

PAULI PROBLEM IN DIMENSION 4

ABSTRACT. The finite-dimensional Pauli problem asks how many measurements in orthonormal bases determine every pure state of a d -level quantum system up to a global phase. Four bases always suffice, and the answer is known to be three for $d = 2$ and four for $d = 3$ and $d \geq 5$. Dimension four remained open because the embedding argument used in other dimensions fails there: the pure-state space $\mathbb{CP}^3$ embeds in $\mathbb{R}^9$, the space of the nine independent probabilities of three bases. We show that three bases do not suffice in $\mathbb{C}^4$, so exactly four are needed. The same argument shows that no ten vectors in $\mathbb{C}^4$ do phase retrieval. Since eleven vectors are known to suffice, the smallest phase-retrieval frame and the smallest rank-one POVM distinguishing all pure states in $\mathbb{C}^4$ both have eleven elements. All three lower bounds follow from one statement: every six-dimensional real space of traceless Hermitian 4×4 matrices with a common isotropic vector, that is, a nonzero e with $e^*Te = 0$ for all T in the space, contains a nonzero matrix of rank at most two. We prove it with complex K -theory: otherwise, an odd unitary map on the five-sphere would have a K^1 -class that is nonzero by antipodal symmetry but vanishes because of the isotropic vector.

1. INTRODUCTION

In 1933, Pauli asked whether the position and momentum distributions of a particle determine its wave function up to a constant phase [1, p. 98]. They do not: distinct wave functions can share both distributions [2]. The question has a natural finite-dimensional version. Measuring the pure state $\rho_x = xx^*$ of a unit vector $x \in \mathbb{C}^d$ in an orthonormal basis $B = \{\varphi_1, \dots, \varphi_d\}$ produces the probabilities

$$p_j(x) = |\varphi_j^*x|^2 = \text{tr}(\varphi_j\varphi_j^*\rho_x), \quad j = 1, \dots, d.$$

These probabilities fix the moduli of the coordinates of x in the basis B but not their relative phases, which must be recovered by measuring in further bases. Let b_d be the minimum number of orthonormal bases of $\mathbb{C}^d$ whose measurement statistics distinguish every pair of distinct pure states. We require uniqueness for every state; reconstructing almost every state, as in [3], is a different problem.

Wright conjectured that three bases suffice in every dimension, as reported by Vogt in 1978 [2]. Moroz claimed a proof in 1983. In an erratum, he explained that the proof does not work for all pure states and credited Gromov with an argument showing that at least four bases are needed in large dimensions; see [4, Section 1] for this history. The argument is topological: measurement data that distinguish all pure states embed the

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pure-state manifold $\mathbb{CP}^{d-1}$ in a Euclidean space whose dimension is the number of independent probabilities, which is $3(d-1)$ for three bases. Moroz and Perelomov spelled it out and excluded three bases for $d \geq 9$ [6, Proposition 1], and the bounds of Heinosaari, Mazzarella, and Wolf exclude them for $d \geq 5$ [7, Theorem 6]. In the other direction, four generic bases distinguish all pure states [8, Theorem 2.1], and Jaming gave an explicit construction of four such bases in every dimension [5], presented in detail by Carmeli, Heinosaari, Schultz, and Toigo [4, Section 5 and Proposition 4]. Carmeli et al. collected these results, together with arguments for small dimensions, into the bounds [4, Sections 1 and 6]

$$(1.1) \quad b_2 = 3, \quad b_d = 4 \quad (d = 3 \text{ or } d \geq 5), \quad b_4 \in \{3, 4\}.$$

Among dimensions $d \geq 3$, dimension four is the only one that the embedding argument cannot reach. Three bases of $\mathbb{C}^4$ produce at most nine independent probabilities, and $\mathbb{CP}^3$ does embed smoothly in $\mathbb{R}^9$ [9, Theorem (b), $n = 3$]. Excluding three bases in $\mathbb{C}^4$ therefore requires an argument that uses the form of the measurement, not only the topology of the pure-state space. An earlier partial result of this kind excludes bases of a special form: four product bases $\{u_i \otimes w_j\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$, each built from two orthonormal bases of $\mathbb{C}^2$, do not distinguish all pure states [4, Section 7].

