

ON PROPERTIES OF PLANCHEREL ALGEBRAS

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ABSTRACT. We proved the predictions by Ben-Zvi, Sakellaridis, and Venkatesh that the Plancherel algebra PL_X is commutative and its loop-rotated version $\mathrm{PL}_{X,h}$ is flat over $k[\hbar]$.

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1. INTRODUCTION

The Plancherel algebra records the local multiplicities of the spherical Hecke action on a variety with reductive group action. In the relative Langlands framework of Ben-Zvi, Sakellaridis, and Venkatesh [BZSV24], its expected spectral description relates these multiplicities to a dual Hamiltonian space. For hyperspherical cotangent spaces, their conjectures predict two fundamental properties: commutativity of the unrotated algebra [BZSV24, Conjecture 8.1.8] and flatness of loop-rotated algebra over $k[\hbar]$ [BZSV24, Conjecture 8.5.2]. These predictions are restated by Liu and Wang in [LW26, Conjecture 4.36(1)–(2)].

1.1. Main results. Let $F = \overline{\mathbb{F}}_q$, and let k be a characteristic-zero ℓ -adic coefficient field, with $\ell \neq \mathrm{char} F$. We work in the sheaf theory and super

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linear algebra conventions of [LW26]. Let G be a connected reductive group over F , and write $\check{G}$ for its Langlands dual over k . Put

$$\begin{aligned}\mathcal{O} &= F[[t]], & \mathcal{K} &= F((t)), & T_{\text{rot}} &= \mathbb{G}_m, \\ R &= H^*(BT_{\text{rot}}, k) = k[\hbar], & |\hbar| &= 2.\end{aligned}$$

Loop rotation acts by $t \mapsto at$. For a smooth affine G -variety X , let $\text{PL}_{X,\hbar}$ and PL_X denote its rotated and unrotated Plancherel algebras. We regard these as algebras in the presentable derived category $\mathcal{C}$ of rational super $\check{G}$ -representations, using convention of [LW26, §4.3.3]. Their precise definition, including their multiplication, is recalled in Section 2.2.

Theorem 1.1 (Flatness). *Let $H \subset G$ be a connected reductive subgroup, let Y be a finite-dimensional H -representation, and set $X = G \times^H Y$. Then $\text{PL}_{X,\hbar}$ is flat over R . If k is a splitting coefficient field for $\check{G}$, its graded cohomology is a free R -module, and there is a noncanonical quasi-isomorphism of underlying R -module complexes*

$$(1.1) \quad \text{PL}_{X,\hbar} \simeq \bigoplus_{\alpha \in I} R[-d_\alpha].$$

Here flatness refers to the graded cohomology over the parameter ring. It implies that multiplication by $\hbar$ is injective on cohomology and that derived specialization has no additional Tor contribution. The splitting in (1.1) concerns modules and does not assert formality of the algebra.

Taking $Y = 0$ gives every homogeneous space G/H with H connected reductive. Taking $H = G$ gives every finite-dimensional G -representation.

Theorem 1.2 (Commutativity). *Let X be a smooth affine G -variety. There is a unital commutative differential graded algebra B_X in rational super $\check{G}$ -representations and an equivalence*

$$\text{PL}_X \simeq B_X \quad \text{in } \text{Alg}_{E_1}(\mathcal{C}).$$

In particular, the equivalence preserves the internal-endomorphism multiplication, unit, and $\check{G}$ -action.

Commutativity uses the symmetry $(-1)^{rs+\epsilon\eta}$ on elements of cohomological degrees r, s and super parities ϵ, η . Theorem 1.2 asserts the existence of a commutative dg model; it does not identify that model with its cohomology algebra or assert that the choice is canonical.

Corollary 1.3 (Hyperspherical cotangent spaces). *Let X be smooth affine, and suppose that T^*X , with its canonical Hamiltonian G -action, is hyperspherical with the untwisted distinguished polarization and characteristic restrictions of Section 2.3. Then $\text{PL}_{X,\hbar}$ is flat over $k[\hbar]$, and PL_X is commutative. Thus the flatness and commutativity predictions of [BZSV24, Conjectures 8.1.8 and 8.5.2] hold.*

The geometric input for the corollary is the presentation $X = G \times^H Y$ supplied by the polarized hyperspherical datum. We recall this input in Section 2.3, including its interpretation in positive characteristic.

1.2. Methods and relation to previous work. For an irreducible representation V of $\check{G}$, let $\mathcal{T}_V$ be its Satake complex and let δ_X be the constant sheaf on $X_{\mathcal{O}}$, extended to $X_{\mathcal{K}}$. The corresponding rotated multiplicity is

$$M_{V,h} = R\mathrm{Hom}_{\mathrm{Sat}_{X,h}}(\mathcal{T}_V * \delta_X, \delta_X).$$

For $X = G \times^H Y$, the equivalence $[X/G] \simeq [Y/H]$ expresses this complex through the space of triples for (H, Y) . On bounded Hecke support, the space has a finite model stratified by vector bundles over $H_{\mathcal{O}}$ -orbits in Gr_H . The Satake parity theorem gives a single parity on all strata. The rotation stabilizers have even equivariant cohomology, free over R , and the resulting localization sequences split.

This extends the filtration argument of Braverman–Finkelberg–Nakajima [BFN18, Lemmas 2.5–2.6] to Satake coefficients. The identification of the multiplicities is based on the finite-model calculation of Ben-Zvi–Sakellaridis–Venkatesh [BZSV24, Proposition 8.3.1]. We record both its rotation equivariance and the ordinary Verdier duality normalization needed for the parity argument.

For commutativity, we use the factorizable Hecke module of [BZSV24, Proposition 16.2.8] together with the coefficient factorization categories of [Lys24]. Relative rigidity allows the internal endomorphisms of the basic object to commute with diagonal restriction and exterior factorization. Taking the total tensor product of the coefficient labels produces a factorization associative family of $\mathcal{C}$ -valued sheaves. Finite-object Verdier duality compares its singleton algebra with PL_X , through an opposite algebra and the Cartan pullback.

The geometric factorization construction is an input to this argument. The introduction to [BZSV24, §16] notes that some technical details of that formalism are not verified there, and Proposition 16.2.8 is presented without a detailed proof. Our commutativity result depends on this input; we do not reconstruct the underlying sheaf theory of factorizable loop spaces here.

The passage from the geometric finite-field setting to topological factorization takes place in the categories generated by closed coordinate diagonals. We compare these categories, with their six-functor diagrams, across characteristic and with Betti sheaves. A compact-support construction then gives a unital E_3 -enhancement whose specified associative restriction is the required internal-endomorphism algebra. This is the fusion–convolution mechanism of [Noc20, Lur17], applied with the coefficient action and units retained. Finally, relative formality of $E_1 \rightarrow E_3$ [FW20, Theorem D'] supplies the commutative dg model. Relative formality is needed here to control the given associative product.

1.3. Organization. Section 2 fixes the conventions and geometric input. Sections 3–5 prove Theorem 1.1. Sections 6 and 7 construct and compare the factorization family. Section 8 proves Theorem 1.2 and Corollary 1.3. All numbered references to preprints use the versions specified in the bibliography.

2. CONVENTIONS AND GEOMETRIC INPUT

2.1. Sheaves, coefficients, and signs. We use the geometric sheaf theory of [LW26, §§2.2–2.4 and 4.3]. For a connected reductive group J , write

$$J_{\mathcal{O}} = J(F[[t]]), \quad J_{\mathcal{K}} = J(F((t))), \quad \mathrm{Gr}_J = J_{\mathcal{K}}/J_{\mathcal{O}};$$

these expressions also denote the corresponding arc and loop functors. We use left actions. Passing from the right-action notation of [LW26] to this notation does not change the constructions.

Let $\mathcal{V} = D(k\text{-supervect})$, and let $\mathcal{C} = \mathrm{Rep}_k(\check{G})$ denote the presentable derived category of rational super representations. The symmetry on homogeneous objects of cohomological degrees r, s and super parities ϵ, η has sign

$$(2.1) \quad (-1)^{rs+\epsilon\eta}.$$

All commutative and opposite algebras use this symmetry. Tensor products of coefficient and sheaf categories are balanced over $\mathcal{V}$. In particular, adjoining the finite parity grading is part of the coefficient theory; the six operations preserve this grading and exterior products use (2.1). Mapping objects in the multiplicity formulas are enriched over $\mathcal{V}$, so they retain both super parities. Their even parts are the ordinary k -valued mapping complexes. We abbreviate universally locally acyclic by ULA; its categorical formulation is recalled in the proof of Lemma 6.2.

