

Primitive Integer Solutions of $x^4 + 3y^3 + 2z^3 = 0$

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Abstract

We determine the primitive integer solutions of $x^4 + 3y^3 + 2z^3 = 0$. They are exactly $(1, -1, 1)$ and $(-1, -1, 1)$, where primitivity means $\gcd(x, y, z) = 1$. A fourth-power descent in $\mathbb{Q}(\sqrt[3]{12})$ produces four plane quartics. A congruence modulo 4 excludes two of them for primitive input. We determine the rational points on the other two by elliptic descent, finite reduction sieves, and elliptic Chabauty. For the final quartic, an explicit nonzero Cassels–Tate pairing sharpens a rank-three descent bound to rank one. All finite certificates and their verification programs accompany the proof.

1. The theorem and the cubic-field descent

Theorem. The complete set of integer solutions satisfying

$$x^4 + 3y^3 + 2z^3 = 0, \quad \gcd(x, y, z) = 1$$

is

$$(x, y, z) = (1, -1, 1), \quad (-1, -1, 1).$$

This is the $H = 56$ equation in Table 4 of Grechuk’s public list [G].

Put $X = x$, $Y = -y$, and $Z = -z$. Then

$$2Z^3 + 3Y^3 = X^4. \tag{1}$$

A primitive solution has $X \neq 0$ and $\gcd(Y, Z) = 1$. If X were even, then Y would be even and Z odd, giving valuation one at 2 on the left. Thus X, Y are odd. If $3 \mid X$, then $3 \mid Z$, $3 \nmid Y$, and the left side has valuation one at 3. Consequently

$$\gcd(X, 6) = 1. \tag{2}$$

Let

$$\begin{aligned} L = \mathbb{Q}(s), \quad s^3 = 12, \quad t = s^2/2, \\ \pi = 2 + s + t, \quad \rho = 3 + s + t, \quad \nu = 1 + 3s - 3t. \end{aligned}$$

The multiplication and norm formulas are

$$\begin{aligned} s^2 &= 2t, & st &= 6, & t^2 &= 3s, \\ N(a + bs + ct) &= a^3 + 12b^3 + 18c^3 - 18abc. \end{aligned} \tag{3}$$

The integer ring is $\mathcal{O}_L = \mathbb{Z}[1, s, t]$, of discriminant -972 . At 2, $t^3 = 18$ is Eisenstein; at 3, $s^3 = 12$ is Eisenstein. These uniformizer bases give local maximality, and the discriminant excludes other index primes. The field has class number one. Indeed, the Minkowski bound is below 9, and every prime ideal of norm at most 8 is principal: the required generators are π , ρ , and $5 + 2s + 2t$, of norms 2, 3, 5. The polynomial is irreducible modulo 7.

The full unit group is $\{\pm \nu^n : n \in \mathbb{Z}\}$, with

$$\nu^{-1} = 55 + 24s + 21t.$$

Here is an elementary certificate for the unit assertion. A positive unit $1 < \eta < \nu^{-1} < 166$ has its two complex absolute values below 1. Fourier inversion, using $16/7 < s < 23/10$, bounds its integral coefficients by $0 \leq a \leq 55$, $0 \leq b \leq 24$, $0 \leq c \leq 21$. The only norm-one elements in this box are 1 and ν^{-1} . Multiplication by powers of ν proves the assertion. These field certificates are regenerated by `prior_D2/verify_mixed_descent.py`.

Set $\alpha = 2Z + sY$. Its norm is $4X^4$. At the unique prime over 2, its valuation is exactly 2; at 3 it is a unit. At any other prime ideal dividing it, both Y, Z are residue units and $s = -2Z/Y$ belongs to the prime field. Thus the prime ideal has degree one, and at most one prime above the same rational prime divides α . Its valuation is divisible by four. Class number one and the unit assertion therefore give

$$2Z + sY = \lambda_j \beta^4, \quad \lambda_j = \pi^2 \nu^j, \quad \beta = a + bs + ct, \quad j \in \{0, 1, 2, 3\}. \tag{4}$$

The unit sign is positive by the norm. Primitivity implies $\gcd(a, b, c) = 1$, and β is a unit above 2, 3. In particular, a is odd, $3 \nmid a$, and

$$|X| = |N\beta|. \tag{5}$$

Let D_j be the projective quartic defined by the vanishing of the t -coefficient in $\lambda_j(a + bs + ct)^4$.

