $P_{<k} = \prod_{i < k} P_i$.

Fix a formula φ defining membership in B , with its ordinal parameters understood. In W_0 , choose for each k the set $\mathcal{Q}_k$ of all nice $P_{<k}$ -names forced to be reals, and the set $\mathcal{E}_k$ of all dense subsets of $P_{<k}$ in W_0 . For $k \geq m$, $\vec{\alpha} \in \delta^d$, and $\vec{\sigma} \in \mathcal{Q}_k^d$, compute in $\text{RO}(P)^{W_0}$ the value

$$b_{k, \vec{\alpha}, \vec{\sigma}} = \|\varphi(\check{\vec{\alpha}}, \vec{\sigma}, \dot{A}, \dot{z})\|.$$

This value belongs to the initial complete subalgebra $\text{RO}(P_{<k})^{W_0}$. To see this, the rowwise tail automorphisms fix all the displayed parameters, including the filtration name. Their weak homogeneity implies that whenever (p, q) decides the formula, its initial component $(p, 1)$ decides it the same way. Thus the Boolean value is a join of initial conditions.

The entire sequence of these tables, together with the sequences $(\mathcal{Q}_k)$ and $(\mathcal{E}_k)$ and the forcing presentations, is one set in W_0 , hence a parameter available in T . Represent the initial Boolean values by regular open subsets of $P_{<k}$. Their evaluation under a generic filter H is then the test $H \cap b_{k, \vec{\alpha}, \vec{\sigma}} \neq \emptyset$.

Inside T , consider filters $H \subseteq P_{<k}$ such that

- (i) H meets every member of $\mathcal{E}_k$;
- (ii) $H \restriction m = g_m$;
- (iii) for every $j \leq k$, the values of $\mathcal{Q}_j$ under $H \restriction j$ form exactly $\mathcal{A}(j)$.

All these are tests on sets in T . In particular, (i) expresses actual W_0 -genericity without an internal predicate for W_0 . The actual filter $G \restriction k \in T$ satisfies all three conditions.

Every filter satisfying these conditions has $W_0[H] = N_k$. Both extensions have the real set R_k , and each is generated over W_0 by a real coding its finite collapse generic. Each generating real therefore belongs to the other extension, proving equality. Consequently the actual tail $G_{\geq k}$ is generic over $W_0[H]$, and

$$W_0[H][G_{\geq k}] = V.$$

This new full generic presentation gives exactly the same filtration $\mathcal{A}$. Condition (iii) gives equality at stages through k , and equality of the k th models gives equality at all later stages. The presentation also gives the same z , by (ii).

For a tuple $((\alpha_i, r_i) : i < d) \in (\delta \times \mathcal{R})^d$, membership in B is therefore equivalent to the following test in T : there exist $k \geq m$, a filter H satisfying (i)–(iii), and $\vec{\sigma} \in \mathcal{Q}_k^d$ such that

$$\sigma_i^H = r_i \quad (i < d), \quad H \cap b_{k, \vec{\alpha}, \vec{\sigma}} \neq \emptyset.$$

The preceding comparison shows that every witness certifies membership in B . Conversely, the actual finite generics and their real names supply witnesses for every tuple in B . Separation in T proves $B \in T$. No sequence of choices of the filters H is used. $\square$

Theorem 13.3 (Relative model identification). *In the stated ground-model setting,*

$$W_0(\mathcal{R}, \mathcal{A}) = S_{\text{FL}}.$$

GCH is not needed for this identification; GA and $V = \text{HOD}$ are used only to supply the stable ground codes of Lemma 2.2.

Proof. By Lemma 13.1, S_{FL} is a transitive ZF inner model containing W_0 and $\mathcal{A}$. We show that it is contained in every such inner model T .

Take $x \in S_{\text{FL}}$, put $Y = \text{tc}(\{x\})$, and fix an ambient definition of x from $\mathcal{A}, z$ and ordinals. Every $y \in Y$ has an evaluation code (α, r) with $r \in \mathcal{R}$. The least possible ordinal codes, as y ranges over Y , are bounded by some ordinal δ . In V define

$$C = \{(\alpha, r) \in \delta \times \mathcal{R} : \text{Eval}_{\mathcal{A}}(\alpha, r) \in Y\},$$

where membership requires that the evaluation be defined. On C let E mean equality of evaluation values and let F mean membership between them. These three relations are ambiently definable from $\mathcal{A}, z$ and ordinals, by expanding Y using the chosen definition of x . By Lemma 13.2, $C, E, F \in T$.

Form C/E and its induced membership relation in T . External evaluation identifies this quotient with the transitive set Y , so its relation is well-founded and extensional, also as computed in T . Its Mostowski collapse in T is therefore the actual Y . In particular $x \in T$. This proves the required minimality and the asserted equality. The reconstruction uses neither GCH nor an internal definition of W_0 ; stable ground codes ensure that S_{FL} itself contains the whole ground. $\square$

13.3 Localization and stage descent

The next lemmas transfer definitions between the ambient universe and finite ZFC stages. They do not require the hereditary model to compute the mantle.

Lemma 13.4 (Tail localization). *Let $z, S \in N_k$. Suppose $B \subseteq S$ is defined in V by a fixed formula from $\mathcal{A}, z$, ordinals, and specified parameters in N_k . Then $B \in N_k$, with a definition there from those parameters and $R_0, \dots, R_{k-1}$. If S is definable from the other parameters in N_k , it need not be retained as an additional parameter.*

Proof. Factor the product after its first k rows. In N_k the original ground is its mantle, by Lemma 2.2; the tail forcing is therefore definable from k and its ground cardinal sequence. The filtration name uses check names for $R_0, \dots, R_{k-1}$, the intrinsic set of reals of N_k at position k , and the canonical real-set names for later finite tail extensions. Every rowwise tail automorphism preserves this name and the specified ground parameters. Consequently weak homogeneity decides the formula at the top condition for each $s \in S$. Otherwise, conditions forcing opposite answers could be moved to compatible conditions without changing the parameters. Separation in N_k , using the forcing relation for the fixed formula, gives B with the asserted definition. Any abbreviations involving ambient definability or evaluation are expanded before taking this forcing relation. $\square$

Lemma 13.5 (Lifting a stage definition). *A set ordinal-definable in N_k from a real $u \in R_k$ and $R_0, \dots, R_{k-1}$ belongs to $\text{OD}_{\mathcal{A}, u}^V$.*

Proof. In V , replace N_k by the uniformly definable class in (98), put $R_i = \mathcal{A}(i)$, and relativize the fixed defining formula to that class. The original ground is ambiently definable and k is an ordinal parameter. $\square$

For $k \geq 1$ set

$$D_k = \{g \in R_k : R_k \subseteq W_0[g]\}.$$

This nonempty set is intrinsic to N_k , using its definable original ground. Its members are exactly the maximal presentations $N_k = W_0[g]$. Indeed, a real generating N_k belongs to R_k , and the defining inclusion for D_k puts that real in $W_0[g]$.