Cohomological shifts satisfy $\mathcal{H}^i(\mathcal{F}[r]) = \mathcal{H}^{i+r}(\mathcal{F})$. For finite Satake labels we use the super normalization of [LW26], denoted by $\mathcal{T}_V$. We write $\langle d \rangle = \Pi^d[d](d/2)$ for the simultaneous parity shift, cohomological shift, and Tate twist; the half twists are chosen as in [LW26]. Over a splitting coefficient field, for a simple Satake object write $\mathcal{T}_V = \mathcal{S}_V \otimes \mathcal{L}_V$, where $\mathcal{S}_V$ is the ordinary perverse intersection complex and $\mathcal{L}_V$ is the invertible coefficient line encoding the fixed super parity and Tate twist. It is placed in cohomological degree zero. We keep this line explicit when passing from the ordinary IC calculation to the normalized Plancherel multiplicity.

The parameter $\hbar$ is the degree-two class of the standard character of T_{rot} . All rotation-equivariant identifications respect this class. On each finite model, arc-group equivariance is computed using a sufficiently deep finite jet quotient; the omitted congruence kernel is pro-unipotent.

2.2. The Plancherel multiplication. Let $i : X_{\mathcal{O}} \hookrightarrow X_{\mathcal{K}}$, and set $\delta_X = i_* k_{X_{\mathcal{O}}}$, with $G_{\mathcal{O}}$ -equivariance. Write Sat_X for the corresponding equivariant $*$ -sheaf category on $X_{\mathcal{K}}$, and $\mathrm{Sat}_{X, \hbar}$ for its loop-rotation-equivariant version. The Satake realization functor from $\mathcal{C}$ acts on these categories by convolution. For a finite representation V , the unrotated algebra is characterized by

$$(2.2) \quad R\mathrm{Hom}_{\mathcal{C}}(V, \mathrm{PL}_X) \simeq R\mathrm{Hom}_{\mathrm{Sat}_X}(\mathcal{T}_V * \delta_X, \delta_X).$$

The rotated analogue uses $G_{\mathcal{O}} \rtimes T_{\mathrm{rot}}$ -equivariance and is R -linear. Equivalently, one first forms the internal endomorphism object in the derived Satake category and then applies the right adjoint to the Satake realization functor,

as in [LW26, §4.3.3]. This uses a monoidal functor from the derived category of the Satake heart; it does not require that functor to be fully faithful on derived mapping complexes.

As in [LW26, §4.3.3], forgetting loop rotation gives derived specialization

$$(2.3) \quad \mathrm{PL}_{X,\hbar} \otimes_R^{\mathbf{L}} k \simeq \mathrm{PL}_X.$$

Here R denotes the polynomial dg model of the cochains on BT_{rot} , with zero differential and the specified class $\hbar$; the universal first Chern class gives this model. Consequently, the freeness theorem gives $H^*(\mathrm{PL}_X) \simeq H^*(\mathrm{PL}_{X,\hbar})/\hbar$.

The product is composition of evaluations. If $a : \mathcal{T}_V * \delta_X \rightarrow \delta_X[r]$ and $b : \mathcal{T}_W * \delta_X \rightarrow \delta_X[s]$, their product is the composite

$$(2.4) \quad \begin{aligned} \mathcal{T}_{V \otimes W} * \delta_X &\simeq \mathcal{T}_V * (\mathcal{T}_W * \delta_X) \xrightarrow{\mathcal{T}_V * b} (\mathcal{T}_V * \delta_X)[s] \\ &\xrightarrow{a[s]} \delta_X[r+s]. \end{aligned}$$

The identity of δ_X is the unit. Thus (2.2) specifies an associative algebra, including its multiplication and unit.

We distinguish this constant basic object from the dualizing basic object $\delta_X^!$ in the !-sheaf theory of [BZSV24]. The latter gives an internal-endomorphism algebra $A_X^!$. Their comparison, as associative algebras, is established in Lemma 6.3.

2.3. The induced presentation. We use distinguished polarizations in the sense of [BZSV24, §3.7], with the cotangent conventions recalled in [LW26, §2.6]. In particular, X is smooth affine and the Hamiltonian structure on T^*X is the canonical one. The geometric input for Corollary 1.3 is the presentation

$$(2.5) \quad X \simeq G \times^H Y,$$

where $H \subset G$ is connected reductive and Y is a finite-dimensional H -representation. We use left actions, so $G \times^H Y$ denotes the quotient by $(g, y) \cdot h = (gh, h^{-1}y)$. This is the same presentation as $Y \times^H G$ in the right-action notation of [LW26, §2.6].

In characteristic zero the structural results behind (2.5) are [BZSV24, Theorem 3.6.1 and §3.7]. In the finite-field setting, we use the definition by structural data, with the characteristic restrictions imposed there; see [BZSV24, §2.4 and Definitions 3.9.3, 3.9.5]. Here an untwisted distinguished polarization is supplied by an untwisted polarized datum over the characteristic-zero base ring and its reduction. In particular, the untwisted condition is imposed on the datum, before reduction. We recall why this construction has the form (2.5).

The general polarized construction has $X = Y \times^{HU} G$, with a Whittaker character on U ; see [BZSV24, (3.33)]. In Hamiltonian induction, moment values restrict, after conjugation, to this character on $\mathfrak{u}$. A nonzero character excludes zero from the image of the moment map. Over characteristic zero, the canonical moment map of T^*X vanishes on the zero section. The character

must therefore vanish. In the even $\mathfrak{sl}_2$ construction, [BZSV24, Lemma 3.7.3] then forces the datum to be trivial, so that $U = 1$. The untwisted polarized datum therefore has trivial Whittaker part, and the same is true after base change to the geometric finite field. The polarized formula reduces to (2.5). Connectedness of H is part of the polarized hyperspherical structure; see [BZSV24, Proposition 3.7.4 and Definition 3.9.3]. Thus the finite-field use of (2.5) comes from these structural data, without an application of the characteristic-zero slice theorem.

2.4. Satake parity. Let J be connected reductive and let ν be a dominant coweight. Normalize the intersection complex IC_ν of $\overline{\mathrm{Gr}}_J^\nu$ to be perverse. For every geometric point ξ , the Satake parity theorem states that

$$(2.6) \quad \mathcal{H}^j(i_\xi^* \mathrm{IC}_\nu) = 0 \quad \text{unless} \quad j \equiv \dim \mathrm{Gr}_J^\nu \pmod{2}.$$

The restriction to each J_O -orbit has constant cohomology sheaves, and the assertion holds for the usual loop-rotation-equivariant lift.

See [Lus83, §11(c)] and [Zhu17, §5.6, (5.6.2) and the ensuing stalk formula]. In the latter formula, the possible stalk degrees are $2i - \langle 2\rho_J, \nu \rangle$, which gives (2.6) in the étale setting. The characteristic-zero assumption here is on the coefficients. It is not an assumption on the geometric ground field.

3. EQUIVARIANT PARITY AND FINITE FILTRATIONS

For an algebraic group A acting on a finite-type scheme Z and $\mathcal{F} \in D_A^b(Z)$, set

$$(3.1) \quad \mathcal{C}_A(Z, \mathcal{F}) := R\Gamma_A(Z, \mathbb{D}_Z \mathcal{F}) = R\mathrm{Hom}_A(\mathcal{F}, \omega_Z).$$

Here $\mathbb{D}_Z$ is ordinary Verdier duality on Z , equipped with its induced equivariant structure. It is not Verdier duality on the quotient stack. The definition applies to an arbitrary constructible complex, without a local constancy assumption on Z .

Lemma 3.1 (Equivariant splitting on an orbit). *Let $O = A/K$, where K is connected, and suppose that $H^{\mathrm{odd}}(BK, k) = 0$. Let $\mathcal{E} \in D_A^b(O)$ have constant cohomology sheaves concentrated in degrees congruent to p modulo 2. If W_j denotes its degree- j stalk cohomology, there is a noncanonical equivariant decomposition*

$$(3.2) \quad \mathcal{E} \simeq \bigoplus_{j \equiv p \pmod{2}} \underline{W_{j_O}}[-j].$$

Proof. The quotient stack $[O/A]$ is BK . An equivariant constructible local system on O corresponds to a representation of the component group of K . Since K is connected, these local systems are constant.

We split the cohomological truncation triangles by induction on the number of nonzero cohomology sheaves. Let m be the largest degree with $W_m \neq 0$.