1.1 Two cases collapse modulo four

In the basis $1, s, t$, the four multipliers are

j	$[1]\lambda_j$	$[s]\lambda_j$	$[t]\lambda_j$
0	16	7	6
1	-2	1	0
2	-20	-5	12
3	286	-173	42

For $j = 0, 3$, expansion gives the polynomial congruence

$$[\lambda_j(a + bs + ct)^4]_t \equiv 2a^4 \pmod{4}. \quad (6)$$

For odd a this coefficient cannot vanish. Hence primitive solutions come only from D_1 or D_2 .

The proof below establishes

$$\boxed{D_1(\mathbb{Q}) = \{(1 : 0 : 0)\}}, \quad \boxed{D_2(\mathbb{Q}) = \{(2 : 1 : 1), (3 : 1 : 1)\}}. \quad (7)$$

The coefficients of D_1 , after dividing by 4, are

$$\begin{aligned} F(a, b, c) = & 2a^3b - 2a^3c - 6a^2b^2 + 9a^2c^2 + 36ab^2c - 36abc^2 \\ & + 6b^4 - 24b^3c + 36bc^3 - 9c^4. \end{aligned} \quad (8)$$

2. The elliptic maps

For a point on D_j , write $\lambda_j\beta^4 = 2Z + sY$, and set

$$n = N\beta, \quad v = n/\beta^3, \quad w = \frac{sY - 2Z}{sY + 2Z}.$$

Taking norms gives $16v^4 = \lambda_j^3(1 + 3w^2)$. Consequently

$$\phi_j : D_j \longrightarrow E_j : \quad V^2 = U^3 - 9\lambda_j U, \quad (U, V) = (12\lambda_j^{-1}v^2, 18wv). \quad (9)$$

Put

$$h = n/\beta = (a^2 - 6bc) + (3c^2 - ab)s + (2b^2 - ac)t.$$

On the projective Weierstrass model the homogeneous coordinates are

$$[12\lambda_j^{-1}h^2\beta^2 : 18\lambda_j^{-1}(sY - 2Z)h : \beta^6]. \quad (10)$$

At a split good prime, two conjugates of β cannot vanish on D_j : they would force $Y = Z = 0$ and then all three conjugates to vanish. If the chosen conjugate of β vanishes, h and the middle coordinate in (10) are nonzero; otherwise the last coordinate is nonzero. Thus the maps have good integral reduction, including at apparent poles of (9). The pullback of the invariant differential is

$$\phi_j^* \frac{dU}{2V} = \frac{2dv}{3\lambda_j w}. \quad (11)$$

3. The decisive rank-one calculation for D_1

Since $\lambda_1 = s - 2$, put $k = 9(s - 2) > 0$ in the real embedding. The curves connected by the standard two-isogeny are

$$E : V^2 = U^3 - kU, \quad E' : V^2 = U^3 + 4kU. \quad (12)$$

Write α_E and $\alpha_{E'}$ for the two U -coordinate descent images, with the usual coefficient class at $(0, 0)$. Every class in either image has even valuations outside the primes over 2, 3. The class-number and unit calculations reduce all possibilities to

$$\mathcal{V} = \langle -1, \nu, \pi, \rho \rangle \subset L^*/L^{*2}, \quad \dim_{\mathbb{F}_2} \mathcal{V} = 4. \quad (13)$$

3.1 The dual image has dimension one

At 3, use the valuation normalized by $v_s(s) = 1$. The coefficient $4k$ has valuation 6, with leading residue 1. For a nonzero U -coordinate of $E'(L_3)$, put $m = v_s(U)$. If $m < 3$, the leading term is U^3 ; if $m > 3$, it is $4kU$. In each case the square condition forces m even and the leading unit of U to be a square. If $m = 3$, the leading units of U^2 and $4k$ add to 2 modulo 3; the right side has odd valuation 9, impossible. Thus every local descent class of E' at 3 is trivial.

In the basis (13), the local square-class labels at 3 are $(1, 0, 1, 2)$, where the first bit denotes the nonsquare unit and the second bit denotes odd valuation. At the real place U is positive. These two conditions leave only 1 and ν . The torsion point of E' has class $4k = (6\pi)^2\nu$. Therefore

$$\alpha_{E'} = \{1, \nu\}. \quad (14)$$

On E , let

$$T = (0, 0), \quad P = (12 + 6s + 6t, 90 + 36s + 36t). \quad (15)$$

Their descent classes are $-\nu$ and ρ , respectively, because

$$-k = -(3\pi)^2\nu, \quad U(P) = \rho s^2.$$

They are independent in $\mathcal{V}$. Thus $\dim \alpha_E \geq 2$. We prove equality with one explicit Cassels–Tate pairing.