Lemma 13.6 (Stage lowering). *Fix $m \geq 1$ and z with $N_m = W_0[z]$. Let $X \in \text{OD}_{\mathcal{A}, z}^V$ be finite and nonempty. Suppose $k > m$ and every member of X belongs to $\text{OD}_{\mathcal{A}, r}^V$ for some $r \in R_k$. Then, for every maximal presentation $N_{k-1} = W_0[u]$,*

$$X \subseteq \text{OD}_{\mathcal{A}, u}^V.$$

Proof. Choose $g_0 \in D_k$. Stable ground codes make every real of R_k ordinal-definable in $N_k = W_0[g_0]$ from g_0 ; the definition lifts by Lemma 13.5. Thus every member of X is ambiently definable from $\mathcal{A}, g_0$ and ordinals. Choose a finite ordinal list $\alpha_0, \dots, \alpha_{d-1}$ such that

$$\{\text{Eval}_{\mathcal{A}}(\alpha_i, g_0) : i < d\} = X.$$

In V , form

$$B = \{(i, g) \in d \times D_k : \text{Eval}_{\mathcal{A}}(\alpha_i, g) \text{ is defined and belongs to } X\},$$

and, for $(i, g), (j, h) \in B$, put

$$(i, g) E (j, h) \iff \text{Eval}_{\mathcal{A}}(\alpha_i, g) = \text{Eval}_{\mathcal{A}}(\alpha_j, h).$$

Expand X by its original definition from $\mathcal{A}, z$ and ordinals, without using X as a ground parameter. Expand the evaluation and D_k definitions as well. Applying Lemma 13.4 to subsets of $d \times R_k$ and its square puts B, E in N_k , ordinal-definably from $z, R_0, \dots, R_{k-1}$. Here R_k and D_k are intrinsic to the stage.

The quotient partition $\mathcal{P} = B/E$ belongs to N_k and has exactly $|X|$ nonempty cells. Each cell has one evaluation value, and the values at g_0 show that every member of X occurs. The quotient cells are the same in N_k and V , because both use the same sets B, E and all their elements. In particular this is a finite family of sets of coded reals in N_k , even if its evaluation values lie outside N_k .

All the parameters $z, R_0, \dots, R_{k-1}$ belong to $N_{k-1} = W_0[u]$. Its stable ordinal codes make $\mathcal{P}$ an $\text{OD}(u)$ family in N_k . The latter is a collapse extension of N_{k-1} at its ω_1 , by Lemma 13.1. Homogeneity gives an invariant name for $\mathcal{P}$, and Theorem 6.1 gives invariant names enumerating its cells. In N_k , the preceding ground $W_0[u]$ and its names are definable from u and ordinals by Lemma 2.2. Thus Lemma 2.3 makes each cell $\text{OD}(u)$ in N_k . This uses no GA or HOD assumption on N_{k-1} .

Apply Lemma 13.5 to the definition of a cell. In V the cell defines its unique common evaluation value, using the fixed ordinal list $(\alpha_i)_{i < d}$. No real representative of the cell is selected. These values exhaust X , proving the conclusion. $\square$

13.4 Uniform parameters, finite choice, and infinite sets

Theorem 13.7 (Uniform finite-family definability). *If $\langle F_n : n < \omega \rangle \in S_{\text{FL}}$ is a sequence of finite sets, there is one real $z \in \mathcal{R}$ such that*

$$\bigcup_{n < \omega} F_n \subseteq \text{OD}_{\mathcal{A}, z}^V.$$

Proof. The whole sequence has an ambient definition from $\mathcal{A}$, ordinals, and one real. Choose $m \geq 1$ containing that real and a maximal presentation $N_m = W_0[z]$. By stable ground codes and Lemma 13.5, the original real is ambiently definable from $\mathcal{A}, z$ and ordinals. The sequence therefore has such a definition too.

Fix n with $F_n \neq \emptyset$. Its finitely many members belong to the hereditary model; choose their real definition parameters and one stage $k \geq m$ containing them. If $k > m$, apply Lemma 13.6 repeatedly, keeping the original definition of F_n from $\mathcal{A}, z$ at every step. This finite descent reaches R_m . Every real there is ordinal-definable from z in N_m , and its definition lifts. Thus $F_n \subseteq \text{OD}_{\mathcal{A}, z}^V$. For each n only finitely many choices and descent steps are used; no sequence of stage bounds is chosen. $\square$

Corollary 13.8. $S_{\text{FL}} \models \text{AC}_{\omega}^{\text{fin}}$.

Proof. For a sequence of nonempty finite sets take z from Theorem 13.7. In V , select from each set the value with least ordinal evaluation code from $(\mathcal{A}, z)$. The sequence itself has an allowed definition, so the resulting choice function belongs to $\mathcal{D}_{\text{FL}}$. Its entries and their transitive closures are already in S_{FL} ; hence the function belongs to S_{FL} . Thus the ambient selection defines a function in S_{FL} without an internal application of Replacement to ambient evaluation. $\square$

Theorem 13.9. *Every infinite set in S_{FL} has a countably infinite subset in S_{FL} .*

Proof. Take $x \in S_{\text{FL}}$ infinite and $z \in \mathcal{R}$ such that $x \in \text{OD}_{\mathcal{A}, z}^V$. Define in V

$$X_k = \{a \in x : \exists r \in R_k \quad a \in \text{OD}_{\mathcal{A}, z, r}^V\}.$$

These sets cover x . Their definitions are uniform from $\mathcal{A}, z$ and the definition of x , and all their elements belong to S_{FL} . Thus both the sets X_k and their sequence belong to S_{FL} .

If some X_k is infinite, choose one enumeration $e : \omega \rightarrow R_k$ in S_{FL} . For each $a \in X_k$, take the least i for which a is ambiently definable from $(\mathcal{A}, z, e(i))$, followed by its least ordinal definition code from these parameters. Lexicographic comparison of these pairs well-orders X_k , since it is an ordinal sum of well-orders of codes. The relation is ambiently definable from $\mathcal{A}, z, e$; substitute an allowed definition of e and join its real parameter with z . Its elements are built from members of x , so the well-order belongs to S_{FL} . Its first ω elements, formed inside that ZF model, give the required injection into x .

