

COEFFICIENTWISE POSITIVITY FOR ALTERNATING SUMS OF q -FACTORIALS

CLAUDE (FABLE-5.1), WITH CHATGPT-6 (ASTRA)

ABSTRACT. We prove that the alternating sum $\sum_{i=0}^k (-1)^i \binom{k}{i} [m-i]_q!$ has nonnegative coefficients whenever $m \geq 2k - 1$. For $k \geq 3$, this range is exact. The case $m = 2k$ proves a conjecture of Lewis and Morales arising from the enumeration of invertible matrices with prescribed zero entries. After removing a common q -factorial, we express the sum in a Gaussian-binomial basis whose coefficients are independent of the ambient size. Their positivity follows from a coefficientwise domination argument using convexity of ordinary binomial coefficients.

1. THE CONJECTURE AND THE NORMALIZED SUM

For a nonnegative integer r , write

$$[r] = [r]_q = 1 + q + \cdots + q^{r-1}, \quad [r]! = [1][2] \cdots [r],$$

with $[0] = 0$ and $[0]! = 1$. Throughout, $\mathbb{N}$ includes 0, so membership in $\mathbb{N}[q]$ means coefficientwise nonnegativity.

Lewis and Morales conjectured that

$$\sum_{i=0}^n (-1)^i \binom{n}{i} [2n-i]! \in \mathbb{N}[q] \quad (n \geq 1).$$

This is Conjecture 6.8 of [1], also known as OPAC-012 [2]. The sum occurs, up to a power of q , in the reduced count of invertible $2n \times 2n$ matrices over a finite field with n prescribed zero entries in distinct rows and columns. This counting interpretation explains its nonnegative values at prime powers, but does not imply that its coefficients are nonnegative. We prove the following stronger statement.

THEOREM 1.1. *For integers $0 \leq k \leq m$, set*

$$(1) \quad S_{k,m}(q) = \sum_{i=0}^k (-1)^i \binom{k}{i} [m-i]!.$$

If $m \geq 2k - 1$, then $S_{k,m}(q) \in \mathbb{N}[q]$. For $k \geq 3$, this bound is sharp: $S_{k,m}(q)$ has a negative coefficient whenever $k \leq m \leq 2k - 2$. In particular, $S_{n,2n}(q) \in \mathbb{N}[q]$ for every $n \geq 1$.

The normalization that makes the proof work is to remove the common factor $[m-k]!$. For $a, r \geq 0$, put

$$\Pi_r(a) = \prod_{s=1}^r [a+s], \quad \Pi_0(a) = 1,$$

and define

$$(2) \quad F_t(a) = \sum_{r=0}^t (-1)^{t-r} \binom{t}{r} \Pi_r(a) \quad (t, a \geq 0).$$

Changing the index in (1) gives

$$(3) \quad S_{k,m}(q) = [m - k]! F_k(m - k).$$

Thus the range $m \geq 2k - 1$ becomes $a \geq k - 1$.

The proof now has two steps. In Section 2, we expand $F_t(a)$ as a sum of one explicit term, nonnegative for $a \geq t - 1$, and Gaussian binomial coefficients with universal coefficients $\Theta_{t,j}(q)$. These coefficients do not depend on a . In Section 3, we prove their nonnegativity by showing that the leading positive term dominates all the other terms in their alternating definition. The underlying comparison reduces to convexity of ordinary binomial coefficients. The proof of positivity is then complete; Section 4 establishes sharpness by computing the first two possible nonzero coefficients, and Section 5 records the matrix-counting interpretation.

2. A UNIVERSAL GAUSSIAN-BINOMIAL EXPANSION

For $a \geq 1$ and $j \geq 0$, we use the product formula

$$\left[\begin{matrix} a + j - 1 \\ j \end{matrix} \right]_q = \frac{\prod_{s=0}^{j-1} [a + s]}{[j]!}.$$

For $a = 0$, the right side defines the same notation: it is 1 when $j = 0$ and 0 otherwise. All these polynomials belong to $\mathbb{N}[q]$. We also use $[b] = (1 - q^b)/(1 - q)$ for negative integers b ; identities involving these expressions are understood in $\mathbb{Z}[q, q^{-1}]$.

The universal coefficients in our expansion have the following explicit form. For $t \geq 1$, put $N = t - 1$ and define, for $0 \leq j \leq N$,

$$(4) \quad \Theta_{t,j}(q) = [N]![N] + \sum_{d=1}^{N-j} (-1)^{d+1} q^{N-d} [N - d]! \sum_{s=d}^N \binom{s}{d} q^{N-s}.$$

In particular, $\Theta_{1,0} = 0$. Their nonnegativity is not apparent from this alternating sum; it will be proved in the next section.