After splitting the lower truncation, the connecting morphism has components in

$$\mathrm{Hom}_{D(BK)}(W_m[-m], W_j[-j+1]) = \mathrm{Hom}_k(W_m, W_j) \otimes_k H^{m-j+1}(BK, k).$$

Both m and j have parity p , so $m-j+1$ is odd. The displayed group vanishes. The triangle therefore splits off $W_m[-m]$, completing the induction. $\square$

For the rest of this section, put $A = H_{\mathcal{O}} \rtimes T_{\mathrm{rot}}$, where H is connected reductive. Choose a maximal torus and a positive root system of H . For a dominant coweight λ , write

$$O_{\lambda} = H_{\mathcal{O}} t^{\lambda} H_{\mathcal{O}} / H_{\mathcal{O}} \subset \mathrm{Gr}_H, \quad L_{\lambda} = Z_H(\lambda).$$

Lemma 3.2 (Rotation stabilizers). *The stabilizer K_{λ} of $t^{\lambda} H_{\mathcal{O}}$ in A is connected and has a split quotient*

$$(3.3) \quad K_{\lambda} \longrightarrow L_{\lambda} \times T_{\mathrm{rot}}$$

with pro-unipotent kernel. Consequently there is an isomorphism of R -algebras

$$(3.4) \quad H^*(BK_{\lambda}, k) \simeq H^*(BL_{\lambda}, k) \otimes_k R.$$

The factor $H^(BL_{\lambda}, k)$ is concentrated in even degrees.*

Proof. Pure rotation fixes the coset $t^{\lambda} H_{\mathcal{O}}$, since $(at)^{\lambda} = t^{\lambda} \lambda(a)$. Thus

$$K_{\lambda} = (H_{\mathcal{O}} \cap t^{\lambda} H_{\mathcal{O}} t^{-\lambda}) \rtimes T_{\mathrm{rot}}.$$

Evaluation at $t = 0$ maps the first factor onto $P_{-\lambda}$, with pro-unipotent kernel. Indeed, the coordinate of the root subgroup for α must be divisible by $t^{\max(0, \langle \alpha, \lambda \rangle)}$. The Levi quotient of $P_{-\lambda}$ is L_{λ} . Rotation acts trivially after evaluation, and constant loops in L_{λ} split this quotient and commute with rotation. This proves (3.3) and the connectedness assertion.

In the sheaf theory under consideration, a unipotent group has no positive-degree classifying-stack cohomology. Moreover, with characteristic-zero coefficients, $H^*(BL_{\lambda}, k)$ is the Weyl-invariant polynomial algebra on the degree-two cohomology of a maximal torus. It is therefore even. These facts give (3.4). The splitting respects projection to T_{rot} , so the isomorphism is over the specified ring R . The calculation may be made in a sufficiently deep finite jet quotient, whose discarded congruence kernel is unipotent. $\square$

Lemma 3.3 (Freeness from a finite filtration). *Let Z be a finite-type A -scheme with a finite filtration by invariant closed subschemes. Suppose that each stratum E_i is an A -equivariant vector bundle over some O_{λ} . Let $\mathcal{F} \in D_A^b(Z)$. Assume that $\mathcal{F}|_{E_i}$ is pulled back from a complex on O_{λ} satisfying Lemma 3.1, with the same parity p for every stratum. Then $H^*(C_A(Z, \mathcal{F}))$ is a graded free R -module, concentrated in degrees congruent to p modulo 2.*

Proof. Let $d_i = \dim E_i$. By equivariant splitting and smooth Verdier duality,

$$\mathbb{D}_{E_i}(\mathcal{F}|_{E_i}) \simeq \bigoplus_{j \equiv p \pmod{2}} W_{ij}^{\vee} \otimes_k \underline{k}_{E_i}[j + 2d_i](d_i).$$

Equivariant homotopy invariance for $E_i \rightarrow O_\lambda$ identifies $H_A^*(E_i, k)$ with $H^*(BK_\lambda, k)$. Lemma 3.2 therefore proves the desired freeness and parity for each stratum.

Consider one step of the filtration, with closed inclusion $i : Z' \hookrightarrow Z''$ and open complement $j : E \hookrightarrow Z''$. Verdier duality gives

$$\begin{aligned} i^! \mathbb{D}_{Z''}(\mathcal{F}|_{Z''}) &\simeq \mathbb{D}_{Z'}(\mathcal{F}|_{Z'}), \\ j^* \mathbb{D}_{Z''}(\mathcal{F}|_{Z''}) &\simeq \mathbb{D}_E(\mathcal{F}|_E). \end{aligned}$$

The localization triangle hence takes the form

$$\begin{aligned} (3.5) \quad C_A(Z', \mathcal{F}|_{Z'}) &\longrightarrow C_A(Z'', \mathcal{F}|_{Z''}) \\ &\longrightarrow C_A(E, \mathcal{F}|_E) \xrightarrow{+1}. \end{aligned}$$

By induction, the cohomology of both outer terms is concentrated in parity p . All connecting maps change degree by one and thus vanish. We obtain a short exact sequence of graded R -modules. Its quotient is graded free, so homogeneous lifts of a homogeneous basis split the sequence. Induction on the filtration proves the lemma. $\square$

Remark 3.4. The use of $\mathbb{D}_Z \mathcal{F}$ in (3.1) is what makes (3.5) compatible with ordinary restrictions of the coefficient complex. The argument does not require $\mathcal{F}$ to be locally constant on all of Z .

4. A FINITE MODEL FOR PLANCHEREL MULTIPLICITIES

Assume $X = G \times^H Y$ as in Theorem 1.1, and first take a splitting coefficient field for $\check{G}$. For $V \in \text{Irr}(\check{G})$, let $\mathcal{S}_V = \text{IC}_V$ be the ordinary perverse Satake sheaf, so that $\mathcal{T}_V = \mathcal{S}_V \otimes \mathcal{L}_V$ as in Section 2.1. With equivariance understood, set

$$\delta_X = i_* k_{X_{\mathcal{O}}}, \quad i : X_{\mathcal{O}} \hookrightarrow X_K,$$

and define

$$(4.1) \quad N_{V,h} := R\text{Hom}_{\text{Sat}_{X,h}}(\mathcal{S}_V * \delta_X, \delta_X).$$

The actual V -multiplicity uses the fully normalized kernel:

$$\begin{aligned} (4.2) \quad M_{V,h} &:= R\text{Hom}_{\text{Sat}_{X,h}}(\mathcal{T}_V * \delta_X, \delta_X) \\ &\simeq \mathcal{L}_V^\vee \otimes_k N_{V,h}. \end{aligned}$$

Thus freeness of $N_{V,h}$ is equivalent to freeness of the normalized multiplicity $M_{V,h}$. The kernel of $\text{Aut}(D) \rightarrow T_{\text{rot}}$, for the formal disk $D = \text{Spec } F[[t]]$, is pro-unipotent. Thus (4.1) may be computed with $G_{\mathcal{O}} \rtimes T_{\text{rot}}$ -equivariance; this replacement preserves $\hbar$.

Consider the space of triples

$$(4.3) \quad \mathcal{R}_{H,Y} = \{(hH_{\mathcal{O}}, v) \in \text{Gr}_H \times Y_{\mathcal{O}} : h^{-1}v \in Y_{\mathcal{O}}\},$$

and the map $q : \mathcal{R}_{H,Y} \rightarrow \text{Gr}_G$ induced by $H \hookrightarrow G$.

Lemma 4.1 (Finite multiplicity model). *Let $S \subset \text{Gr}_H$ be the reduced inverse image of $\text{supp } \mathcal{S}_V$ under $\text{Gr}_H \rightarrow \text{Gr}_G$. Then S is a finite union of $H_{\mathcal{O}}$ -orbits. For all sufficiently large N , the formula*

$$(4.4) \quad Z_N = \{(hH_{\mathcal{O}}, \bar{v}) \in S \times (Y_{\mathcal{O}}/t^N Y_{\mathcal{O}}) : v \in hY_{\mathcal{O}}\}$$

defines a finite-type scheme. Writing $q_N : Z_N \rightarrow \text{Gr}_G$ for the induced map and $n = \dim Y$, there is an R -linear identification

$$(4.5) \quad N_{V, \hbar} \simeq R\Gamma_{H_{\mathcal{O}} \times T_{\text{rot}}}(Z_N, \mathbb{D}_{Z_N} q_N^* \mathcal{S}_V)[-2Nn](-Nn).$$

These normalized complexes are independent of the choice of sufficiently large N .