Otherwise every X_k is finite. Inside S_{FL} , the set of linear orders on X_k is nonempty and finite, with its unique empty order when $X_k = \emptyset$. Apply Corollary 13.8 to choose one order for each k . Listing the sets X_k in these orders gives a partial map on $\omega \times \omega$ with range x . Retaining first occurrences in a fixed enumeration gives an injection $\omega \rightarrow x$, since x is infinite. $\square$

By Theorem 13.3, both choice conclusions hold in the relative Feferman–Levy model $M_0 = W_0(\mathcal{R}, \mathcal{A})$.

14 Hereditary definability and choice

We now consider hereditary definability in extensions obtained by adding generalized Cohen subsets. The parameter bound is the successor of the closure cardinal, while the number of columns may be larger.

Definition 14.1. For regular λ , write $\mathcal{D}_\lambda = \text{OD}_{H_\lambda}^V$, also denoted $\text{OD}_{<\lambda}$ in this setting. Its hereditary part is $\text{HOD}_{H_\lambda}^V$, with the parameter convention of section 12.1.

Throughout the generalized Cohen results, unless another width is specified, let κ be regular uncountable in L , let $\Lambda > \kappa$ be a cardinal of L , and put

$$\begin{aligned} P &= \text{Add}(\kappa, \Lambda)^L, & V &= L[G], & \lambda &= (\kappa^+)^L = \kappa^+, \\ A &= \mathcal{P}(\kappa)^V, & S^* &= \text{HOD}_{H_\lambda^V}^V, & \mathcal{D} &= \text{OD}_{H_\lambda^V}^V. \end{aligned} \tag{99}$$

This is the setting of section 12.3, since $L \models \kappa^{<\kappa} = \kappa$; we retain its forcing notation.

As in Lemma 2.2, the proofs of the hereditary-model and arithmetic results work over a fixed $W_0 \models \text{ZFC} + \text{GCH} + \text{GA} + (V = \text{HOD})$, using its stable ordinal codes. Taking $W_0 = L$ gives the stated results. The support and ultrafilter theorems require only a GCH ground.

14.1 The direct hereditary-definability theorem

Theorem 14.2. *In the setting (99), the class S^* is an inner model of ZF. Every member of H_λ^V belongs to an original coordinate extension on at most κ columns. Moreover:*

- (i) $L(\mathcal{P}(\kappa)^V) \subseteq S^*$ and $\mathcal{P}(\kappa)^{S^*} = \mathcal{P}(\kappa)^V$.
- (ii) Every function in V with domain an ordinal at most κ and range contained in S^* belongs to S^* .
- (iii) No uniform ultrafilter on κ belongs to $\mathcal{D}$. In particular, S^* has no uniform ultrafilter on κ .

Proof. Lemma 12.9 gives ZF, parameter capture, (i), and (ii). Assertion (iii) is the following theorem. $\square$

Theorem 14.3. *Let $W_0 \models \text{ZFC} + \text{GCH}$, let κ be regular uncountable in W_0 , and let $\Lambda > \kappa$ be any ground cardinal. In $V = W_0[G]$, where G is $\text{Add}(\kappa, \Lambda)^{W_0}$ -generic, no uniform ultrafilter on κ belongs to $\text{OD}_{H_{\kappa^+}}^V$.*

Proof. Put $\lambda = \kappa^+$. Suppose U is a uniform ultrafilter on κ defined from $z \in H_\lambda$ and ordinal parameters. Choose a name for z supported on a ground set E of at most κ columns, as in Lemma 12.8(ii). Choose $p \in G$ forcing the definition to have a unique value that is a uniform ultrafilter, and enlarge E to include the columns of p . Fix $\beta \in \Lambda \setminus E$ and let c_β be the generic subset of κ on that column.

Choose $q \in G$, $q \leq p$, deciding whether c_β or its complement belongs to the named ultrafilter; call the selected name $\dot{A}$. Put

$$D_q = \{\xi < \kappa : (\beta, \xi) \in \text{dom}(q)\}.$$

This ground set has size less than κ . The ground forcing automorphism τ that flips precisely the bits (β, ξ) with $\xi \notin D_q$ fixes q , p , and the parameter name. By uniqueness of the definition, it fixes the ultrafilter name modulo forced equality below p . Hence

$$q \Vdash \dot{A} \in \dot{U} \wedge \tau \dot{A} \in \dot{U}.$$

Outside D_q these two sets are complementary. Their intersection is therefore forced to have size less than κ , contrary to uniformity. □

Corollary 14.4. *In the setting (99), every ambient set of at most κ elements of S^* belongs to S^* and has an enumeration in S^* . The model S^* satisfies DC_κ . Every nontrivial Boolean algebra in S^* of ambient cardinality at most κ has an ultrafilter in S^* , although S^* fails the Boolean prime ideal theorem.*

Proof. Choose an ambient enumeration and apply Theorem 14.2(ii); take its range inside S^* . The same argument puts any ambient dependent-choice sequence of length κ into S^* . This also proves the version in which each choice depends on the entire preceding sequence, since all shorter sequences of members of S^* are available there.

For a small Boolean algebra choose an ultrafilter in V and enumerate it. Its members lie in S^* , so the enumeration and its range do too. Finally, subsets of κ of size less than κ are the same in V and S^* . One direction is absolute, and the other follows by putting an ambient short enumeration into S^* . The $\text{co-} < \kappa$ filter is consequently a proper filter in S^* . If the Boolean prime ideal theorem held there, this filter would extend to a uniform ultrafilter, contradicting Theorem 14.2(iii). □

14.2 Sharpness at the next cardinal

The next theorem shows that the small-family bound cannot be increased. It holds over every GCH ground, without a ground definability assumption.

Theorem 14.5. *Let $W_0 \models \text{ZFC} + \text{GCH}$, let κ be regular uncountable in W_0 , and put $\lambda = \kappa^+$. In $V = W_0[G]$, where G is $\text{Add}(\kappa, \lambda)^{W_0}$ -generic, the family $\mathcal{U}_\kappa$ of all uniform ultrafilters on κ is ordinal-definable and satisfies*

$$|\mathcal{U}_\kappa|^V = \lambda^+, \quad \mathcal{U}_\kappa \cap \text{OD}_{H_\lambda}^V = \emptyset.$$

In particular, over L the conclusion of Theorem 8.15 fails at the next cardinal after λ .