The same gap appears in complex phase retrieval. There the measurement vectors $a_1, \dots, a_N \in \mathbb{C}^d$ are arbitrary, and the intensity measurements $|a_j^* x|^2$ must determine every $x \in \mathbb{C}^d$ up to a global phase; we then say that the vectors do phase retrieval and call $\{a_j\}$ a *phase-retrieval frame*. In quantum terms, a rank-one positive operator-valued measure (POVM) is a measurement whose effects are rank-one operators $a_j a_j^*$ summing to the identity. Every phase-retrieval frame can be normalized to such a POVM without changing its number of vectors, and conversely (Lemma 5.1). Finkelstein showed that a rank-one POVM distinguishing all pure states needs at least $3d - 2$ outcomes, which is ten for $d = 4$ [10, Theorem II]. Bandeira, Cahill, Mixon, and Nelson predicted that fewer than $4d - 4$ vectors never do phase retrieval and that $4d - 4$ generic vectors do [11, Section 2.2]. The second statement was proved by Conca, Edidin, Hering, and Vinzant, for every number $N \geq 4d - 4$ of generic vectors [13, Theorem 1.1]. The first fails in $\mathbb{C}^4$: Vinzant's frame of eleven vectors, one fewer than $4d - 4 = 12$, determines every vector of $\mathbb{C}^4$ up to phase [14, Theorem 1]. Before this work, the smallest phase-retrieval frame in $\mathbb{C}^4$ was thus known to have ten or eleven vectors. Wang and Shang showed that three bases distinguishing all pure states in $\mathbb{C}^4$ would yield a rank-one POVM with ten outcomes that also does so [15, Theorem 2 and Section 3.3]. Excluding rank-one POVMs with ten outcomes would thus also exclude three bases.

In this work we close both gaps.

Theorem 1.1 (Minimal numbers of bases, vectors, and outcomes in dimension four). *Exactly four orthonormal bases are needed to distinguish all pure states in $\mathbb{C}^4$; that is, $b_4 = 4$. Exactly eleven vectors are needed for complex phase retrieval in $\mathbb{C}^4$, and exactly eleven outcomes are needed for a rank-one POVM that distinguishes all pure states in $\mathbb{C}^4$.*

The upper bounds come from Jaming's four-basis construction and Vinzant's eleven-vector frame; our contribution is the three lower bounds. They follow from a statement about the Hermitian matrices orthogonal to all measured effects. Let $\text{Herm}(4)$ be the real vector space of Hermitian 4×4 matrices, equipped with the Hilbert–Schmidt inner product $\langle A, B \rangle = \text{tr}(AB)$, and let $\text{Herm}_0(4)$ be its traceless subspace. Two states give the same statistics exactly when their difference is orthogonal to every measured effect [7, Proposition 2]. We call a nonzero vector $e \in \mathbb{C}^4$ a *common isotropic vector* of a real subspace $V \subseteq \text{Herm}(4)$ if $e^*Te = 0$ for every $T \in V$.

Theorem 1.2 (Common isotropic vector obstruction). *Every six-dimensional real subspace $V \subset \text{Herm}_0(4)$ with a common isotropic vector contains a nonzero matrix of rank at most two.*

The hypotheses of Theorem 1.2 are exactly what the measurements supply. The effects of three bases, or of a rank-one POVM with at most ten outcomes, span a subspace of $\text{Herm}(4)$ of dimension at most ten that contains the identity and a rank-one effect ee^* . Its orthogonal complement therefore has dimension at least $16 - 10 = 6$, consists of traceless matrices, and has e as a common isotropic vector. A nonzero traceless matrix of rank at most two has the form $\lambda(xx^* - yy^*)$ with orthonormal x, y and $\lambda \neq 0$. The matrix supplied by Theorem 1.2 thus exhibits two orthogonal pure states with identical statistics. In Section 2 we carry out this reduction and show that a single rank-one operator in the measured span already suffices.

The common isotropic vector is what lets us exclude dimension six. Without it, six is the best bound we know. Causin showed that a real subspace of $\text{Herm}(4)$ in which every nonzero matrix has rank at least three has dimension at most six [19, Proposition 3.4 and Theorem 3.5]. For traceless subspaces, the same bound follows from the topological lower bound of Heinosaari, Mazzarella, and Wolf [7, Theorem 6] together with Lemma 2.1. Theorem 1.2 shows that dimension six is not attained by a traceless subspace with a common isotropic vector, and the dimension six cannot be lowered (Remark 6.1). Whether dimension six is attained by some traceless subspace is equivalent to the open question of whether a general ten-outcome POVM can distinguish all pure states in $\mathbb{C}^4$ (Section 6).