Proof. We first identify the correspondence and its coefficient complex, and then verify the finite bounds and the normalization.

The correspondence. Put $\mathcal{Q}_X = [X_{\mathcal{O}}/G_{\mathcal{O}}]$. The double-regular relative Grassmannian is the fiber product

$$(4.6) \quad \mathcal{Z}_X := \mathcal{Q}_X \times_{[X_{\mathcal{K}}/G_{\mathcal{K}}]} \mathcal{Q}_X.$$

It classifies two G -bundles on the disk with regular X -sections and an isomorphism on the punctured disk that respects the sections. The equivalence $[X/G] \simeq [Y/H]$ sends a G -bundle with an X -section to its H -reduction with a Y -section. Applying this description on the disk and punctured disk gives

$$(4.7) \quad \mathcal{Z}_X \simeq [\mathcal{R}_{H,Y}/H_{\mathcal{O}}].$$

In these coordinates, the G -Hecke map is induced by viewing $h \in H_{\mathcal{K}}$ as an element of $G_{\mathcal{K}}$. The identifications are identities of moduli stacks and therefore descend from trivializations of the bundles. They also commute with loop rotation.

The coefficient complex. Let $p : \mathcal{Z}_X \rightarrow \mathcal{Q}_X$ be the projection to the target section. The convolution correspondence and adjunction identify (4.1) with

$$(4.8) \quad R\text{Hom}(q^* \mathcal{S}_V, p^! \underline{k}_{\mathcal{Q}_X})$$

on $\mathcal{Z}_X$, with rotation equivariance. Here $q^* \mathcal{S}_V$ denotes pullback of the equivariant Satake kernel by the Hecke map.

For clarity, the adjunction calculation can be made at a finite truncation. Restrict the source of the convolution correspondence to the regular-arc locus and call the resulting one-sided correspondence E . Let $a : E \rightarrow [X_{\mathcal{K}}/G_{\mathcal{O}}]$ be its target map and $\tilde{q}$ its Hecke map. Base change along $i : \mathcal{Q}_X \rightarrow [X_{\mathcal{K}}/G_{\mathcal{O}}]$ gives a Cartesian square

$$\begin{array}{ccc} \mathcal{Z}_X & \xrightarrow{j} & E \\ p \downarrow & & \downarrow a \\ \mathcal{Q}_X & \xrightarrow{i} & [X_{\mathcal{K}}/G_{\mathcal{O}}]. \end{array}$$

Since the source sheaf is constant, convolution is $a_! \tilde{q}^* \mathcal{S}_V$. Consequently,

$$\begin{aligned}
 R\mathrm{Hom}(a_! \tilde{q}^* \mathcal{S}_V, i_* \underline{k}_{\mathcal{Q}_X}) &\simeq R\mathrm{Hom}(i^* a_! \tilde{q}^* \mathcal{S}_V, \underline{k}_{\mathcal{Q}_X}) \\
 &\simeq R\mathrm{Hom}(p_! j^* \tilde{q}^* \mathcal{S}_V, \underline{k}_{\mathcal{Q}_X}) \\
 &\simeq R\mathrm{Hom}(q^* \mathcal{S}_V, p^! \underline{k}_{\mathcal{Q}_X}).
 \end{aligned}
 \tag{4.9}$$

In particular, the coefficient is the ordinary pullback of the Satake kernel, even when it is not locally constant.

Restricting to bounded Hecke support and finite arc truncations makes these the usual six-functor operations. Their compatibility with the infinite-dimensional definition is the finite-model calculation of [BZSV24, Proposition 8.3.1, (8.11), and §8.3.6]. Every map and every adjunction or base-change isomorphism in (4.9) is rotation equivariant, so the identification is over R .

Finite bounds. Choose a faithful G -representation W . The support of $\mathcal{S}_V$ is finite dimensional, so the associated lattices admit a uniform bound

$$t^b W_{\mathcal{O}} \subset g W_{\mathcal{O}} \subset t^{-b} W_{\mathcal{O}}$$

for some b . If $t^\lambda \in \mathrm{Gr}_H$ maps into this support, then every weight χ of $W|_H$ satisfies $|\langle \chi, \lambda \rangle| \leq b$. Faithfulness implies that these weights span the rational character space of a maximal torus of H . Hence only finitely many dominant coweights λ can occur. The closed inverse image S is thus a finite union of spherical strata and is of finite type. Passing to its reduced structure does not change the constructible sheaf category.

Choose N such that $t^N Y_{\mathcal{O}} \subset h Y_{\mathcal{O}}$ for every $h H_{\mathcal{O}} \in S$; the finite list of coweights makes this possible. The condition in (4.4) is then independent of the representative v of $\bar{v}$. It imposes finitely many linear conditions on the jet of v , and gives the finite-type scheme Z_N . For sufficiently large $m \geq N$, the action on this model factors through the jet group $H_m = H(\mathcal{O}/t^m)$.

Duality and normalization. Under (4.7), the target $\mathcal{Q}_X$ becomes $[Y_{\mathcal{O}}/H_{\mathcal{O}}]$. Write $Y_N = Y_{\mathcal{O}}/t^N Y_{\mathcal{O}}$. Its finite model $[Y_N/H_m]$ is smooth of dimension $Nn - \dim H_m$. Thus the term $p^! \underline{k}$ in (4.8) is the dualizing complex of the source stack, shifted and twisted by minus twice, and minus once, this dimension.

On the other hand, the pullback of the stack dualizing complex from $[Z_N/H_m]$ to Z_N is $\omega_{Z_N}[-2 \dim H_m](-\dim H_m)$. The two group-dimension contributions cancel, leaving exactly

$$\mathbb{D}_{Z_N} q_N^* \mathcal{S}_V[-2Nn](-Nn)$$

inside equivariant cohomology. Including T_{rot} in both quotients cancels its dimension in the same way. This proves (4.5), with ordinary Verdier duality on Z_N .

Independence of the truncation. For N sufficiently large, the map $Z_{N+1} \rightarrow Z_N$ is an affine bundle of rank n , and $q_{N+1}^* \mathcal{S}_V$ is the ordinary pullback of $q_N^* \mathcal{S}_V$. Smooth Verdier duality contributes $[2n](n)$. This is canceled by the change in the normalization $[-2Nn](-Nn)$. Equivariant homotopy invariance therefore

identifies the normalized complexes at consecutive levels. Increasing m changes only unipotent equivariance and has no effect. This proves the asserted independence and completes the proof. $\square$

Remark 4.2. Changing the normalization of the Satake action changes (4.5) by a uniform shift and twist depending on V , which does not affect freeness. We use only the module identification of [BZSV24, Proposition 8.3.1]. The comparison of algebra multiplications discussed in [BZSV24, Remark 8.3.2] is not needed.

5. FREENESS OVER THE ROTATION PARAMETER

Proposition 5.1. *Under the hypotheses of Theorem 1.1, and over a splitting coefficient field, the graded R -module $H^*(M_{V,h})$ is free for every $V \in \text{Irr}(\check{G})$.*

Proof. Use the finite model of Lemma 4.1. For an $H_{\mathcal{O}}$ -orbit $O_{\lambda} \subset S$, let $Z_{N,\lambda}$ denote its inverse image in Z_N . The map $Z_{N,\lambda} \rightarrow O_{\lambda}$ is a vector bundle with fiber

$$(5.1) \quad (Y_{\mathcal{O}} \cap t^{\lambda} Y_{\mathcal{O}}) / t^N Y_{\mathcal{O}}.$$

Indeed, the stabilizer acts linearly on this space, and the bundle is obtained by translating it along O_{λ} . This description is equivariant under $H_{\mathcal{O}} \rtimes T_{\text{rot}}$. If $Y = \bigoplus_{\chi} Y_{\chi}$ is the weight decomposition, then its rank is

$$(5.2) \quad Nn - \sum_{\chi} \max(\langle \chi, \lambda \rangle, 0) \dim Y_{\chi}.$$

This is the lattice-intersection calculation of [BFN18, Lemma 2.2]. We use a fixed target lattice in (4.4); using the source lattice instead reverses the sign of λ in (5.2).

Let ν be the highest coweight corresponding to V , and put

$$p_V = \langle 2\rho_G, \nu \rangle \bmod 2.$$

The map $O_{\lambda} \rightarrow \text{Gr}_G$ lands in a single $G_{\mathcal{O}}$ -orbit. By (2.6), the pullback of $\mathcal{S}_V$ to O_{λ} has cohomology only in parity p_V . Its equivariant cohomology sheaves are constant, since its stabilizer is connected by Lemma 3.2. That lemma also gives the vanishing of odd stabilizer cohomology. Lemma 3.1 consequently splits this pullback equivariantly. Pulling the decomposition back along $Z_{N,\lambda} \rightarrow O_{\lambda}$ verifies the coefficient hypothesis of Lemma 3.3, with parity p_V independent of λ .