3.2 The two covering classes used in the pairing

For $d \in L^*/L^{*2}$, the first covering is

$$C_d : \quad r^2 = du^4 - \frac{k}{d}v^4. \quad (16)$$

It maps to E by $U = d(u/v)^2$, $V = dru/v^3$, and its Jacobian is E' . Let S be the corresponding isogeny Selmer group. It is contained in $\mathcal{V}$.

Both $d = -1$ and $d = \pi$ are everywhere locally soluble. Here are explicit Hensel witnesses at the two ramified places; in every row $v = 1$. Triples are coefficients in $1, s, t$.

d	place	u	approximate root r_0	$v(r_0)$	$v(r_0^2 - \text{RHS})$
-1	2	1	(25, 27, 31)	0	18
-1	3	0	(21, 15, 24)	3	12
π	2	1	(28, 8, 25)	1	21
π	3	0	(12, 21, 0)	3	12

At 2 the valuation is normalized by $v_t(t) = 1$; at 3 by $v_s(s) = 1$. Each error exceeds twice the derivative valuation, so Hensel's lemma gives an actual root. The respective root-error bounds are 15, 9, 17, 9. Outside 2, 3 the quartics have smooth genus-one reduction; the Hasse bound and Hensel's lemma give local points. Real points are immediate from $k, \pi > 0$, and the complex place poses no restriction.

The four classes $-\nu, \rho, -1, \pi$ are independent and belong to S . Consequently $S = \mathcal{V}$ has dimension four.

3.3 An explicit pushout function

Set

$$A = 3 - 3s + 3t, \quad B = 9 + 6s - 6t.$$

The key identity is

$$A^2 + B^2 = 25k. \tag{17}$$

On $C_{-1} : r^2 = -u^4 + kv^4$, define

$$g = Au^2 + Br - 5kv^2, \quad f = g/v^2.$$

Then

$$g(Au^2 + Br + 5kv^2) = -(Ar - Bu^2)^2. \tag{18}$$

The conic $U^2 + R^2 = kV^2$ has the point $(A : B : 5)$, and $g = 0$ is its tangent pulled back by $(U : R : V) = (u^2 : r : v^2)$. Hence $\text{div}(f) = 2\mathfrak{d}$ for a degree-zero divisor $\mathfrak{d}$ of order two. This class is nonzero: if f were a constant times a square, its square root would have poles bounded by the two points at infinity, whose Riemann–Roch space is spanned by $1, u/v$. Its square could not have the nonzero Br/v^2 term. The Jacobian E' has exactly one nonzero rational two-torsion point, since $U^2 + 4k$ has no root in the real embedding. Thus f is a pushout function for that point.

The Cassels–Tate pairing on S is alternating, its radical contains α_E , and its value on $(-1, \pi)$ is the product of Hilbert symbols $(f(P_{\mathfrak{p}}), \pi)_{\mathfrak{p}}$ [F, Theorem 2.1 and Section 3]. We now compute that product exactly.

3.4 The local Hilbert-symbol certificate

Above 2. Use $u = v = 1$ and the first Hensel row. Substituting its approximate root gives

$$g_0 = 462 - 213s + 456t, \quad v_t(g_0) = 2.$$

The exact root differs from r_0 by valuation at least 15, so this substitution preserves the required square class. Let

$$\begin{aligned} N_1 &= (-2 - 2s - 2t)^2 - \pi = 50 + 19s + 15t, \\ N_2 &= (-2 - s - t)^2 - \pi = 14 + 6s + 5t. \end{aligned}$$

Both are norms from $L_2(\sqrt{\pi})$. The quotient $g_0/(N_1 N_2)$ is a unit whose coefficient vector modulo 8 is $(1, 2, 6)$. It is the square of $(1, 3, 2)$ in $\mathcal{O}_L/8\mathcal{O}_L$. Since $v_t(8) = 9 > 2v_t(2) = 6$, this square root lifts. The error in g preserves the same residue modulo 8 after normalization. Thus f is a local norm and the Hilbert symbol is $+1$.

Above 3. Use $u = 0, v = 1$ and the second Hensel row. Then

$$g_0 = 603 - 216s + 270t, \quad v_s(g_0) = 6,$$

and g_0/s^6 reduces to 1. Its square root lifts. The root-error bound preserves its valuation and leading residue, so f itself is a square in L_3 . The symbol is again $+1$.