Proof idea. Suppose that every nonzero matrix in V has rank at least three. We compute one topological invariant in two ways: the class in $K^1(S^5) \cong \mathbb{Z}$ of an odd unitary-valued map on the unit sphere of V . The model case is the Dirac map. Let $\Gamma_1, \dots, \Gamma_5 \in \text{Herm}_0(4)$ be pairwise anticommuting matrices with $\Gamma_a^2 = I_4$, for instance

$$(1.2) \quad \Gamma_k = \sigma_1 \otimes \sigma_k \quad (k = 1, 2, 3), \quad \Gamma_4 = \sigma_2 \otimes I_2, \quad \Gamma_5 = \sigma_3 \otimes I_2,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. On the unit sphere $S^5 \subset \mathbb{R}^6$, the map

$$(1.3) \quad \beta(x) = \sum_{a=1}^5 x_a \Gamma_a + ix_0 I_4$$

is unitary and odd. Because β is odd with values in $U(4)$, its class is nonzero (Lemma 3.1): a null-homotopy would make four copies of the complexified

tautological line bundle over $\mathbb{RP}^6$ stably trivial, which Atiyah's computation $\widetilde{K}^0(\mathbb{RP}^6) \cong \mathbb{Z}/8$ forbids. The matrix size four and the sphere dimension five enter the argument at this point. On the equator $x_0 = 0$, the map β is Hermitian with eigenvalues ± 1 , each of multiplicity two. [Lemma 3.2](#) shows that $[\beta]$ can be nonzero only if the negative spectral bundle over this four-sphere has nonzero degree-four Chern character.

For the space V we build the same structure intrinsically. A small odd perturbation f of the cubic $\frac{1}{3} \operatorname{tr}(T^3)$ takes the place of x_0 . If $\operatorname{tr} T = \operatorname{tr} T^3 = 0$, the characteristic polynomial of T is even, so the eigenvalues of T come in pairs $\pm \mu$, and $\operatorname{rank} T \geq 3$ excludes $\mu = 0$. Hence, for a small perturbation, every T on the zero set Y of f is invertible with two positive and two negative eigenvalues. It follows that $T + if(T)I_4$ is invertible on the whole unit sphere of V , and its unitary polar factor $u(T)$ is odd, so $[u] \neq 0$. The common isotropic vector e now enters. Since $e^*Te = 0$, the vector e lies neither in the positive nor in the negative spectral subspace of T . Its projections $P_{\pm}(T)e$ onto these subspaces are therefore nowhere-vanishing sections of the two spectral bundles over Y . These sections force the degree-four Chern character to vanish, hence $[u] = 0$, a contradiction. The Dirac map escapes this contradiction because $\Gamma_1, \dots, \Gamma_5$ have no common isotropic vector ([Remark 3.3](#)).

The topological input consists of three standard computations in complex K -theory: Atiyah's calculation $\widetilde{K}^0(\mathbb{RP}^6) \cong \mathbb{Z}/8$, injectivity of the Chern character on $K^1(S^5)$, and the description of the Bott periodicity isomorphism by clutching functions. We isolate these computations in [Section 3](#); the rest of the proof consists of explicit spectral deformations and a Chern-class calculation.

Organization. [Section 2](#) reduces the lower bounds in [Theorem 1.1](#) to [Theorem 1.2](#). [Section 3](#) states the K -theoretic input and proves two lemmas, one on odd unitary maps and one localizing the K^1 -class of a unitary polar factor at a hypersurface. [Section 4](#) proves [Theorem 1.2](#). [Section 5](#) completes the proof of [Theorem 1.1](#) by relating phase-retrieval frames to rank-one POVMs. [Section 6](#) shows that [Theorem 1.2](#) is sharp and discusses general POVMs and open problems.