The finite collection of orbits in S is closed under specialization. Order the orbits compatibly with closure and take inverse images in Z_N of the resulting closed unions. This gives a finite invariant closed filtration with strata $Z_{N,\lambda}$. Lemma 3.3 shows that

$$H^* R\Gamma_{H_{\mathcal{O}} \rtimes T_{\text{rot}}}(Z_N, \mathbb{D}_{Z_N} q_N^* \mathcal{S}_V)$$

is graded free over R . The uniform shift and twist in (4.5) preserve freeness, which proves freeness of $N_{V,h}$. Tensoring with the invertible line $\mathcal{L}_V^{\vee}$ in (4.2) proves the proposition for the actual Plancherel multiplicity $M_{V,h}$. $\square$

Remark 5.2. The constant-coefficient filtration of [BFN18, Lemmas 2.5–2.6] provides the geometric pattern for the proof. For Satake coefficients, the additional ingredients are equivariant splitting of the restricted kernel, parity independent of the stratum, and the duality identity that produces (3.5).

Proof of Theorem 1.1. First suppose that k is a splitting coefficient field for $\check{G}$. Since $\check{G}$ is reductive in characteristic zero, its rational representations are semisimple. The same argument applies separately in the two super parities. The inner-Hom characterization gives, as R -module complexes with $\check{G}$ -action,

$$(5.3) \quad \mathrm{PL}_{X,\hbar} \simeq \bigoplus_{V \in \mathrm{Irr}(\check{G})} V \otimes_k M_{V,\hbar}.$$

Here V ranges over ordinary irreducible representations, regarded as even; its multiplicity is a complex of super vector spaces. Thus both super parities are included in the multiplicity, without counting each V twice. These are the multiplicities in (4.2), obtained by the right-adjoint construction of [LW26, §4.3.3]. Equivalently, test the underlying object against finite representations and their parity reversals. The parameter $\hbar$ has trivial $\check{G}$ -action and acts in the multiplicity factors.

By Proposition 5.1, tensoring with V and taking the direct sum in (5.3) gives

$$(5.4) \quad H^*(\mathrm{PL}_{X,\hbar}) \text{ graded free over } R.$$

Choose a homogeneous R -basis $\{e_\alpha\}$ and, in a dg R -module model, choose a cocycle representing each basis element. Sending the generator of $R[-\deg e_\alpha]$ to the chosen cocycle gives an R -linear map

$$\bigoplus_{\alpha} R[-\deg e_\alpha] \longrightarrow \mathrm{PL}_{X,\hbar}$$

that induces the isomorphism (5.4). It is therefore a quasi-isomorphism. This is an assertion about the underlying module complex; no compatibility with multiplication is required. It gives flatness, injectivity of multiplication by $\hbar$, and the absence of additional Tor in specialization at $\hbar = 0$.

For an arbitrary coefficient field k , extend to a splitting field k' . The finite-model construction (4.5) commutes with extension of coefficients. Flatness then descends along the faithfully flat map $k[\hbar] \rightarrow k'[\hbar]$. $\square$

6. FACTORIZATION AND INTERNAL ENDOMORPHISMS

We now assume only that X is smooth and affine. The purpose of this section is to construct a factorization associative family with singleton algebra $A_X^!$, and to compare that algebra with PL_X . Throughout this section, categorical tensor products are balanced over $\mathcal{V}$.

6.1. The coefficient factorization category. Take $\Sigma = \mathbb{A}_F^1$, and put $R_I = \mathrm{Shv}(\Sigma^I; \mathcal{V})$, with exceptional tensor product $\otimes^!$ and unit ω_{Σ^I} . The notation R_I refers to a sheaf category and is distinct from the parameter ring $R = k[\hbar]$. We use the closed-diagonal construction of [Lys24, §§2.1–2.2 and 2.5]. It applies to Ind-constructible ℓ -adic sheaves. Its formulation for a projective curve gives the present construction by restriction from $\mathbb{P}^1$.

For a finite nonempty set I , let $\mathrm{Tw}(I)$ have objects $s = (I \twoheadrightarrow J \twoheadrightarrow K)$. A morphism from s_1 to s_2 consists of compatible surjections $J_1 \twoheadrightarrow J_2$ and $K_2 \twoheadrightarrow K_1$. To s associate

$$D_s = \mathcal{C}^{\otimes J} \otimes \mathrm{Shv}(\Sigma^K),$$

viewed as an R_I -module category supported on the closed diagonal $\Delta_K : \Sigma^K \hookrightarrow \Sigma^I$. The transition functor tensors coefficient labels along $J_1 \rightarrow J_2$ and then pushes forward along $\Sigma^{K_1} \hookrightarrow \Sigma^{K_2}$. Define

$$(6.1) \quad \mathcal{C}_I = \operatorname{colim}_{s \in \mathrm{Tw}(I)} D_s$$

in R_I -module categories.

The construction has coherent diagonal and factorization equivalences

$$(6.2) \quad \Delta_\alpha^! \mathcal{C}_I \simeq \mathcal{C}_J, \quad \mathcal{C}_I|_{U_\beta} \simeq (\boxtimes_b \mathcal{C}_{I_b})|_{U_\beta}.$$

Here $\alpha : I \twoheadrightarrow J$, and U_β is the locus where coordinates in different blocks I_b are disjoint. Pullback of a module category means extension of scalars through exceptional pullback on base sheaves. For the equivalences and the pointwise symmetric monoidal structures, see [Lys24, §§2.1.11–2.1.18, 2.2.9–2.2.17, and 2.5].

Lemma 6.1 (Total tensor product of labels). *There are continuous, base-linear, strong symmetric monoidal functors*

$$(6.3) \quad Q_I : \mathcal{C}_I \longrightarrow \mathrm{Shv}(\Sigma^I) \otimes \mathcal{C}$$

compatible with exceptional diagonal restriction, factorization, permutations, and insertion of unit-labelled points. For a singleton, Q_I is the identity under the natural identification.

Proof. Let $m_J : \mathcal{C}^{\otimes J} \rightarrow \mathcal{C}$ be total tensor product. On the term D_s , define

$$(6.4) \quad Q_s = m_J \otimes \Delta_{K,!}.$$

These functors form a cocone: coefficient transitions use $m_{J_2} m_{J_1/J_2} = m_{J_1}$, and geometric transitions use functoriality of proper pushforward. The proper projection formula gives base linearity, so the colimit defines Q_I .

For a diagonal map $f : \Sigma^{I'} \rightarrow \Sigma^I$, push out the index diagram along $I \rightarrow I'$, obtaining $I' \rightarrow J' \rightarrow K'$. Then $\Sigma^{K'} = \Sigma^K \times_{\Sigma^I} \Sigma^{I'}$, and proper exceptional base change gives

$$(6.5) \quad f^! \Delta_{K,!} \simeq \Delta_{K',!} f_K^!.$$

After contracting coefficient labels, their total tensor product is still m_J . These are the pushout-index maps defining (6.2); the cofinality equivalences therefore identify $f^!Q_I$ with $Q_{I'}f^!$.

For monoidality, take indices $s_a = (I \rightarrow J_a \rightarrow K_a)$, for $a = 1, 2$, and write $\Sigma^{K_0} = \Sigma^{K_1} \times_{\Sigma^I} \Sigma^{K_2}$. If $r_a : \Sigma^{K_0} \rightarrow \Sigma^{K_a}$, then

$$(6.6) \quad \Delta_{K_1,!}F_1 \otimes^! \Delta_{K_2,!}F_2 \simeq \Delta_{K_0,!}(r_1^!F_1 \otimes^! r_2^!F_2).$$

The product in (6.1) uses this common-support diagram and the coefficient slots of $J_1 \amalg J_2$. Total tensor product gives $m_{J_1}(-) \otimes m_{J_2}(-)$, proving monoidality on the colimit terms and hence on $\mathcal{C}_I$. Associativity and symmetry follow from the corresponding six-functor constraints and the symmetry of $\mathcal{C}$.