Proof. The family is definable from the ordinal κ . The absence of its members from $\text{OD}_{H_\lambda}^V$ is exactly Theorem 14.3.

For every infinite κ in any ZFC universe, the following argument gives $|\mathcal{U}_\kappa| = 2^{2^\kappa}$. There is a family $(A_x)_{x \subseteq \kappa}$ of subsets of a set of size κ such that every finite Boolean combination of distinct members has size κ . Take the underlying set

$$I = \{(s, H, \gamma) : s \in [\kappa]^{<\omega}, H \subseteq \mathcal{P}(s), \gamma < \kappa\}$$

and put

$$A_x = \{(s, H, \gamma) \in I : x \cap s \in H\}.$$

For finitely many distinct x 's, choose finite s distinguishing their restrictions. Choosing H gives any desired membership pattern, and varying γ gives κ points with that pattern. Since $|I| = \kappa$, transport the family to κ .

Every binary assignment to this 2^κ -sized family, together with the $\text{co-} < \kappa$ filter, generates a proper filter. Extend it to an ultrafilter using choice in V . It is uniform because it contains the $\text{co-} < \kappa$ filter, and distinct assignments give distinct ultrafilters. The reverse bound counts all subsets of $\mathcal{P}(\kappa)$, so

$$|\mathcal{U}_\kappa| = 2^{2^\kappa}.$$

The forcing preserves cardinals. Its $< \kappa$ -closure preserves those at most κ , and its κ^+ -chain condition preserves those at least κ^+ . Its λ distinct columns and the nice-name bound $(\lambda^\kappa)^{W_0} = \lambda$ give $(2^\kappa)^V = \lambda$. Finally a nice name for a subset of λ is a subset of the fixed ground set of pairs consisting of an ordinal below λ and a condition. This set has size λ . Ground GCH bounds the number of these names by $(2^\lambda)^{W_0} = (\lambda^+)^{W_0}$. Cardinal preservation and Cantor's theorem give the matching lower bound in V . Hence

$$|\mathcal{U}_\kappa|^V = 2^{2^\kappa} = 2^\lambda = \lambda^+,$$

which proves the claim. $\square$

15 Saturation in generalized Cohen extensions

15.1 Copies of small definable structures

For the following corollary, specialize (99) to width $\Lambda = \lambda = \kappa^+$.

Corollary 15.1. *Every OD_{H_λ} structure of cardinality at most λ in a fixed countable language has an ambiently isomorphic copy in S^* .*

Proof. Its domain belongs to $\mathcal{D}$ and is pointwise in $\mathcal{D}$ by Theorem 8.15. Apply Lemma 12.2. $\square$

The isomorphism is asserted in the ambient universe. In particular, ambient cardinality and saturation pass to the copy; a well-order of its domain need not belong to S^* .

15.2 The generalized Cohen category environment

Apply section 12.3 with $\kappa > \omega$, width Λ , and $\mathcal{S} = S^*$. The good bases form the set D of Definition 12.10. For every ground Cohen tree $\mathcal{T} \cong \kappa^{<\kappa}$, the associated ideals are closed under κ unions and satisfy the finite-product category theorem, local decision, and conull uniformization into arbitrary codomains. The next argument uses these properties on a tree of arithmetic residues.

15.3 The generalized arithmetic obstruction

Return to the setting of (99), with arbitrary $\Lambda > \kappa$. Saturation is computed in V ; κ^+ -saturation realizes types over at most κ parameters.

Theorem 15.2. *Let κ be regular uncountable in L , let $\Lambda > \kappa$ be any cardinal of L , and let*

$$V = L[G], \quad G \subseteq \text{Add}(\kappa, \Lambda)^L \text{ be generic}, \quad S^* = \text{HOD}_{H_{\kappa^+}^V}^V.$$

No model of $I\Delta_0$ belonging to S^ is κ^+ -saturated in V . There is no restriction on the cardinality of the model. The same nonexistence statement holds for models belonging to $L(\mathcal{P}(\kappa)^V)$.*

We prove the theorem over the fixed stable ground W_0 from the convention following (99). We embed a ground arithmetic model into a hypothetical saturated model. We work in their canonical exponential cuts, while realizing all types in the original models. The category argument gives a small coinital family above a cofinal hull, which contradicts saturation.

15.4 A ground arithmetic model and its cofinal hull

Suppose that $N \in S^*$ is a κ^+ -saturated model of $I\Delta_0$. Put $C = C(N)$, using Lemma 12.5. Arithmetic calculations below take place in the EFA model C , in its conservative exponential language with bounded induction. No saturation of C is assumed. Write $\text{Bit}(b, i)$ for

$$\exists u \leq b \exists v < 2^i (b = (2u + 1)2^i + v).$$

Bounded induction proves division with remainder and the exponent laws. It also constructs the XOR of two strings below 2^m . Induct on m with the bounded assertion that some $z < 2^m$ has the prescribed XOR digits at all $i < m$. At a successor step append the next digit by retaining z or adding 2^m . The same induction proves uniqueness from the digits, hence associativity, self-inverseness, and compatibility with reduction modulo a smaller power of two. These identities hold at every internal position, including nonstandard ones. We will also use

$$b < 2^d, \quad i \geq d \quad \implies \quad \neg \text{Bit}(b, i).$$

All arithmetic searches below are explicitly bounded.

The complete theory $T_0 = \text{Th}(N)$ is a real. The forcing adds no reals, so $T_0 \in W_0$. In W_0 choose a κ -saturated $M_0 \models T_0$ of cardinality κ . To construct it, start with a model of size κ and build an elementary chain of length κ . At a successor stage realize every type over every parameter subset of size less than κ from the preceding model. There are at most κ such types, since $\kappa^{<\kappa} = \kappa$ and the language is countable. Compactness and downward Löwenheim–Skolem give the next model of size κ . Regularity ensures that every small parameter set in the union is contained in a stage. The union is therefore κ -saturated. This construction and its choices take place in W_0 .

Let $C_0 = C(M_0)$, computed in W_0 . It is unchanged in V , by absoluteness of satisfaction for the ground set-sized structure. In W_0 , for any $D \subseteq C_0$ of size less than κ , the type

$$\{T_n(x) : n < \omega\} \cup \{a < x : a \in D\}$$

is finitely satisfiable in C_0 . Saturation of M_0 realizes it in C_0 . The analogous type above all standard numerals supplies a nonstandard point. Therefore $|C_0| = \text{cf}(C_0) = \kappa$.