Above 5. The factorization is

$$s^3 - 12 \equiv (s - 3)(s^2 + 3s + 4) \pmod{5}.$$

At the degree-one prime, $s \equiv 8 \pmod{25}$ and $t \equiv 7 \pmod{25}$. Thus $A \equiv 0, B \equiv 15 \pmod{25}$, while $k \equiv 4 \pmod{5}$. The local point $u = 0, v = 1, r \equiv 2 \pmod{5}$ gives $g/5 \equiv 2 \pmod{5}$: its valuation is one. Also $\pi \equiv 2 \pmod{5}$ is nonsquare. The Hilbert symbol is -1 . At the degree-two prime, $\pi \equiv 2s = (1 + 2s)^2$ in $\mathbb{F}_5[s]/(s^2 + 3s + 4)$, so the symbol is $+1$.

Every other place. At the real place $\pi > 0$, and complex symbols are trivial. At an odd prime outside $2, 3, 5$, normalize u, v integrally with at least one a unit. The element k is a unit because $N(k) = 2916$. If v is a unit and g vanishes modulo the prime, the other factor in (18) is a unit, so $v(g)$ is even. If v is not a unit, u is a unit; g cannot vanish, by $A^2 + B^2 = 25k$. Hence $v(f)$ is even in every case. As π is also a unit, the symbol is $+1$.

The product is therefore

$$\boxed{\langle -1, \pi \rangle = -1.} \tag{19}$$

3.5 Rank and odd index

A nonzero alternating pairing on a four-dimensional space has rank at least two. Since the two independent actual classes $-\nu, \rho$ belong to its radical, the radical has dimension exactly two. Hence $\dim \alpha_E = 2$. Together with (14), the two-isogeny rank formula gives

$$\boxed{\text{rank } E(L) = 2 + 1 - 2 = 1.} \quad (20)$$

The reductions at degree-one primes over 5, 11 have orders 4, 12. They bound global torsion by four. The only nonzero rational two-torsion point is T , since $k = (3\pi)^2\nu$ is nonsquare. A half of T would have $U^2 = -k$ by the doubling formula, impossible in the real embedding. Thus $E(L)_{\text{tors}} = \{O, T\}$.

The natural map $E(L)/2E(L) \rightarrow \alpha_E$ is a surjection between two-dimensional spaces. The classes of P, T span the target, so they span the source. Therefore $\langle P, T \rangle$ has finite odd index in $E(L)$.

4. The complete 331 sieve and local uniqueness on D_1

The prime 331 splits completely in L , with roots

$$s \equiv 7, 107, 217 \pmod{331}. \quad (21)$$

Each elliptic reduction has $332 = 4 \cdot 83$ points. The reduction of P in the product has order 166; adding T gives a subgroup H of order 332. Its quotient in the product has order

$$332^3/332 = 110224 = 16 \cdot 83^2.$$

The point $[4]P$ reduces in the three factors to

$$(276, 40), \quad (118, 216), \quad (94, 179). \quad (22)$$

These are nonzero. If the odd index of $\langle P, T \rangle$ were divisible by 83, then $P = 83R + eT$ for a global R , and $[4]P$ would reduce to zero in all factors. Thus the index is prime to 83. It is consequently prime to $|E(\mathbb{F}_{331})^3/H|$, and

$$\text{red}_{331} E(L) = H. \quad (23)$$

Complete enumeration of all $331^2 + 331 + 1 = 109893$ projective residue points finds 332 on D_1 . Under (10), only

$$(1 : 0 : 0) \quad (24)$$

maps into H . Polynomial evaluation and independent multiplication in $\mathbb{F}_{331}[s]/(s^3 - 12)$ give the same projective list. All $3 \cdot 332^2 = 330672$ elliptic additions agree in affine and Jacobian representations. The rational formula, homogeneous formula and interpolation of the three conjugates give identical images, including the apparent poles.

It remains to prove uniqueness in the disk of (24). The quartic is smooth modulo 331: the equation and its partial derivatives generate the unit ideal in each of its three affine charts. In the chart $a = 1$, use c as parameter. At the center, the partial derivative with respect to b is 8, and the tangent vector is $(0, 1, 1)$.