2. FROM MEASUREMENTS TO SUBSPACES OF HERMITIAN MATRICES

In this section we reduce the lower bounds in [Theorem 1.1](#) to [Theorem 1.2](#). Let $\mathcal{M} \subseteq \operatorname{Herm}(4)$ be the real span of the measured observables. Let $K = \mathcal{M}^{\perp}$ be its orthogonal complement with respect to the Hilbert–Schmidt inner product: every $T \in K$ satisfies $\operatorname{tr}(AT) = 0$ for all $A \in \mathcal{M}$. Two states give the same data precisely when their difference lies in K . For pure states, this yields the following rank criterion [[7](#), Proposition 4 and Corollary 1]; see also [[11](#), Lemma 9] for phase retrieval and [[4](#), Proposition 2] for bases.

Lemma 2.1 (Rank criterion). *Let $\mathcal{M} \subseteq \operatorname{Herm}(4)$ be a real subspace containing I_4 , and let $K = \mathcal{M}^{\perp}$. The expectations of the operators in $\mathcal{M}$ distinguish all pure states if and only if K contains no nonzero matrix of rank at most two.*

Proof. If distinct pure states give equal data, then $0 \neq \rho_x - \rho_y \in K$ has rank at most two. Conversely, orthogonality to I_4 makes every matrix in K traceless. By the spectral theorem, a nonzero traceless Hermitian matrix of rank at most two has the form $T = \lambda(xx^* - yy^*)$, where x, y are orthogonal unit vectors and $\lambda \neq 0$. If $T \in K$, the states xx^* and yy^* therefore give identical data. $\square$

A rank-one operator $ee^* \in \mathcal{M}$ forces $e^*Te = \text{tr}(ee^*T) = 0$ for every $T \in K$, so e is a common isotropic vector of K . Together with a dimension count, this isotropy is all that [Theorem 1.2](#) requires.

Proposition 2.2 (Reduction to a single rank-one operator). *Suppose $\mathcal{M} \subseteq \text{Herm}(4)$ contains I_4 and a nonzero rank-one positive operator. If $\dim_{\mathbb{R}} \mathcal{M} \leq 10$, its expectations do not distinguish all pure states.*

Proof. Write the rank-one operator as $w ee^*$, with $w > 0$ and $\|e\| = 1$. Since $\dim_{\mathbb{R}} \text{Herm}(4) = 16$, the orthogonal complement K has dimension at least six. Choose a six-dimensional subspace $V \subseteq K$. Orthogonality to I_4 and $w ee^*$ gives $\text{tr} T = 0$ and $e^*Te = 0$ for every $T \in V$. [Theorem 1.2](#) supplies a nonzero matrix of rank at most two in $V \subseteq K$, so by [Lemma 2.1](#) the expectations do not distinguish all pure states. $\square$

For three bases $B_\ell = \{\varphi_{\ell 1}, \dots, \varphi_{\ell 4}\}$, with $\ell = 1, 2, 3$, write $P_{\ell j} = \varphi_{\ell j} \varphi_{\ell j}^*$. Each basis resolves the identity, so the measured span contains I_4 and satisfies

$$(2.1) \quad P_{\ell 4} = I_4 - \sum_{j=1}^3 P_{\ell j}, \quad \dim_{\mathbb{R}} \text{span}_{\mathbb{R}} \{P_{\ell j}\} \leq 1 + 3(4 - 1) = 10.$$

By (2.1), the effects of three bases span at most ten dimensions. Likewise, if a rank-one POVM has at most ten nonzero effects, these effects span at most ten dimensions and sum to I_4 . Both spans contain a rank-one positive operator, so [Proposition 2.2](#) yields the following consequence of [Theorem 1.2](#).

Corollary 2.3 (Lower bounds for bases and rank-one POVMs). *Neither three orthonormal bases of $\mathbb{C}^4$ nor a rank-one POVM on $\mathbb{C}^4$ with at most ten nonzero effects can distinguish all pure states.*

We prove [Theorem 1.2](#) in [Section 4](#), after developing the topological tools in [Section 3](#). The matching upper bounds and the statement for phase retrieval follow in [Section 5](#).

3. THE TOPOLOGICAL INPUT

The contradiction in [Section 4](#) concerns a unitary map on S^5 whose K -theory class is nonzero by antipodal symmetry but vanishes by a spectral-bundle calculation. In this section we prove the two corresponding lemmas from basic properties of K -theory and Chern classes, together with three standard computations in complex topological K -theory.

We first recall the objects involved. The group $K^0(M)$ consists of formal differences of complex vector bundles over M . The group $K^1(M)$ consists of homotopy classes of unitary-valued maps on M , where two maps are identified if they become homotopic after adjoining identity blocks. Pointwise multiplication of unitary maps adds their classes [[16](#), Lemma 2.4.6].