Enumerate M_0 in V and use saturation of N to extend partial elementary maps, obtaining an elementary embedding

$$e : M_0 \longrightarrow N.$$

At each stage fewer than κ previously chosen images occur as parameters. The graph of e has κ hereditary entries in S^* , so it belongs to S^* by Theorem 14.2. Put $M = e[C_0] \subseteq C$. This inclusion follows by transferring each standard formula T_n . The image M is closed under arithmetic and exponentiation, since e preserves the arithmetic operations and the graph E . No elementarity between the two exponential cuts is needed. Define

$$K = \{b \in C : \exists m \in M (b < m)\}, \quad A_N = C \setminus K. \quad (100)$$

This is a cut in C , and all its displayed objects belong to S^* . The initial segment K contains M and is closed under successor, addition, and $b \mapsto 2^b$. Apply each of these operations to bounds in M to obtain the required new bounds. Saturation of N realizes

$$\{T_n(d) : n < \omega\} \cup \{m < d : m \in M\}.$$

Every finite fragment has a witness in M . Fix a realization $d \in C$. Thus $d \in A_N$, and 2^d exists in C . Every element of A_N is above every element of M , and

$$h \in A_N \implies \lfloor h/2 \rfloor \in A_N. \quad (101)$$

Indeed, if the half belonged to K , closure under addition and successor would put $h \leq 2\lfloor h/2 \rfloor + 1$ in K .

15.5 The Cohen tree of coherent residues

In W_0 , choose an increasing cofinal sequence $(c_\alpha : \alpha < \kappa)$ in C_0 , with c_0 nonstandard and

$$c_\beta - c_\alpha > n \quad (\alpha < \beta < \kappa, n < \omega). \quad (102)$$

To construct it, enumerate C_0 in order type κ . At stage β require an element above the next enumerated element and above $c_\alpha + n$ for every $\alpha < \beta$ and $n < \omega$. Include every $T_n(x)$. The resulting type has size below κ and every finite part has a witness in C_0 . Saturation of M_0 realizes it in C_0 . At stage zero include all inequalities asserting nonstandardness.

Let $\mathcal{T}_{\text{res}}$ consist of the sequences $r = \langle r_\alpha : \alpha < \delta \rangle$, $\delta < \kappa$, with entries in M_0 satisfying

$$0 \leq r_\alpha < 2^{c_\alpha}, \quad r_\beta \bmod 2^{c_\alpha} = r_\alpha \quad (\alpha < \beta < \delta). \quad (103)$$

Every displayed residue lies in C_0 by its upper bound. The calculations take place in that EFA cut, and the tree order is extension.

Lemma 15.3. *In W_0 , the tree $\mathcal{T}_{\text{res}}$ is isomorphic to $\kappa^{<\kappa}$.*

Proof. Unions of increasing chains of length below κ are nodes. The tree has cardinality at most κ , since $|M_0| = \kappa^{<\kappa} = \kappa$. For a node r of length δ , its immediate successors are the solutions in M_0 of

$$b < 2^{c_\delta}, \quad b \bmod 2^{c_\alpha} = r_\alpha \quad (\alpha < \delta).$$

There are κ solutions. To prove the lower bound, append inequalities excluding any prescribed set of fewer than κ elements. The type still has size below κ . Its exponent values are named

parameters, so its congruences are formulas in the ordinary arithmetic language of M_0 . A finite subsystem of the congruences reduces to the one at its largest index α . If there are n exclusions, the $n + 1$ distinct elements $r_\alpha + j2^{c_\alpha}$, $j = 0, \dots, n$, satisfy that congruence and lie below 2^{c_δ} by (102). At least one survives the exclusions. If no congruence occurs, nonstandardness of c_δ gives the same conclusion. Saturation of M_0 realizes the full type; the upper bound puts the realization in C_0 . Thus the solution set cannot have size below κ , and its size is at most $|M_0| = \kappa$.

Choose a bijection from κ onto each successor set in W_0 . Use them recursively at successor levels and take unions at limit levels. This gives the required tree isomorphism. Surjectivity at a limit level follows from the coherent inverse images of all proper initial segments. $\square$

Put $X = [\mathcal{T}_{\text{res}}]^V$. The ground tree and its short nodes are unchanged in V , and its branches are coded by subsets of κ . Apply the environment of sections 12.3.2 to 12.3.4 to this tree. Write $\mathcal{I}$ for its ideal on X and $\mathcal{I}_2$ for the ideal on X^2 . Nonempty cylinders and rectangles are positive, binary functions admit local decisions, and Theorem 12.18 applies to every relation in S^* .

Lemma 15.4. *There is a function $F : X \rightarrow C$ in S^* with $F(r) < 2^d$ for every $r \in X$, and a set $X_0 \subseteq X$ with $X \setminus X_0 \in \mathcal{I}$ such that for $r \in X_0$,*

$$F(r) < 2^d, \quad F(r) \bmod 2^{e(c_\alpha)} = e(r_\alpha) \quad (\alpha < \kappa). \quad (104)$$

Proof. For any branch r , the displayed conditions form a finitely satisfiable ordinary-language type of size κ in N , using the displayed exponent values as named parameters. For a finite subsystem, the residue at the greatest mentioned index is itself a witness below 2^d . Saturation therefore gives a witness for each branch. The bound below $2^d \in C$ puts every witness in C by initiality. This defines a relation in S^* with all sections nonempty, since its ambient witnesses are elements of C and therefore belong to S^* . Apply Theorem 12.19, and extend its partial uniformizer by 0 off its domain. The bound $F(r) < 2^d$ therefore holds on all of X . $\square$

Write $\text{Ext}(r)(i) \in \{0, 1\}$ for the truth value of $C \models \text{Bit}(F(r), i)$. Coordinatewise bitwise XOR of residues makes X a Boolean group. Reduction modulo a smaller power of two commutes with XOR, so the operation preserves coherence and translations act on the tree. If $r, r + t \in X_0$, their residues determine the relation between these two bit strings at *every* position $i \in K$. Choose α with $i < e(c_\alpha)$ and use (104) at that modulus. The EFA identity for XOR computes the bits throughout that interval in C , including positions $i \notin M$.