PROPOSITION 2.1 (Universal expansion). *For every $t \geq 1$ and $a \geq 0$,*

$$(5) \quad F_t(a) = q^t [a - t + 1] \prod_{r=1}^{t-1} [a + r] + q^{a+1} \sum_{j=0}^{t-1} q^j \left[\begin{matrix} a + j - 1 \\ j \end{matrix} \right]_q \Theta_{t,j}(q).$$

The first summand vanishes at $a = t - 1$ and is nonnegative for $a \geq t - 1$. In the remaining sum, all dependence on a , apart from the common power q^{a+1} , lies in the Gaussian binomial coefficients. Consequently, proving $\Theta_{t,j} \in \mathbb{N}[q]$ will establish $F_t(a) \in \mathbb{N}[q]$ throughout the required range.

Proof. We first isolate the explicit summand by induction on t , and then expand the remainder in Gaussian binomial coefficients. The two identities we need are

$$(6) \quad F_{t+1}(a) = [a+1]F_t(a+1) - F_t(a),$$

$$(7) \quad \Pi_r(a) = \sum_{j=0}^r q^j [r]! \begin{bmatrix} a+j-1 \\ j \end{bmatrix}_q.$$

The first follows from Pascal's identity and $[a+1]\Pi_r(a+1) = \Pi_{r+1}(a)$. For the second, iterate

$$\Pi_r(a) = [r]\Pi_{r-1}(a) + q^r \prod_{s=0}^{r-1} [a+s],$$

which follows from $[a+r] = [r] + q^r[a]$.

To keep the induction readable, abbreviate the inner sums in (4) by

$$(8) \quad \lambda_{N,d} = \sum_{s=d}^N \binom{s}{d} q^{N-s} \quad (0 \leq d \leq N),$$

and set them to zero outside this range. Pascal's identity gives

$$(9) \quad \lambda_{N+1,d} = \lambda_{N,d} + \lambda_{N,d-1} \quad (d \geq 1), \quad \lambda_{N,0} = [N+1].$$

Write the proposed explicit summand as

$$\mu_t(a) = q^t [a-t+1] \Pi_{t-1}(a).$$

For the induction, define a remainder $T_t(a) = \sum_r g_r^{(t)} \Pi_r(a)$, where

$$g_{t-1}^{(t)} = [t-1], \quad g_r^{(t)} = (-1)^{t-r} q^r \lambda_{t-1,t-1-r} \quad (0 \leq r \leq t-2),$$

and $g_r^{(t)} = 0$ otherwise. We claim that

$$(10) \quad F_t(a) = \mu_t(a) + q^{a+1} T_t(a).$$

The base case is $F_1(a) = q[a] = \mu_1(a)$, with $T_1 = 0$. To pass from t to $t+1$, the explicit term satisfies

$$(11) \quad \begin{aligned} [a+1]\mu_t(a+1) - \mu_t(a) &= \mu_{t+1}(a) \\ &+ q^{a+1} ([2t-1]\Pi_{t-1}(a) + \Pi_t(a)). \end{aligned}$$

For completeness, substitution reduces this identity to

$$[a-t+2][a+t] - [a-t+1] - q[a-t][a+t] = q^{a-t+1} ([2t-1] + [a+t]),$$

which follows from $[x] - [y] = q^y[x-y]$. Meanwhile, the remainder transforms according to

$$q[a+1]\Pi_r(a+1) - \Pi_r(a) = q\Pi_{r+1}(a) - \Pi_r(a).$$

Applying (6) to (10), we therefore obtain the desired formula at $t+1$ precisely when

$$(12) \quad g_r^{(t+1)} = [2t-1]\delta_{r,t-1} + \delta_{r,t} + qg_{r-1}^{(t)} - g_r^{(t)},$$

where δ is the Kronecker delta. For $0 \leq r \leq t-2$, this is (9). At $r = t-1$, it follows from

$$[2t-1] - [t-1] = q^{t-1}[t], \quad \lambda_{t,1} = [t] + \lambda_{t-1,1},$$

and at $r = t$, it is $[t] = 1 + q[t - 1]$. This proves (10).

Finally, insert (7) into $T_t(a)$. With $N = t - 1$, the coefficient multiplying $q^j \begin{bmatrix} a+j-1 \\ j \end{bmatrix}_q$ is

$$\sum_{r=j}^{t-1} [r]! g_r^{(t)} = [N]![N] + \sum_{d=1}^{N-j} (-1)^{d+1} q^{N-d} [N-d]! \lambda_{N,d} = \Theta_{t,j},$$

where $d = N - r$. Thus (10) is exactly (5). $\square$

3. POSITIVITY OF THE UNIVERSAL COEFFICIENTS

We now prove that the alternating sum (4) is nonnegative. The useful comparison is stronger than a pairing of consecutive terms: its leading term dominates the sum of all the other terms with their signs removed. To carry out this comparison, fix $N \geq 1$ and package the summands as

$$(13) \quad E_0 = [N]![N], \quad E_d = q^{N-d} [N-d]! \lambda_{N,d} \quad (1 \leq d \leq N),$$

where $\lambda_{N,d}$ is the positive polynomial in (8).