Concretely, $[g] = 0$ means that $g \oplus I_k$ can be deformed continuously to a constant map for some k . Since $U(n)$ is connected, basepoints play no role. We write $\underline{\mathbb{C}}^r$ for the trivial bundle of rank r , and use integral cohomology for Chern classes and rational cohomology for the Chern character. The bundle calculation uses only the degree-four component of the Chern character; for every complex vector bundle $E \rightarrow M$ [17, Section 4.1],

$$(3.1) \quad \text{ch}^{(4)}(E) = \frac{1}{2}(c_1(E)^2 - 2c_2(E)) \in H^4(M; \mathbb{Q}).$$

Chern classes are natural under pullback and satisfy the Whitney sum formula $c(E \oplus F) = c(E)c(F)$ [17, Theorem 3.2]. The odd Chern character $\text{ch}_{\text{odd}} : K^1(M) \rightarrow H^{\text{odd}}(M; \mathbb{Q})$ is obtained by identifying $K^1(M)$ with $\widetilde{K}^0(\Sigma(M_+))$, applying the Chern character, and using the suspension isomorphism $\widetilde{H}^{k+1}(\Sigma(M_+); \mathbb{Q}) \cong H^k(M; \mathbb{Q})$ [17, discussion preceding Proposition 4.5]. Here M_+ denotes M with a disjoint basepoint, and $\text{ch}_{\text{odd}}^{(5)}$ denotes the degree-five component of ch_{odd} . We use the following three computations.

- (i) *The projective-space calculation.* Let L be the complexification of the tautological real line bundle over $\mathbb{RP}^6$. Then

$$(3.2) \quad \widetilde{K}^0(\mathbb{RP}^6) \cong \mathbb{Z}/8, \quad [L] - [\underline{\mathbb{C}}] \text{ is a generator.}$$

Here $\widetilde{K}^0(\mathbb{RP}^6)$ denotes reduced K -theory, the subgroup of $K^0(\mathbb{RP}^6)$ consisting of classes of virtual rank zero. The calculation (3.2) is part of Atiyah's computation of the complex K -theory of real projective spaces [16, Corollary 2.7.6 and Proposition 2.7.7, pp. 105–107]. By Atiyah's exact sequence, $K^0(\mathbb{RP}^7)$ is a quotient of the representation ring $R(\mathbb{Z}/2) = \mathbb{Z}[\rho]/(\rho^2 - 1)$, and its reduced part is cyclic of order eight. It is therefore generated by the image of $\rho - 1$, which is $[L] - [\underline{\mathbb{C}}]$ under $K_{\mathbb{Z}/2}(S^7) \cong K(\mathbb{RP}^7)$. Restriction $K^0(\mathbb{RP}^7) \rightarrow K^0(\mathbb{RP}^6)$ is an isomorphism.

- (ii) *Detection on the sphere.* The odd Chern character is injective on S^5 :

$$(3.3) \quad \text{ch}_{\text{odd}}^{(5)} : K^1(S^5) \hookrightarrow H^5(S^5; \mathbb{Q}).$$

This follows from Bott periodicity and the Chern-character calculation on even-dimensional spheres, transferred to odd spheres by suspension [17, Theorem 2.11, Proposition 4.3, and the discussion preceding Proposition 4.5].

- (iii) *Clutching and Bott periodicity.* Let $E = \text{im } P$ be a subbundle of a trivial bundle over a compact space M , where P is the orthogonal projection onto E . Let $r : [-1, 1] \rightarrow U(1)$ be a loop with $r(-1) = r(1) = 1$ and winding number ± 1 . Acting by this phase on E and by the identity on its complement gives the unitary map

$$(3.4) \quad v(y, t) = I + (r(t) - 1)P(y).$$

The map is the identity at both ends of the interval, so it descends to the space

$$\Sigma(M_+) = (M \times [-1, 1]) / (M \times \{-1, 1\}),$$

in which all boundary points are collapsed to one point. The degree-five odd Chern character of $[v]$ is

$$(3.5) \quad \text{ch}_{\text{odd}}^{(5)}([v]) = \pm \sigma(\text{ch}^{(4)}(E)),$$