15.6 Translations and a bound above the cut

For $\alpha < \kappa$, define ground branches $a_\alpha, a_\infty \in X$ by

$$(a_\alpha)_\beta = 2^{\min(c_\alpha, c_\beta)} - 2^{c_0}, \quad (a_\infty)_\beta = 2^{c_\beta} - 2^{c_0}, \quad t_\alpha = a_\alpha + a_\infty. \quad (105)$$

All these arithmetic calculations and the entire branches belong to W_0 . The pattern a_α has ones on $[c_0, c_\alpha)$, and a_∞ has ones at positions at least c_0 . For every cylinder U , translation by t_α fixes U for all sufficiently large α . If U specifies a node of length δ , then the β th residue of t_α is zero for $\beta < \delta \leq \alpha$. These translations are ground tree automorphisms, so preserve $\mathcal{I}$ by Proposition 12.15.

For $t = a_\alpha$ or $t = a_\infty$, let $\theta_t(i)$ denote in C the characteristic function of $[e(c_0), e(c_\alpha))$ or of $[e(c_0), \infty)$, respectively. The residue identities give

$$\text{Ext}(r + t)(i) = \text{Ext}(r)(i) \oplus \theta_t(i) \quad (i \in K) \quad (106)$$

whenever $r, r + t \in X_0$. The union of the two exceptional sets where coding fails belongs to $\mathcal{I}$.

Lemma 15.5 (Local bound above the cut). *For such a t and every nonempty basic cylinder U , there are $H \in A_N$ and a nonempty basic $U' \subseteq U$ such that (106) holds simultaneously at every $i < H$, for $r \in U'$ outside one member of $\mathcal{I}$.*

Proof. For each r , search for the least failure of (106) at a position $i < d$, and take d if there is none. The failure predicate is bounded in the exponential language, with the two integer codes and interval endpoints as parameters, so EFA supplies this bounded minimum. When both codes are valid, the identity throughout K puts the resulting value in A_N . Replace it by d if it belongs to K . This defines a function $h : X \rightarrow A_N$ in S^* ; outside the coding exceptions the identity holds below $h(r)$. Neither the external function F nor the cut K occurs in the formula to which bounded minimization is applied.

Apply the general form of Lemma 12.23 to this function h . Its hypotheses are supplied by Proposition 12.15, Corollary 12.17, and Theorem 12.18 and (101). Removing the coding exceptions gives the claim. $\square$

15.7 A small coinital family and the omitted type

For each $\alpha < \kappa$, let $u_\alpha(r)$ be the least position $e(c_\alpha) < i < d$ at which

$$\text{Ext}(r)(i) \neq \text{Ext}(r + a_\alpha)(i).$$

This is a bounded search. Since both codes are below 2^d , their bits vanish at every position $i \geq d$, so the search finds every possible disagreement above $e(c_\alpha)$. Use a formal value ∞ , greater than every element of C , if there is no such position. For a basic cylinder U , put

$$I_{\alpha,U} = \{b \in A_N : \{r \in U : u_\alpha(r) \geq b\} \in \mathcal{I}\}. \quad (107)$$

Each $I_{\alpha,U}$ is a final segment of A_N .

Lemma 15.6. *Every $I_{\alpha,U}$ is proper, and*

$$A_N = \bigcup_{\alpha < \kappa, U \text{ basic}} I_{\alpha,U}.$$

Proof. This is the final-segment argument of Lemma 12.25, with $(C, \omega, M \setminus \omega, n + 1)$ replaced by $(X, K, A_N, e(c_\alpha))$. We verify the steps in this setting. Lemma 15.5 for a_α gives $u_\alpha \geq H$ on a positive subcylinder, proving properness. If b misses the union, apply that lemma to a_∞ and choose $i \in A_N$ with $i + 1 < \min(b, H)$, using (101). Locally decide the bit $\text{Ext}(r + a_\infty)(i)$ and choose α so that t_α fixes the resulting cylinder. Translation invariance makes the bits at $r + a_\alpha$ and $r + a_\infty$ equal there modulo $\mathcal{I}$, while the latter differs from the bit at r . Thus $u_\alpha \leq i$ locally, $i + 1 \in I_{\alpha,U'}$, and finality puts b in that segment, a contradiction. All thresholds $e(c_\alpha)$ lie in K , so i is above each of them, as required in the cited proof. $\square$

Proof of Theorem 15.2. Use the constructions above for a hypothetical saturated $N \in S^*$. There are κ pairs (α, U) , since the cylinder basis has size κ . In V , choose $b_{\alpha,U} \in A_N \setminus I_{\alpha,U}$ for each pair. The sequence belongs to S^* by Theorem 14.2. It is coinital in A_N , since $b \in I_{\alpha,U}$ implies $b_{\alpha,U} < b$ by finality.

Consider the type

$$\{T_n(x) : n < \omega\} \cup \{m < x : m \in M\} \cup \{x < b_{\alpha,U} : \alpha < \kappa, U \text{ basic}\}. \quad (108)$$

It has at most κ parameters and is finitely satisfiable. Indeed, choose an element of M above the finitely many lower bounds; it satisfies every T_n , and every upper bound lies in A_N , hence is above all of M . A realization by κ^+ -saturation of N lies in C , is above every member of M , and hence lies in A_N , but is below its coinital family. This is impossible.

Finally, Lemma 12.1 gives $L(\mathcal{P}(\kappa)^V) \subseteq S^*$, so the same nonexistence conclusion holds in that inner model. $\square$

15.8 The exact saturation threshold and definable presentations

Corollary 15.7. *Let $W_0 \models \text{ZFC} + \text{GCH}$, let κ be regular uncountable in W_0 , and let $V = W_0[G]$ for $\text{Add}(\kappa, \Lambda)^{W_0}$, where $\Lambda > \kappa$ is any ground cardinal. Every completion of $I\Delta_0$ has a model in $L(\mathcal{P}(\kappa)^V)$ of cardinality κ which is κ -saturated in V . In the setting of Theorem 15.2, this is also a model in S^* . Thus κ -saturation is possible in both inner models, whereas κ^+ -saturation is impossible.*

Proof. The completion belongs to W_0 because there are no new reals. The elementary-chain construction above gives a ground κ -saturated model M_0 with domain the ordinal κ . Every parameter set of size below κ from M_0 has a ground enumeration, by the absence of new short ordinal sequences. Every type over such a set is also ground, since the countable language supplies fewer than κ formulas with those parameters. The ground witnesses therefore show that M_0 remains κ -saturated in V . Using the canonical ordinal pairing functions, its countably many relations and operation graphs have one subset-of- κ code. That code and its decoded structure belong to $L(\mathcal{P}(\kappa)^V)$; Lemma 12.1 puts this model in S^* as well. $\square$