The unit is represented by unit labels and ω_{Σ^I} in the term $I = J = K$, and maps to the target unit. Over a disjoint-block locus, the indices and their tensor products split blockwise. Inserting points inserts unit labels. These descriptions also give all the refinement compatibilities. $\square$

The functors Q_I retain the rational $\check{G}$ -action and the super symmetry. The use of $\mathcal{C}_I$ is essential: forming internal Homs in a fixed $\text{Rep}(\check{G}^I)$ and then restricting diagonally would introduce coinduction at collisions and would not give (6.5).

6.2. The factorizable Satake action. Factorizable constructible Satake gives coherent base-linear monoidal functors from $\mathcal{C}_I$ to the spherical Hecke categories over Σ^I . They have the diagonal compatibilities of (6.2), preserve exterior factorization, and send the unit to the trivial modification; see [Lys24, Proposition 4.1.15 and §§7.3.8–7.3.17]. The exterior factorization functors in the constructible target are fully faithful.

We fix the normalization as follows. On finite labels use $\mathcal{T}_V$, including its super parity. On Σ^K , normalize the ULA multifusion family to have unit ω_{Σ^K} : if $P_{V,K}$ is the super-normalized global perverse family, use $P_{V,K}\langle |K| \rangle$. The multifusion and refinement identities of [Zhu17, Proposition 5.4.2 and Remark 5.4.12, (5.4.13)–(5.4.14)] then identify exceptional restriction at a collision with tensoring the labels, with this normalization already included. In particular, for unit labels the normalized perverse family is $\Pi^{|K|}k_{\Sigma^K}[|K|](|K|/2)$, so the additional $\langle |K| \rangle$ gives precisely ω_{Σ^K} . The factorization symmetry includes the central-parity correction of [Lys24, §7.3.17]; in the super convention fixed above, this is part of the symmetry (2.1).

On a term $(I \rightarrow J \rightarrow K)$ of (6.1), the action functor sends finite labels and a base sheaf F to the closed-diagonal pushforward of

$$(6.7) \quad \mathcal{T}_{(\otimes_{j \rightarrow k} V_j)_{k \in K}, K} \otimes_{R_K} F.$$

Associativity handles coefficient transitions; proper projection formula and fusion restriction handle geometric transitions. The common-support calculation (6.6), together with convolution of Satake families, proves pointwise monoidality. On disjoint loci the construction is exterior product. The

finite-label fusion constraints are determined by intermediate extension from the disjoint locus, as in [Lys24, §7.3.16]; this gives their coherence with convolution in the specified super normalization.

We use the ind-finite, or renormalized, Hecke category of [BZSV24, Definition B.6.1]. Finite labels and constructible base sheaves give kernels of bounded Beilinson–Drinfeld Schubert support, descending to finite jet quotients. They lie in the small finite-sheaf category whose Ind completion defines the renormalized category. By [Lys24, Lemma 2.6.12], the localized finite labels compactly generate $\mathcal{C}_I$. Defining the action on these labels and extending continuously gives the renormalized action with the same constraints. For finitely many kernels, a sufficiently thick sum of graph divisors provides a common jet bound.

The second geometric input is [BZSV24, Proposition 16.2.8]. For smooth affine X , it supplies a faithful unital factorization module $\mathcal{M}_I$ for the Hecke categories, with dualizing basic object $\delta_I = \delta_I^!$. Its ULA condition may be read over R_I : the relative rigidity of the Hecke category and the equivalence of the two ULA conditions are [BZSV24, Lemma 16.2.6 and Proposition B.9.7(2)]. Here faithfulness means that exterior product on disjoint loci is fully faithful. We use the algebraic assertion, which includes the étale theory.

Lemma 6.2 (Factorization of internal endomorphisms). *The functor*

$$T_I : \mathcal{C}_I \longrightarrow \mathcal{M}_I, \quad c \longmapsto \text{Sat}_I(c) * \delta_I$$

has a continuous base-linear right adjoint T_I^R . The internal-endomorphism algebras

$$(6.8) \quad B_I = T_I^R(\delta_I)$$

form a unital Cartesian factorization associative algebra. Its singleton fiber is $A_X^!$, with the composition product.

Proof. Tensor product of rational representations has a continuous right adjoint given by coinduction along the diagonal, with its projection formula. The right adjoint to the unit $\mathcal{V} \rightarrow \mathcal{C}$ is taking invariants, and is continuous because $\tilde{G}$ is reductive in characteristic zero. These adjunctions make $\mathcal{C}$ relatively rigid over $\mathcal{V}$, also with super coefficients. By [Lys24, Corollary 2.7.7 and Appendix G], $\mathcal{C}_I$ is relatively rigid over R_I .

The categorical ULA property is the existence of a continuous linear right adjoint to action on the given object [BZSV24, Definition B.9.4]. For a category relatively rigid over R_I , being ULA over R_I is equivalent to being ULA over that category [BZSV24, Proposition B.9.7(2)]. Applied to δ_I , this gives a continuous $\mathcal{C}_I$ -linear right adjoint, and in particular a continuous R_I -linear right adjoint. Composition of evaluations defines the unital associative product on B_I .

Extension of scalars along a diagonal preserves the adjunction $T_I \dashv T_I^R$. By the geometric and coefficient identifications, its left member becomes T_J .

Uniqueness of right adjoints, with their counits, gives

$$(6.9) \quad \Delta^! B_I \simeq B_J$$

as associative algebras. The continuity and base linearity above ensure that this is an extension of the full adjunction.

On a disjoint-block locus, let $u : \boxtimes_b \mathcal{M}_{I_b} \rightarrow \mathcal{M}_I$ be exterior product. There

$$T_I = u \circ \boxtimes_b T_{I_b}, \quad \delta_I = u(\boxtimes_b \delta_{I_b}).$$

Full faithfulness of u , followed by the tensor product of the continuous adjunctions, identifies the evaluation Hom functor with that represented by $\boxtimes_b B_{I_b}$. Consequently,

$$(6.10) \quad B_I|_{U_\beta} \simeq (\boxtimes_b B_{I_b})|_{U_\beta}.$$

Evaluations and their compositions agree under this identification. The coherence of (6.9)–(6.10) follows from the same adjunction and factorization diagrams.

Insertion of an identity modification preserves the basic object. Applying it to evaluations gives the unit-insertion maps; on a separated additional point it inserts the identity endomorphism. These maps agree with the associative unit. At a singleton the algebra is therefore characterized by

$$(6.11) \quad R\mathrm{Hom}_{\mathcal{C}}(V, A_X^!) \simeq R\mathrm{Hom}_{\mathcal{M}^!}(\mathcal{T}_V \star! \delta_X^!, \delta_X^!),$$

with multiplication given by composition of evaluations. $\square$

6.3. Comparison of the basic objects.

Lemma 6.3 (Finite-object duality). *Let σ be the symmetric monoidal Cartan pullback on $\mathcal{C}$ in the chosen Satake convention. There is an equivalence of unital associative algebras*

$$(6.12) \quad \mathrm{PL}_X \simeq \sigma^{-1}((A_X^!)^{\mathrm{op}}).$$

Proof. Use contravariant Verdier duality on finite objects in the $!$ - and $*$ -sheaf theories. At finite levels, ordinary Verdier duality exchanges exceptional and ordinary pullback and commutes with proper pushforward. These identities are compatible with pro-smooth arc-tail transitions and bounded ind-proper support transitions. They also apply to finite jet quotient presentations and their equivariant descent diagrams. This is the finite-object duality of [BZSV24, §7.2.3 and §§B.7.1–B.7.2]. The functor from renormalized sheaves to the usual descent-defined sheaf category is fully faithful on the small finite-sheaf category [BZSV24, Definition B.6.1]; see also [LW26, Example 4.17]. Mapping complexes between these objects can therefore be computed at finite levels.

Only the basic object and its convolutions by finitely many finite Satake kernels enter the calculation. The ULA property over the coefficient category makes the basic object compact in the ind-finite category. Action by a finite Satake kernel has continuous right adjoint given by its rigid dual, and hence preserves compactness. These objects have finite Schubert bounds and finite

jet presentations as in [BZSV24, §§8.2.4 and 8.3.6]. Duality carries $\delta_X^!$ to the constant basic object $\delta_X^* = \delta_X$. This is a duality on the finite objects used here; the comparison of complete loop-space categories discussed in [BZSV24, Remark 7.5.4] is a separate assertion.