where $\sigma : H^4(M; \mathbb{Q}) \rightarrow \tilde{H}^5(\Sigma(M_+); \mathbb{Q})$ is the suspension isomorphism. To see this, write $z = r(t)$, so that $v = zP + (I - P)$ is a clutching function over $M \times S^1$. The bundle it clutches over $M \times S^2$ is $[E, z] \oplus [E^\perp, 1] \cong E * H \oplus E^\perp * 1$, where H is the canonical line bundle over S^2 [17, Section 2.1, p. 42, Examples 1 and 3]. Since $E * 1 \oplus E^\perp * 1$ is trivial, the reduced class of this bundle is $(H - 1) * [E]$. Under the identification of K^1 with reduced K^0 of the suspension [16, Lemma 2.4.6], the class $[v]$ therefore corresponds to $\pm(H - 1) * [E]$, the image of $[E]$ under the Bott periodicity isomorphism [17, Theorem 2.11]. Formula (3.5) follows because $\text{ch}((H - 1) * x) = c_1(H) \times \text{ch}(x)$ and $c_1(H)$ generates $H^2(S^2; \mathbb{Z})$, together with the definition of the odd Chern character by suspension [17, Proposition 4.2, the proof of Proposition 4.3, and the discussion preceding Proposition 4.5].

The first lemma uses antipodal symmetry to build a bundle over $\mathbb{RP}^6$, to which we apply (3.2).

Lemma 3.1 (An odd map has nonzero K^1 -class). *If $g : S^5 \rightarrow U(4)$ is continuous and satisfies $g(-x) = -g(x)$, then $[g] \neq 0$ in $K^1(S^5)$.*

Proof. Suppose that $[g] = 0$, and choose k and a homotopy h from $g \oplus I_k$ to a constant map. Then $G(sx) = h(x, 1 - s)$, for $x \in S^5$ and $s \in [0, 1]$, extends $g \oplus I_k$ to a continuous map $G : D^6 \rightarrow U(4 + k)$. Regard $\mathbb{RP}^6$ as the disk D^6 with antipodal boundary points identified. In this model, $L^{\oplus 4} \oplus \underline{\mathbb{C}}^k$ is the quotient of $D^6 \times \mathbb{C}^{4+k}$ by the boundary identifications

$$(x, z) \sim (-x, Dz), \quad D = \text{diag}(-I_4, I_k).$$

On the boundary, oddness gives $G(-x) = DG(x)$, which is exactly the compatibility needed for the columns of G to descend to this quotient. Since G is unitary, these columns are $4 + k$ sections that are linearly independent in every fiber. The bundle $L^{\oplus 4} \oplus \underline{\mathbb{C}}^k$ is therefore trivial, so

$$(3.6) \quad 4([L] - [\underline{\mathbb{C}}]) = 0 \quad \text{in } \widetilde{K}^0(\mathbb{RP}^6).$$

Relation (3.6) contradicts (3.2), because $[L] - [\underline{\mathbb{C}}]$ has order eight. $\square$

The second lemma reduces the vanishing of the K^1 -class of a unitary polar factor to a Chern-character calculation on a regular hypersurface.

Lemma 3.2 (Localization at a regular zero set). *Let $X \cong S^5$, let $f : X \rightarrow \mathbb{R}$ be smooth with 0 a regular value, and suppose $Y = f^{-1}(0)$ is nonempty. Let $B : X \rightarrow \text{Herm}(4)$ be smooth and invertible on Y , and let $E_- \rightarrow Y$ be its negative spectral bundle. If u_B is the unitary polar factor of $B + ifI_4$, then*

$$\text{ch}^{(4)}(E_-) = 0 \implies [u_B] = 0 \text{ in } K^1(X).$$

Proof. Deformation to the form (3.4). The matrix $B + ifI_4$ is invertible: a real eigenvalue of B can cancel if only when $f = 0$, where B is invertible. We deform its polar factor into the form (3.4) near Y . Since X is compact and 0 is a regular value, $df \neq 0$ on $f^{-1}([-\varepsilon, \varepsilon])$ for small $\varepsilon > 0$, and the flow of