Corollary 15.8. *In the setting of Theorem 15.2, if $\Lambda = \kappa^+$, there is no $\text{OD}_{H_{\kappa^+}}^V$ presentation of an $I\Delta_0$ model of ambient cardinality at most κ^+ that is κ^+ -saturated in V . In particular, there is no such presentation of a model of true arithmetic. More generally, at any $\Lambda > \kappa$, an arbitrary-size $\text{OD}_{H_{\kappa^+}}^V$ presentation whose domain is pointwise $\text{OD}_{H_{\kappa^+}}^V$ cannot be κ^+ -saturated.*

Proof. For $\Lambda = \kappa^+$, apply Theorem 8.15 and Corollary 15.1 to obtain an ambiently isomorphic copy in S^* . Theorem 15.2 rules it out. For the final assertion, the definition-code construction of Lemma 12.2 applies directly to the pointwise definable domain. That construction uses only closure of the definability class under finite parameters and ZF in its hereditary part, both available by Theorem 14.2. Its S^* -copy contradicts the same theorem. $\square$

Corollary 15.9. *Let κ be any infinite regular cardinal of L . The generic extension $V = L[G]$ by $\text{Add}(\kappa, \kappa^+)^L$ preserves all cardinals and cofinalities and satisfies GCH. In V , every complete consistent extension of $I\Delta_0$ has a κ^+ -saturated model of cardinality κ^+ , but none of these models has an $\text{OD}_{\mathcal{P}(\kappa)}$ copy. In particular there is no $\text{OD}_{\mathcal{P}(\kappa)}$ κ^+ -saturated elementary extension of $\mathbb{N}$ of cardinality κ^+ .*

Proof. The forcing has size κ^+ , is $< \kappa$ -closed, and is κ^+ -cc. These properties preserve all cardinals and cofinalities. No subsets of infinite cardinals below κ are added. For each cardinal $\tau \geq \kappa$, a nice name for a subset of τ uses τ antichains, each of size at most κ . The number of such names in L is at most

$$((\kappa^+)^{\kappa})^{\tau} = (\kappa^+)^{\tau} = \tau^+.$$

The ground subsets give the reverse bound, so every instance of GCH is preserved. Consequently $(\kappa^+)^{<\kappa^+} = \kappa^+$, and the standard saturated-model construction supplies the asserted models of every complete countable theory extending $I\Delta_0$.

For $\kappa = \omega$, Corollary 12.31 excludes the definable copies. For uncountable κ , apply Corollary 15.8, since every subset of κ belongs to H_{κ^+} . As in Remark 1.1, the proof applies to every original ZFC ground satisfying GCH, GA, and $V = \text{HOD}$. $\square$

Remark 15.10. In L itself, its canonical well-order makes the usual saturated-model construction definable, giving an OD κ^+ -saturated elementary extension of $\mathbb{N}$ of size κ^+ for every infinite κ . The regularity restriction in the corollary is exact. If κ is any singular cardinal of L , a cardinal- and cofinality-preserving extension satisfying GCH still makes κ a singular strong limit. Then Corollary 12.42 supplies such an OD model and rules out the analogous negative conclusion.

15.9 Trace coding and finite-support models

The following observations hold in ZFC, independently of forcing. In PA, decode z as $(b, c - 2)$, $c \geq 2$, and define $\text{ap}(z, m) = \lfloor b/c^m \rfloor \bmod c$. This total operation interpolates at finitely many arbitrary model positions. For distinct $m_1, \dots, m_r$ and prescribed v_i , choose $c > \max_i v_i$ and $b = \sum_{i=1}^r v_i c^{m_i}$. PA verifies the required digits for each standard finite r .

Proposition 15.11. *If κ is infinite, $N \models \text{PA}$ is κ^+ -saturated, and $M \subseteq N$ has size at most κ , then*

$$\text{Ev}(z)(m) = \text{ap}(z, m), \quad \pi(z) = \{m \in M : \text{ap}(z, m) = 1\}$$

map N onto ${}^M N$ and $\mathcal{P}(M)$, respectively. Their equality kernels are

$$\begin{aligned} z E_M^{\text{fun}} w &\iff \forall m \in M [\text{ap}(z, m) = \text{ap}(w, m)], \\ z E_M^{\text{set}} w &\iff \forall m \in M [\text{ap}(z, m) = 1 \leftrightarrow \text{ap}(w, m) = 1]. \end{aligned}$$

Consequently $N/E_M^{\text{fun}} \cong {}^M N$ and $N/E_M^{\text{set}} \cong \mathcal{P}(M)$.

Proof. For $f : M \rightarrow N$, the type $\{\text{ap}(z, m) = f(m) : m \in M\}$ has at most κ parameters and is finitely satisfiable by interpolation. Saturation realizes it. Characteristic functions give the second surjection; the displayed equivalence relations are their kernels. $\square$

Corollary 15.12. *If S is a transitive ZF inner model closed under ambient sequences of length at most κ , and $M, N \in S$ with $|M| \leq \kappa$, then $({}^M N)^S = ({}^M N)^V$ and $\mathcal{P}(M)^S = \mathcal{P}(M)^V$. When N satisfies the preceding proposition, its trace maps, kernels, quotients and induced bijections belong to S .*

Proof. An ambient enumeration of M belongs to S . Composing any $f : M \rightarrow N$ with it proves the first equality, and characteristic functions prove the second. The absolute trace definitions and ZF in S supply the indicated maps and quotients. $\square$

Proposition 15.13. *No $\aleph_1$ -saturated PA model has an expansion generated definably or algebraically by an infinite sequence of distinct order-indiscernibles. In particular an ordinary Skolemized Ehrenfeucht–Mostowski model on such a sequence is not $\aleph_1$ -saturated.*

Proof. Choose a countably infinite monotone subsequence (a_{i_n}) of the indiscernibles. If b is uniquely defined by $\theta(y, a_{j_1}, \dots, a_{j_r})$, the relative order of i_n and the finitely many j_s is eventually constant. Apply indiscernibility to $\exists y [\theta(y, y_1, \dots, y_r) \wedge \text{Bit}(y, z)]$. It makes $\text{Bit}(b, a_{i_n})$ eventually constant. Thus the countable type requiring alternating bits at the a_{i_n} is omitted, although every finite part is realized by binary interpolation. Algebraic generation reduces to unique definition, since each element of a finite definable set is its r th member in the PA order for some standard finite r . $\square$

Indiscernibility is in the generating language. For a quantifier-free EM blueprint, use its Skolemized language with the definable bit function, so the formula after term substitution is atomic. The argument also covers distinct fixed-length tuples. One fixed coordinate differs on every ordered pair and supplies the bit positions. The omitted type uses countably many parameters; the conclusion concerns $\aleph_1$ -saturation, not recursive saturation over a finite tuple.