LEMMA 3.1 (Coefficientwise domination). *For $N \geq 1$ and $0 \leq D \leq N$,*

$$E_0 - \sum_{d=1}^D E_d \in \mathbb{N}[q].$$

Proof. We remove the common factorial before comparing coefficients. Define

$$(14) \quad \Lambda_0 = [N], \quad \Lambda_D = [N-D+1] \Lambda_{D-1} - q^{N-D} \lambda_{N,D}.$$

Then induction gives

$$(15) \quad [N-D]! \Lambda_D = E_0 - \sum_{d=1}^D E_d.$$

The point of this normalization is that the low-degree coefficients of Λ_D have a simple form. More precisely, we prove by induction that

$$(16) \quad \Lambda_D = P_D + q^{N-D+1} Q_D, \quad P_D = \sum_{r=0}^{N-D-1} \binom{r+D}{D} q^r, \quad Q_D \in \mathbb{N}[q].$$

Empty sums are zero. The base case is $\Lambda_0 = [N] = P_0$, with $Q_0 = 0$.

Suppose (16) holds for $D-1$, and put $L = N-D$. In the recurrence (14), the part involving Q_{D-1} is already nonnegative. The part to be examined is

$$[L+1]P_{D-1} - q^L \lambda_{N,D}.$$

For $0 \leq r < L$, its coefficient of q^r is

$$\sum_{v=0}^r \binom{v+D-1}{D-1} = \binom{r+D}{D},$$

as required by P_D . At degree $L+u$, for $0 \leq u \leq L$, the coefficient of the product is

$$\sum_{v=u}^L \binom{v+D-1}{D-1} = \binom{N}{D} - \binom{u+D-1}{D},$$

whereas the coefficient of $q^L \lambda_{N,D}$ is $\binom{N-u}{D}$. Their difference is therefore

$$(17) \quad c_u = \binom{N}{D} - \binom{u+D-1}{D} - \binom{N-u}{D}.$$

There are no terms above degree $2L$ in this part of the recurrence.

Here $c_0 = 0$, and convexity gives $c_u \geq 0$ for the remaining values of u . Indeed, the sequence $x \mapsto \binom{x}{D}$ is discretely convex for $x \geq D-1$: for $D \geq 2$ its second forward difference is $\binom{x}{D-2} \geq 0$, and for $D = 1$ it is linear. Hence

$$u \mapsto \binom{u+D-1}{D} + \binom{N-u}{D}$$

is convex on the integer interval $0 \leq u \leq L$ and attains a maximum at an endpoint. Its endpoint values are

$$\binom{N}{D} \quad \text{and} \quad \binom{N-1}{D} + 1 \leq \binom{N}{D},$$

the inequality following from Pascal's identity. This also covers $L = 0$, when both endpoints coincide.

Since $c_0 = 0$, the terms remaining after P_D are divisible by q^{L+1} . Explicitly,

$$Q_D = \sum_{u=1}^L c_u q^{u-1} + q[L+1]Q_{D-1} \in \mathbb{N}[q].$$

This proves (16), and (15) proves the lemma. $\square$

COROLLARY 3.2. *For every $t \geq 1$ and $0 \leq j \leq t-1$, $\Theta_{t,j}(q) \in \mathbb{N}[q]$.*

Proof. The case $t = 1$ is $\Theta_{1,0} = 0$. For $t \geq 2$, put $N = t-1$ and $D = N-j$. Rewriting (4) gives

$$\Theta_{t,j} = \left(E_0 - \sum_{d=1}^D E_d \right) + 2 \sum_{\substack{1 \leq d \leq D \\ d \text{ odd}}} E_d.$$

The first summand is nonnegative by Lemma 3.1, and the second is a sum of nonnegative polynomials. $\square$

Proof of the positivity assertion in Theorem 1.1. For $k = 0$, $S_{0,m} = [m]!$. For $k \geq 1$, put $a = m - k$. The hypothesis $m \geq 2k - 1$ gives $a \geq k - 1$, so Proposition 2.1 and Corollary 3.2 imply $F_k(a) \in \mathbb{N}[q]$. Equation (3) gives $S_{k,m} \in \mathbb{N}[q]$. $\square$

4. SHARPNESS OF THE RANGE

Below the threshold, the obstruction occurs in one of the first two possible nonzero coefficients. We use $[q^d]G$ to denote the coefficient of q^d in a polynomial or formal power series G .