For $\omega = dU/(2V)$, compute the three elliptic logarithms by $\log P = \log([166]P)/166$. The roots in (21) lift modulo 331^3 to

$$13040745, \quad 28244337, \quad 31244300.$$

The normalized logarithm is

$$\frac{\log P}{331} \equiv (99490, 6698, 96118) \pmod{331^2},$$

and reduces to $(190, 78, 128)$. An exact integral annihilator is $(0, L_3, -L_2)$, where L_i are the exact entries of $\log P/331$. It reduces to

$$(0, 128, 253). \tag{25}$$

Direct differentiation of (9) gives the exact coefficient $-s$ at the center in the parameter c . Its three residues are $(324, 224, 114)$, and their dot product with (25) is

$$\boxed{251 \not\equiv 0 \pmod{331}}. \tag{26}$$

All map denominators and elliptic V -coordinates are units in this disk. The rank-one calculation implies that the annihilator vanishes on every global elliptic logarithm. Section 6's local zero lemma therefore gives exactly one rational point in the disk. Equations (23)–(26) prove

$$D_1(\mathbb{Q}) = \{(1 : 0 : 0)\}. \tag{27}$$

5. The D_2 certificate, included in the proof

For completeness this section gives the other rational-point classification in (7), with its executable certificates in `prior_D2/`. Its elliptic curve is

$$E_2 : V^2 = U(U - r)(U + r), \quad r = 3\pi\nu.$$

5.1 Full two-descent

The Kummer map is $(U, U - r)$ modulo squares, with values $(-r^2, -r)$ and $(r, 2r^2)$ at the two generators of full two-torsion. Each factor belongs to $\mathcal{V}$ of (13), so there are 256 global candidate pairs. At 2, the local Kummer image has dimension $[L_2 : \mathbb{Q}_2] + 2 = 5$. At 3 it has dimension two.

Here are three further actual local points at 2, besides torsion. Their right sides have valuation zero, and the last column supplies a unit square root modulo 8 in the basis $1, s, t$.

U	$U(U^2 - r^2)$ modulo 8	square root modulo 8
$(-7, -7, -4)$	$(1, 4, 0)$	$(1, 2, 0)$
$(-7, -6, -6)$	$(1, 7, 6)$	$(1, 2, 3)$
$(-7, -4, -4)$	$(5, 1, 0)$	$(1, 3, 3)$

The five Kummer vectors are independent and exhaust the local image. The two torsion vectors exhaust the image at 3. At infinity, the two signs must agree. With masks in the order $(-1, \nu, \pi, \rho)$, the complete global survivor list is

(0,0) (0,2) (1,13) (1,15)
(6,8) (6,10) (7,5) (7,7)
(10,12) (10,14) (11,1) (11,3)
(12,4) (12,6) (13,9) (13,11)

They are spanned by torsion and the two actual points

$$P_2 = (-2 + 2s - t, 4 + 12s - 12t),$$

$$R_2 = (9 - 6s + 2t, -33 + 18s - 3t).$$

The four global Kummer pairs are (1, 13), (12, 6), (7, 7), (0, 2). Their span has size 16, proving rank two and odd index for these points and full two-torsion. Finite orders 8, 12 at degree-one primes over 5, 11 show that the global torsion is exactly the full two-torsion.

5.2 Two compatible reduction sieves

The primes 211 and 7207 split completely. At both, the elliptic factors have order $p + 1$ and structure $\mathbb{Z}/((p + 1)/2) \times \mathbb{Z}/2$. In the product, the subgroup generated by P_2, R_2 and torsion has the following index:

p	$\#D_2(\mathbb{F}_p)$	product subgroup index	surviving points
211	212	$424 = 8 \cdot 53$	3
7207	7208	$14416 = 16 \cdot 17 \cdot 53$	7

The minors of the free-generator coordinate matrix are (15, 27, 2) modulo 53 at 211, and (10, 10, 3) modulo 17 and (3, 8, 43) modulo 53 at 7207. Their nonvanishing proves saturation at 17, 53. Odd index then proves that the computed product subgroups are the full global reduction images.

At 211, the surviving projective points are

$$(1 : 22 : 49), \quad (1 : 106 : 106), \quad (1 : 141 : 141).$$

Their coordinates in the global free generators modulo 53 are

$$(30, 32), \quad (1, 0), \quad (52, 0).$$

The seven points at 7207 have coordinates

$$(2, 4), (27, 51), (1, 0), (52, 12), (52, 0), (9, 30), (4, 3).$$

Compatibility leaves only (1, 0), (52, 0), eliminating the first 211 disk. The remaining disks are the reductions of (2 : 1 : 1), (3 : 1 : 1). Every projective point is checked; an independently organized C++ field-multiplication enumeration agrees with the polynomial enumeration.