References

- [1] L. D. Beklemishev, *Reflection principles and provability algebras in formal arithmetic*, Russian Math. Surveys **60** (2005), no. 2, 197–268. doi:10.1070/RM2005v060n02ABEH000823.
- [2] I. Ben Yaacov, A. Berenstein, and J. Melleray, *Polish topometric groups*, Trans. Amer. Math. Soc. **365** (2013), 3877–3897. Author manuscript.
- [3] I. Ben Yaacov and J. Melleray, *Grey subsets of Polish spaces*, arXiv:1103.2762; author manuscript, especially Theorem 3.6.
- [4] F. Delbaen, *Conditionally atomless extensions of sigma algebras*, arXiv:2003.09254.
- [5] J. D. Dixon, P. M. Neumann, and S. Thomas, *Subgroups of small index in infinite symmetric groups*, Bull. London Math. Soc. **18** (1986), 580–586. doi:10.1112/blms/18.6.580.
- [6] A. Enayat, *Automorphisms of models of bounded arithmetic*, Fund. Math. **192** (2006), 37–65. doi:10.4064/fm192-1-3.
- [7] P. Erdős, A. Hajnal, and A. Máté, *Chain conditions on set mappings and free sets*, Acta Sci. Math. (Szeged) **34** (1973), 69–79. Article PDF.
- [8] A. Enayat and V. Kanovei, *An unpublished theorem of Solovay on OD partitions of reals into two non-OD parts, revisited*, J. Math. Log. **21** (2021), no. 3, 2150014. arXiv:2001.11058.
- [9] D. H. Fremlin, *Measure Theory, Volume 3: Measure Algebras*, Torres Fremlin, 2002; online edition, Chapters 33 and 38.
- [10] D. H. Fremlin, *Measure Theory, Volume 5: Set-theoretic Measure Theory*, online edition, Chapter 55, especially 556F–556G.
- [11] H. Friedman, *On definability of nonmeasurable sets*, Canad. J. Math. **32** (1980), no. 3, 653–656. doi:10.4153/CJM-1980-051-3.
- [12] Z. Ghadernezhad and A. Villaveces, *The small index property for homogeneous models in AECs*, Arch. Math. Logic **57** (2018), 141–157. arXiv:1710.02920.
- [13] L. A. Harrington, D. E. Marker, and S. Shelah, *Borel orderings*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 293–302. doi:10.1090/S0002-9947-1988-0965754-3.
- [14] E. Glazer, *Explicit models of arithmetic do not have full standard system*, unpublished manuscript, July 2026.
- [15] E. Glazer, *A surjection from square onto power: Is limit Hartogs/Lindenbaum number necessary?*, MathOverflow answer 456549, 2023. Online answer.
- [16] S. Grigorieff, *Intermediate submodels and generic extensions in set theory*, Ann. of Math. (2) **101** (1975), 447–490. doi:10.2307/1970935.
- [17] T. Jech, *Set Theory*, third millennium edition, revised and expanded, Springer, 2003.

- [18] V. Kanovei and V. Lyubetsky, *Countable OD sets of reals belong to the ground model*, Arch. Math. Logic **57** (2018), 285–298. arXiv:1609.01032.
- [19] V. Kanovei and V. Lyubetsky, *A generic property of the Solovay set Σ* , Siberian Math. J. **58** (2017), no. 6, 1012–1014. arXiv:1611.00176.
- [20] V. Kanovei and S. Shelah, *A definable nonstandard model of the reals*, J. Symbolic Logic **69** (2004), 159–164. arXiv:math/0311165.
- [21] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics **156**, Springer, 1995.
- [22] Z. Kostana, *Homogeneity of the Lévy collapse from the perspective of Fraïssé theory*, preprint, 2026. arXiv:2603.06285.
- [23] D. Lascar and S. Shelah, *Uncountable saturated structures have the small index property*, Bull. London Math. Soc. **25** (1993), 125–131. doi:10.1112/blms/25.2.125.
- [24] M. Malicki, *The automorphism group of the Lebesgue measure has no non-trivial subgroups of index $< 2^\omega$* , Colloq. Math. **133** (2013), 169–174. doi:10.4064/cm133-2-2.
- [25] A. Marcone, *Borel quasi-orderings in subsystems of second-order arithmetic*, Ann. Pure Appl. Logic **54** (1991), no. 3, 265–291. doi:10.1016/0168-0072(91)90050-V.
- [26] I. B. Smythe, *Equivalence of generics*, Arch. Math. Logic **61** (2022), 795–812. arXiv:1810.04704.
- [27] G. Melles and S. Shelah, *A saturated model of an unsuperstable theory of cardinality greater than its theory has the small index property*, Proc. London Math. Soc. (3) **69** (1994), 449–463. doi:10.1112/plms/s3-69.3.449.
- [28] C. Rosendal and S. Solecki, *Automatic continuity of group homomorphisms and discrete groups with the fixed point on metric compacta property*, Israel J. Math. **162** (2007), 349–371. arXiv:math/0604575.
- [29] S. G. Simpson, *Subsystems of Second Order Arithmetic*, second edition, Perspectives in Logic, Cambridge University Press, 2009.
- [30] D. Sullivan, B. Weiss, and J. D. Maitland Wright, *Generic dynamics and monotone complete C^* -algebras*, Trans. Amer. Math. Soc. **295** (1986), 795–809. Article PDF.
- [31] T. Usuba, *The downward directed grounds hypothesis and very large cardinals*, J. Math. Log. **17** (2017), no. 2, 1750009. arXiv:1707.05132.
- [32] J. C., *Countable OD sets in a Cohen extension contain only OD members: A proof by GPT-6 Astra*, unpublished notes, version 2, September 2026.
- [33] *Small-index subgroups of the automorphism group of a Cohen algebra*, GPT-6 Astra output provided to the author by Jason Chen, September 8, 2026.
- [34] D. H. Fremlin, *Measure Theory, Volume 4: Topological Measure Spaces*, online edition, Chapter 45, especially 458N–458P.