$\nabla f/|\nabla f|^2$ identifies this set with a tube $Y \times [-\varepsilon, \varepsilon]$ whose normal coordinate is $t = f$. Let $P_-(y)$ be the negative spectral projection of $B(y)$. Replacing the eigenvalues of $B(y)$ by their signs gives $S(y) = \operatorname{sgn} B(y) = I_4 - 2P_-(y)$. Let $\kappa : [-\varepsilon, \varepsilon] \rightarrow [0, 1]$ be smooth, equal to 1 near zero and to 0 near the endpoints. Set $B_1(y, t) = \kappa(t)S(y)$ in the tube and $B_1 = 0$ outside. The straight-line interpolation $B_s = (1 - s)B + sB_1$ keeps $B_s + ifI_4$ invertible. Indeed, off Y the imaginary scalar part is nonzero, and on Y the interpolation from $B(y)$ to $\operatorname{sgn} B(y)$ preserves the sign of every eigenvalue. Thus u_B is homotopic to the polar factor u_1 of $B_1 + ifI_4$.

In the tube, u_1 multiplies the positive spectral subspace by the phase a_+ and the negative one by a_+r , where

$$(3.7) \quad a_+(t) = \frac{\kappa(t) + it}{\sqrt{\kappa(t)^2 + t^2}}, \quad r(t) = \frac{-\kappa(t) + it}{\kappa(t) + it}.$$

The scalar map a_+ extends to $\alpha : X \rightarrow U(1)$ by taking $\alpha = i \operatorname{sgn} f$ outside the tube. Since X is simply connected, α has a continuous real argument and is null-homotopic. Dividing u_1 by α therefore leaves its class in $K^1(X)$ unchanged and gives

$$(3.8) \quad v(y, t) = I_4 + (r(t) - 1)P_-(y)$$

inside the tube and the identity outside. The loop r in (3.7) starts and ends at 1 and has winding number -1 : the argument of $z(t) = \kappa(t) + it$ increases by π , and $r = -\bar{z}/z$.

Collapse to the suspension. Let $q : X \rightarrow \Sigma(Y_+)$ collapse the complement of the tube to the basepoint and rescale the normal interval to $[-1, 1]$. The map (3.8) is the pullback under q of the map (3.4) with $M = Y$, $E = E_-$, and the loop r reparametrized to $[-1, 1]$. By naturality and (3.5),

$$(3.9) \quad \operatorname{ch}_{\text{odd}}^{(5)}([u_B]) = \pm q^* \sigma(\operatorname{ch}^{(4)}(E_-)).$$

If $\operatorname{ch}^{(4)}(E_-) = 0$, the right-hand side of (3.9) vanishes, and injectivity in (3.3) gives $[u_B] = 0$. $\square$

The proof uses neither connectedness of Y nor an orientation convention. The Dirac map (1.3) shows that the hypothesis $\operatorname{ch}^{(4)}(E_-) = 0$ in Lemma 3.2 is a genuine restriction.

Remark 3.3 (The Dirac map). Take $X = S^5 \subset \mathbb{R}^6$, $f(x) = x_0$, and $B(x) = \sum_{a=1}^5 x_a \Gamma_a$, with Γ_a as in (1.2). Then 0 is a regular value of f , the zero set Y is a four-sphere, and $B^2 = I_4$ on Y . Since $B(x)^2 + x_0^2 I_4 = I_4$ on X , the polar factor u_B is the Dirac map β of (1.3). The map β is odd, so $[\beta] \neq 0$ by Lemma 3.1, and Lemma 3.2 forces $\operatorname{ch}^{(4)}(E_-) \neq 0$. (Up to orientation, E_- is the quaternionic Hopf bundle over $S^4 \cong \mathbb{H}\mathbb{P}^1$, whose second Chern class generates $H^4(S^4; \mathbb{Z})$.) Consequently, the span of $\Gamma_1, \dots, \Gamma_5$ has no common isotropic vector, since the argument of Section 4.3 applied to B would otherwise give $\operatorname{ch}^{(4)}(E_-) = 0$. The same conclusion follows directly from the identity $\sum_{a=1}^5 (e^* \Gamma_a e)^2 = \|e\|^4$ for $e \in \mathbb{C}^4$, which one checks by expanding ee^* in the Hilbert–Schmidt orthogonal basis $\{I_4, \Gamma_a, i\Gamma_a \Gamma_b\}_{a < b}$ of $\operatorname{Herm}(4)$.