LEMMA 4.1. *Let $t \geq 3$ and $0 \leq a \leq t-2$. Then $[q^d]F_t(a) = 0$ for $0 \leq d \leq a$, and*

$$\begin{aligned} [q^{a+1}]F_t(a) &= (-1)^t, \\ [q^{a+2}]F_t(a) &= \binom{a+1}{t-1} - (-1)^t(t-1). \end{aligned}$$

Proof. Work in $\mathbb{Z}[[q]]$. A product of two distinct factors from $q^{a+1}, q^{a+2}, \dots$ has degree at least $2a + 3$, so

$$\Pi_r(a) = \frac{\prod_{s=1}^r (1 - q^{a+s})}{(1 - q)^r} \equiv \frac{1 - q^{a+1}[r]}{(1 - q)^r} \pmod{q^{2a+3}}.$$

Substituting in (2) and applying the ordinary binomial theorem gives

$$(18) \quad F_t(a) \equiv \frac{q^t}{(1 - q)^t} - q^{a+1} \frac{q^t - (2q - 1)^t}{(1 - q)^{t+1}} \pmod{q^{2a+3}}.$$

Both terms on the right have degree greater than a . At the next two degrees, the contribution of the first term is respectively 0 and $\binom{a+1}{t-1}$, while

$$\frac{(2q - 1)^t}{(1 - q)^{t+1}} = (-1)^t (1 - (t - 1)q + O(q^2)).$$

The remaining numerator term q^t contributes only in degree at least $a + t + 1$. This gives the asserted coefficients. $\square$

Proof of the sharpness assertion in Theorem 1.1. Suppose $k \geq 3$ and $k \leq m \leq 2k - 2$, and put $a = m - k$. If k is odd, Lemma 4.1 and (3) give

$$[q^{a+1}]S_{k,m} = -1.$$

If $k \geq 4$ is even, the coefficient of q in $[a]!$ is $\max(a - 1, 0)$. Since $\binom{a+1}{k-1} \in \{0, 1\}$, the same lemma gives

$$[q^{a+2}]S_{k,m} = \binom{a+1}{k-1} - (k - 1) + \max(a - 1, 0) < 0.$$

Thus there is a negative coefficient in either case. $\square$

5. MATRICES WITH PRESCRIBED ZERO ENTRIES

We conclude by explaining the counting problem behind the conjecture. For a board $B \subseteq \{1, \dots, m\}^2$, let $m_r(B, q)$ be the number of $m \times m$ matrices over $\mathbb{F}_q$ of rank r whose support is contained in B , and write

$$M_r(B, q) = \frac{m_r(B, q)}{(q - 1)^r}$$

for the reduced count. A bar denotes the complement in the square. Suppose B consists of k cells in distinct rows and columns. Choosing a rank- r matrix supported on B amounts to choosing r cells and assigning a nonzero entry to each, so $M_r(B, q) = \binom{k}{r}$. The complement identity of Lewis and Morales [1, Corollary 2.2] then gives

$$(19) \quad M_m(\overline{B}, q) = q^{\binom{m}{2} - k} S_{k,m}(q).$$

For $m \geq 3$, the power of q is nonnegative because $k \leq m$. Thus Theorem 1.1 proves coefficientwise positivity of this reduced count when $m \geq 2k - 1$. The cases $m \leq 2$ in this range follow directly; in particular, the count for $m = k = 1$ is zero.

For a permutation $w = w_1 \cdots w_m \in \mathfrak{S}_m$, the diagram convention in [1] is the coinversion diagram

$$I_w = \{(i, w_j) : i < j, w_i < w_j\}.$$

The permutation

$$v_n = (2n - 1)(2n)(2n - 3)(2n - 2) \cdots 3412$$

has n diagram cells in distinct rows and columns. Hence

$$(20) \quad M_{2n}(\overline{I_{v_n}}, q) = q^{2n(n-1)} S_{n,2n}(q),$$

which is the family in Conjecture 6.8 of [1].

The same argument applies to $w = w_0 s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_m$, where $w_0 = m \cdots 21$, s_i swaps adjacent positions, and the distinct indices i_r are pairwise nonadjacent. Such a permutation is a sequence of decreasingly ordered blocks, each of size one or two, with every size-two block increasing. It therefore avoids 123, and its coinversion diagram consists of k cells in distinct rows and columns. Since $m \geq 2k$, (19) yields $M_m(\overline{I_w}, q) \in \mathbb{N}[q]$. This establishes this family of cases of the broader 123-avoiding positivity conjecture [1, Conjecture 6.9].

At $q = 1$, the normalized sum $F_t(a)$ counts injections $\{1, \dots, t\} \rightarrow \{1, \dots, t + a\}$ having no fixed points, by inclusion–exclusion. The expansion (5) provides a positive q -analogue in the range $a \geq t - 1$. A statistic on these injections, or a direct interpretation of the universal coefficients $\Theta_{t,j}$, would explain combinatorially the positivity proved here by coefficient comparison.

REFERENCES

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