5.3 Local uniqueness at 211

The three roots are 38, 63, 110, with lifts modulo 211^3

$$614892, \quad 953783, \quad 7825256.$$

The normalized logarithms, using the finite exponent 106, are

$$\begin{aligned} \log(P_2)/211 &\equiv (19201, 14981, 36081) \pmod{211^2}, \\ \log(R_2)/211 &\equiv (9872, 44497, 23895) \pmod{211^2}. \end{aligned}$$

The first row is divisible by 211. Taking the cross product of the exact rows $\log(P_2)/211^2$ and $\log(R_2)/211$ gives an integral annihilator reducing to $(200, 22, 167)$. In the chart $b = 1$, the two pullback coefficients from (11) are

point	parameter	exact coefficient	combined residue
$(2 : 1 : 1)$	$a - 2$	$3 + s + t$	178
$(3 : 1 : 1)$	$a - 3$	$2 + s + 2t/3$	94

The dependent-coordinate derivatives are $-16, -24$, both units. The maps are integral and the elliptic V -coordinates are units on both disks. Section 6 proves uniqueness in each disk, giving

$$D_2(\mathbb{Q}) = \{(2 : 1 : 1), (3 : 1 : 1)\}. \quad (28)$$

6. The common local zero lemma and the logarithm precision

Let p be either 211 or 331. Compose the three elliptic logarithms with the exact integral annihilator. On a surviving disk, center a regular parameter T at the known rational point. The pullback of the invariant differential is integral with unit constant coefficient. Its integral is

$$F(T) = uT + \sum_{n \geq 2} \frac{a_{n-1}}{n} T^n, \quad u \in \mathbb{Z}_p^*, \quad a_{n-1} \in \mathbb{Z}_p, \quad T \in p\mathbb{Z}_p.$$

It vanishes at every rational point in the disk. For $T \neq 0$ and $n \geq 2$,

$$v_p \left(\frac{a_{n-1}}{n} T^n \right) \geq nv_p(T) - v_p(n) > v_p(T).$$

These valuations tend to infinity. The ultrametric inequality implies that the convergent tail cannot cancel the linear term. Hence $F(T) = 0$ forces $T = 0$. This is the all-heights uniqueness argument used in both cases.

For the models $V^2 = U^3 + bU$ and parameter $\tau = -U/V$, the exact formal differential is

$$\frac{dU}{2V} = \frac{d\tau}{\sqrt{1 - 4b\tau^4}}, \quad \log(\tau) = \tau + O(\tau^5).$$

The scalar entering the formal group is prime to p . Thus computations modulo p^3 give the normalized logarithms to the stated precision. The programs compute each value using both Jacobian projective arithmetic and independent division-polynomial recurrences.

7. Reconstruction

By (6), only $j = 1, 2$ can occur for primitive input. The complete point lists (27)–(28) give these primitive integral representatives:

j	(a, b, c)	$N\beta$
1	(1, 0, 0)	1
2	(2, 1, 1)	2
2	(3, 1, 1)	3

A primitive integral representative of a rational projective point is unique up to sign, and the fourth power removes that sign. Conditions (2) and (5) eliminate the last two rows. The first gives

$$2Z + sY = s - 2, \quad (X, Y, Z) = (\pm 1, 1, -1).$$

Undoing the sign changes gives precisely $(x, y, z) = (\pm 1, -1, 1)$. Direct substitution verifies both points. This proves the theorem.

8. Reproduction

Run from the directory containing this note:

```
python run_all.py
```

The requirements are Python 3.10 or later, SymPy, NumPy, and a C++17 compiler. Run with Python assertions enabled. The driver regenerates the certificates from the displayed constants and runs these stages:

program	finite certificate
prior_D2/verify_mixed_descent.py	integer ring, class number, units, E_2 descent
prior_D2/verify_mixed_sieve.py	complete 211, 7207 sieves and compatibility
prior_D2/verify_mixed_chabauty.py	both D_2 local uniqueness certificates
verify_reduction.py	all four quartics, mod-4 exclusions, reconstruction
verify_rank.py	exact pairing witnesses, isogeny images, rank-one data
verify_sieve.py	complete 331 sieve in independent representations
verify_chabauty.py	smoothness, logarithms, and the unit derivative
	251

The mathematical implications use unique ideal factorization, Minkowski's bound, Mordell–Weil, Kummer descent and the two-isogeny rank formula, Hensel's lemma, the Hasse bound, Hilbert symbols, the Cassels–Tate pairing, and elliptic logarithms. Sections 1–7 specify their applications and connect the finite certificates to the all-heights conclusion.

References

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