Orient the bounded Hecke correspondence as in (2.4). Verdier duality exchanges exceptional pullback and tensor product with ordinary pullback and tensor product. The pushforward is proper on the Satake support and is self-dual. Thus

$$(6.13) \quad \mathbb{D}(\mathcal{T}_V \star! F) \simeq (\mathbb{D}\mathcal{T}_V) \star_* \mathbb{D}F.$$

The normalized Satake duality is

$$(6.14) \quad \mathbb{D}\mathcal{T}_V \simeq \mathcal{T}_{\sigma(V^\vee)}.$$

Equivalently, σ is the composite of the contravariant tensor dualities on finite representations and on the normalized Satake heart. Normalized Schubert intersection complexes are Verdier self-dual, and their ULA fusion gives the tensor and symmetry constraints in (6.14), including the super parity. Extending from finite labels gives a symmetric monoidal autoequivalence of $\mathcal{C}$.

Applying duality to (2.2), then rigid adjunction for the Satake kernel, yields

$$(6.15) \quad \begin{aligned} R\mathrm{Hom}_{\mathcal{C}}(V, \mathrm{PL}_X) &\simeq R\mathrm{Hom}_{\mathcal{M}^!}(\delta_X^!, \mathcal{T}_{\sigma(V^\vee)} \star! \delta_X^!) \\ &\simeq R\mathrm{Hom}_{\mathcal{M}^!}(\mathcal{T}_{\sigma(V)} \star! \delta_X^!, \delta_X^!) \\ &\simeq R\mathrm{Hom}_{\mathcal{C}}(\sigma(V), A_X^!). \end{aligned}$$

Finite representations and their parity reversals generate $\mathcal{C}$, so this identifies the underlying objects in (6.12).

To check multiplication, transpose an evaluation by $\mathbb{D}$ and take its rigid mate. This is the map in (6.15). Transposition reverses a composite of evaluations. Taking rigid mates moves the dual label to the input; symmetric interchange identifies the reversed tensor order with the opposite multiplication, with sign (2.1). The same calculation for any finite sequence of evaluations respects associativity coherences, and the identity evaluation transposes to the identity. Hence (6.12) is an equivalence of unital E_1 -algebras. $\square$

6.4. The coefficient-valued family. Apply Lemma 6.1 to Lemma 6.2, and put

$$(6.16) \quad F_I = Q_I(B_I) \in \mathrm{Shv}(\Sigma^I) \otimes \mathcal{C}.$$

This is a unital factorization family with compatible associative products. Translation and the coordinate $t - x$ trivialize its unary construction, giving

$$(6.17) \quad F_{\{1\}} = \omega_{\mathbb{A}^1} \otimes A_X^!.$$

For a diagonal stratum whose blocks are indexed by J , equations (6.9) and (6.10) identify the exceptional restriction with

$$(6.18) \quad \omega_{\text{Conf}_J(\mathbb{A}^1)} \otimes (A_X^1)^{\otimes J}.$$

Here Conf_J is the ordered configuration space of distinct J -labelled points. In particular, these exceptional restrictions have constant coefficients.

7. COMPARISON FOR SHEAVES GENERATED BY DIAGONALS

To extract topological operations from (6.16), we compare a specific class of sheaves across characteristic and with Betti sheaves. The comparison retains the full factorization diagrams and their associative products.

Let $p = \text{char } F$, and let S be the strict henselization of $\text{Spec } \mathbb{Z}_{(p)}$. Choose a geometric generic point and an embedding of its algebraic closure into $\mathbb{C}$. Choose compatible trivializations of the prime-to- p Tate lines over S . We use the resulting comparison only geometrically, without a Frobenius-equivariance assertion.

For a finite set I , let $\mathcal{D}_I(S)$ be the Ind completion of the thick category generated by

$$e_L = i_{L*} k_L, \quad L \subset \mathbb{A}_S^I \text{ a closed coordinate diagonal.}$$

The diagonal for the discrete partition is the entire space. Use the analogous notation for the two geometric fibers and for Betti sheaves on $\mathbb{C}^I$.

Lemma 7.1 (Diagonal comparison). *Restriction to either geometric fiber and the characteristic-zero étale–Betti comparison give equivalences on these generated categories. In all arities together, they intertwine exterior products, exceptional tensor products, diagonal exceptional pullbacks and proper pushforwards, permutations, partial-disjoint open restrictions and extensions, and exceptional pullbacks along coordinate projections. They preserve the exceptional fiber at the origin and all composition and base-change coherences. The same assertions hold after extension of coefficients to $\mathcal{C}$.*

Proof. For closed diagonals L, M , put $N = L \cap M$ and $c = \text{codim}_M N$. Proper exceptional base change and purity for a regular linear immersion give

$$(7.1) \quad R\text{Hom}(e_L, e_M) \simeq R\Gamma(N, k)[-2c](-c) \simeq k[-2c](-c).$$

The last step is affine-space homotopy invariance over the strictly henselian local base. In this linear situation, purity also follows by iterating codimension-one localization and the Kummer class. Intersections of coordinate diagonals are smooth over S , and restriction preserves their fundamental classes and codimensions. Thus the actual restriction maps on (7.1) are quasi-isomorphisms. This proves full faithfulness on the generators, hence on their thick closure and Ind completion. Every target generator is in the image. The characteristic-zero étale–Betti comparison preserves the same fundamental classes and gives the Betti equivalence.

For compatibility, use the common diagram of functors over S . Exterior product and proper diagonal pushforward preserve the generators. If $d : D \hookrightarrow \mathbb{A}^I$ is a diagonal, then

$$(7.2) \quad d^! e_L \simeq i_{L \cap D, D*} k[-2c'](-c'), \quad c' = \text{codim}_L(L \cap D).$$

The same formula applies at the origin. Exceptional pullback along a coordinate projection is smooth pullback with its relative dimension shift. Exceptional tensor product is exterior product followed by diagonal exceptional pullback. Over S , use the relative dualizing normalization, with unit $k[2|I|](|I|)$; this restricts to the dualizing unit on each fiber. All finite generators are ULA over S , and relative Verdier duality preserves their thick closure.

If $j : U \hookrightarrow \mathbb{A}^I$ is a partial-disjoint open, its complement is a finite union of coordinate diagonals. The finite closed-cover Čech resolution and localization express $j_! j^* e_L$ in the thick diagonal category. Verdier duality gives the same assertion for $j_* j^* e_L$. These constructions compare the adjunctions and localization triangles as well. Finite cohomological dimension allows the formulas to commute with arbitrary coefficient complexes and Ind extensions. All transformations come from the common six-functor diagram, so their higher composition compatibilities are preserved. Tensoring the diagram with $\mathcal{C}$ proves the coefficient-valued assertion. $\square$

Lemma 7.2 (A membership criterion). *A coefficient-valued sheaf whose exceptional restrictions to all diagonal strata are constant complexes belongs to $\mathcal{D}_I \otimes \mathcal{C}$.*

Proof. Let $u : S_\pi \hookrightarrow L_\pi$ be a diagonal stratum in its closed diagonal closure. The finite localization construction in the preceding proof expresses $u_! k$ in diagonal generators, and Verdier duality does the same for $u_* k$. Tensoring with any constant coefficient complex gives the assertion for that stratum, with any dualizing normalization.

For a sheaf F , use the localization triangle

$$i_* i^! F \longrightarrow F \longrightarrow j_* j^* F \xrightarrow{+1},$$

where j is the union of the top open strata and i is its closed complement. Induction on the finite number of strata proves the claim. This argument uses the exceptional restrictions in the hypothesis. $\square$

By (6.18), these lemmas apply to F_I . They transport the family, its associative products, and its Cartesian, factorization, and unit diagrams to Betti sheaves

$$(7.3) \quad F_I^B \in \text{Shv}(\mathbb{C}^I) \otimes \mathcal{C}.$$

The singleton exceptional fiber is still $A_X^!$, with the composition product of (6.11).

8. FUSION, CONVOLUTION, AND COMMUTATIVITY

8.1. Extracting the topological algebra. We give the compact-support construction explicitly to track the associative product and unit. The topological Ran space below parametrizes finite nonempty subsets.

Lemma 8.1 (Fusion and convolution). *Let $F = (F_I)$ be a $\mathcal{C}$ -valued unital Cartesian factorization associative family on powers of $\mathbb{C}$, belonging to the categories generated by coordinate diagonals. Suppose that its unary object is $\omega_{\mathbb{C}} \otimes A$, with its given associative product, and its exceptional fiber at a configuration of m distinct points is $A^{\otimes m}$. Then A , with that product, is the specified associative restriction of a unital E_3 -algebra in $\mathcal{C}$.*

Proof. Write $q_I : \mathbb{C}^I \rightarrow \text{Ran}(\mathbb{C})$. Let Fin_{surj} be the category of finite nonempty sets and surjections. Define

$$(8.1) \quad K_F(U) = \text{colim}_{I \in \text{Fin}_{\text{surj}}^{\text{op}}} R\Gamma_c(q_I^{-1}U, F_I).$$

The transition maps are induced by the counits

$$\Delta_! F_J = \Delta_! \Delta^! F_I \longrightarrow F_I.$$

On each finite-dimensional power, compactly supported sections give the covariant Verdier construction from !-sheaves to cosheaves. This construction takes no linear dual of the coefficient object A .

Contraction on disk sectors. Choose disjoint round disks $D_1, \dots, D_m$, with centers $z_1, \dots, z_m$. Consider the independent basic open

$$U = \text{Ran}(D_1) \star \dots \star \text{Ran}(D_m),$$

whose configurations meet every disk. Its preimage in $\mathbb{C}^I$ is the disjoint union of sectors $W_\eta = \prod_a D_a^{\eta^{-1}(a)}$, indexed by surjections $\eta : I \twoheadrightarrow [m]$. Let z_η be the tuple of their centers. The closed-point counit induces

$$(8.2) \quad i_{z_\eta}^! F_I \longrightarrow R\Gamma_c(W_\eta, F_I).$$

This is an equivalence on the entire category generated by diagonals. Indeed, for a generator e_L , both sides vanish if L identifies points in different disks. Otherwise $L \cap W_\eta$ is a product of disks containing its center, and both sides are $k[-2\dim_{\mathbb{C}} L]$; the map is the orientation-class isomorphism. Both functors are continuous, so the calculation extends to arbitrary coefficient complexes and Ind objects.

The Cartesian and factorization identifications make the left-hand side $A^{\otimes m}$. Under these identifications, the transition counits in (8.1) are identity maps. The indexing category of pairs $(I, \eta : I \twoheadrightarrow [m])$, with opposite-surjection transitions, has initial object $([m], \text{id})$. Its nerve is contractible, so

$$(8.3) \quad K_F(U) \simeq A^{\otimes m}.$$

Inclusions of basic opens preserving the clusters induce equivalences: on generators, they preserve the same orientation class. Exterior factorization

and compact-support Künneth give $K_F(U \star V) \simeq K_F(U) \otimes K_F(V)$ for independent opens, with refinement coherences.

Descent. Each $q_I^{-1}U$ is an open subset of the finite-dimensional manifold $\mathbb{C}^I$. The coefficient-valued compact-support cosheaf at this level satisfies hyperdescent for unbounded complexes. Pulling a hypercover of a Ran open back to each finite level, and exchanging its colimit with the colimit in (8.1), proves hyperdescent for K_F . A uniform bound on the finite-level dimensions is not required. This is the finite-level-to-Ran argument of [Noc20, Proposition 4.5, Theorem 4.6, and Appendix B.31–B.33]; here the finite-level input also follows from the contraction calculation and finite covering dimension.

The associative product. Compactly supported sections have the colax comparison

$$(8.4) \quad R\Gamma_c(W, F \otimes^! G) \longrightarrow R\Gamma_c(W, F) \otimes R\Gamma_c(W, G),$$

induced by the proper diagonal counit. On a basic disk sector it is an equivalence. Naturality of (8.2) gives the commutative diagram

$$(8.5) \quad \begin{array}{ccc} i_z^!(F \otimes^! G) & \xrightarrow{\sim} & i_z^! F \otimes i_z^! G \\ \sim \downarrow & & \downarrow \sim \\ R\Gamma_c(W, F \otimes^! G) & \longrightarrow & R\Gamma_c(W, F) \otimes R\Gamma_c(W, G). \end{array}$$

The vertical maps are the contraction equivalences. After the Ran colimit the same conclusion follows from (8.3). Inverting this canonical colax equivalence makes extraction strong symmetric monoidal on basic opens. Its associativity, symmetry, and compatibility with inclusions are inherited from the counits. In particular,

$$R\Gamma_c(D, \omega_D \otimes A) \simeq A$$

on one disk, with its original product and without an additional shift.

Units and additivity. The constructible-factorization-cosheaf theorem [Lur17, Theorem 5.5.4.10], applied first with target $\mathcal{C}$, gives an E_2^{nu} -algebra. By (8.5), its disk operations preserve the associative products, with all their coherences. Hence the action lifts to the category of unital associative algebras in $\mathcal{C}$. The supplied insertion of an identity-labelled point gives a quasi-unit: fusing it with an occupied disk acts as the identity, by the unit-insertion compatibilities of the original family. This is the associative identity endomorphism, so it is compatible with the associative structure.

The quasi-unit theorem [Lur17, Theorem 5.4.4.5] therefore gives a unital E_2 -algebra in unital associative algebras. Dunn additivity [Lur17, Theorem 5.1.2.2] identifies it with a unital E_3 -algebra. The product calculation above identifies its specified associative restriction with A . $\square$

Apply Lemma 8.1 to (7.3). Its hypotheses follow from (6.9)–(6.10) and (6.17)–(6.18), transported by Lemma 7.1. We obtain a unital E_3 -algebra in rational super $\check{G}$ -representations whose specified E_1 -restriction is $A_X^!$.

8.2. Relative formality.

Lemma 8.2 (A commutative model). *Over a characteristic-zero field, the associative restriction of a unital E_3 -algebra in complexes of rational super representations of a reductive group admits a unital commutative dg model.*

Proof. The rational relative formality theorem of Fresse–Willwacher [FW20, Theorem D'] identifies, for $n \geq 3$, the arrow of unitary chain operads

$$C_*(D_1^+; \mathbb{Q}) \longrightarrow C_*(D_n^+; \mathbb{Q})$$

with the arrow

$$(8.6) \quad \text{Ass}^+ \longrightarrow \text{Pois}_n^+$$

through a zigzag of weak equivalences of operad arrows. The latter factors as $\text{Ass}^+ \rightarrow \text{Com}^+ \rightarrow \text{Pois}_n^+$. Thus the theorem compares the inclusion as well as the two operads. The formulation using augmented Λ -operads encodes the zero-arity operation and input-deletion maps; by [FW20, Introduction and Theorem B'], it is equivalent to the unitary formulation with arity-zero value $\mathbb{Q}$. It requires no augmentation from an algebra to $\mathbb{Q}$.

Tensor this relative diagram with k and take $n = 3$. Exactness of scalar extension preserves its weak equivalences. Give each operation complex trivial group action and even super parity. Transport of algebra structures along this diagram takes place inside the symmetric monoidal category of rational super representations. If A is the specified associative restriction of the original E_3 -algebra, it gives a Pois_3^+ -algebra B and an equivalence

$$(8.7) \quad A \simeq B|_{\text{Ass}^+} \quad \text{in } \text{Alg}_{E_1}(\mathcal{C}).$$

Restriction along $\text{Com}^+ \rightarrow \text{Pois}_3^+$ gives B its unital commutative multiplication.

Characteristic-zero operadic rectification provides a strict dg presentation: rational representations of a reductive group are semisimple, tensor product is exact, and finite-group coinvariants are exact. These properties apply to unbounded complexes with the super symmetry. The relative formality diagram ensures that (8.7) preserves the original product and unit. Since transport takes place in the representation category, it also preserves the group action. $\square$

Proof of Theorem 1.2. Lemmas 6.1 and 6.2 construct the unital factorization associative family (6.16), with singleton algebra $A_X^!$. By (6.18) and Lemmas 7.1–7.2, its complete structure transports to Betti sheaves. Lemma 8.1 enhances the composition algebra $A_X^!$ to a unital E_3 -algebra. By Lemma 8.2, there is a unital commutative dg algebra B and an associative equivalence $A_X^! \simeq B$.

By Lemma 6.3,

$$\mathrm{PL}_X \simeq \sigma^{-1}((A_X^!)^{\mathrm{op}}) \simeq \sigma^{-1}(B^{\mathrm{op}}) = \sigma^{-1}(B).$$

A commutative algebra equals its opposite, and σ^{-1} is symmetric monoidal. Taking $B_X = \sigma^{-1}(B)$ proves the theorem, with the product, unit, rational $\check{G}$ -action, and super symmetry specified in (2.2)–(2.4). $\square$

Proof of Corollary 1.3. In the stated conventions, X is smooth and affine. Section 2.3 gives $X \simeq G \times^H Y$, with H connected reductive and Y an H -representation. Theorem 1.1 proves flatness of $\mathrm{PL}_{X,h}$, and Theorem 1.2 gives the commutative dg model of PL_X . $\square$

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