4. PROOF OF THE COMMON ISOTROPIC VECTOR OBSTRUCTION

In this section we prove [Theorem 1.2](#). Let $V \subset \text{Herm}_0(4)$ be a six-dimensional real subspace with a common isotropic vector e , normalized so that $\|e\| = 1$. Assume for contradiction that every nonzero $T \in V$ has rank at least three. We construct an odd unitary polar factor and use e to verify the vanishing condition in [Lemma 3.2](#). Put

$$X = S(V) = \{T \in V : \text{tr}(T^2) = 1\} \cong S^5, \quad c(T) = \frac{1}{3} \text{tr}(T^3).$$

The odd cubic c selects a hypersurface on which the positive and negative spectral bundles both have rank two.

4.1. A regular hypersurface of matrices of signature $(2, 2)$. Newton's identities give

$$(4.1) \quad \det(\lambda I_4 - T) = \lambda^4 - \frac{1}{2} \text{tr}(T^2) \lambda^2 - c(T) \lambda + \det T$$

for every traceless Hermitian 4×4 matrix T . If $c(T) = \det T = 0$, the characteristic polynomial (4.1) is divisible by λ^2 , so by the spectral theorem $\text{rank } T \leq 2$. Our rank assumption therefore makes every $T \in Z = c^{-1}(0) \subset X$ invertible. On Z , the characteristic polynomial is even, so its roots occur in nonzero opposite pairs and T has signature $(2, 2)$. Since invertibility and signature are stable under small perturbations, this signature persists on an open neighborhood $\mathcal{U}$ of Z .

[Lemma 3.2](#) requires 0 to be a regular value, so we perturb the cubic while preserving oddness. For a sufficiently small $R \in V$, put

$$f(T) = c(T) + \langle R, T \rangle.$$

The map $F : X \times V \rightarrow \mathbb{R}$, $F(T, R) = c(T) + \langle R, T \rangle$, is a submersion: at (T, R) , its derivative with respect to R in the direction T is $\langle T, T \rangle = 1$. Parametric transversality therefore makes 0 a regular value of f for arbitrarily small choices of R [[18](#), Chapter 3, Theorem 2.7]. Since $X \setminus \mathcal{U}$ is compact and c does not vanish there, $|c|$ has a positive minimum m on $X \setminus \mathcal{U}$. Since $\langle T, T \rangle = 1$ on X , the Cauchy–Schwarz inequality gives $|\langle R, T \rangle| \leq \|R\|$, where $\|R\| = \langle R, R \rangle^{1/2}$. Choosing R as above with $\|R\| < m$ therefore gives $|f| \geq m - \|R\| > 0$ on $X \setminus \mathcal{U}$, so $Y = f^{-1}(0) \subset \mathcal{U}$. Oddness of f makes Y nonempty. Thus Y is a closed smooth four-manifold consisting of invertible matrices of signature $(2, 2)$.

4.2. An odd unitary map. Adding the imaginary scalar $if(T)I_4$ gives an invertible family with antipodal symmetry. Indeed, when $f(T) \neq 0$, the eigenvalues of $T + if(T)I_4$ have nonzero imaginary part; when $f(T) = 0$, the matrix T is invertible because $T \in Y$. The unitary polar factor is therefore defined everywhere by

$$(4.2) \quad u(T) = (T + if(T)I_4)(T^2 + f(T)^2 I_4)^{-1/2}.$$

Since f is odd, $u(-T) = -u(T)$. Identifying X with S^5 through a linear isometry $V \cong \mathbb{R}^6$, which commutes with $T \mapsto -T$, [Lemma 3.1](#) gives

$$(4.3) \quad [u] \neq 0 \quad \text{in } K^1(X).$$

4.3. Two sections and a Chern-class calculation. The common isotropic vector now forces the same class to vanish. Over Y , let $P_{\pm}(T)$ be the positive and negative spectral projections of T , and let $E_{\pm}$ be their image bundles. The spectral gap at zero makes these smooth rank-two bundles, with

$$(4.4) \quad E_- \oplus E_+ \cong \underline{\mathbb{C}}^4.$$

Both sections $P_-(T)e$ and $P_+(T)e$ are nowhere zero. Indeed, if either vanished, the nonzero vector e would lie in the positive or the negative spectral subspace of T , on which T is definite, contradicting $e^*Te = 0$. Each section splits off a trivial line bundle, so $c_2(E_-) = c_2(E_+) = 0$.

Put $a = c_1(E_-)$ and $b = c_1(E_+)$. The degree-two and degree-four terms of the identity $c(E_-)c(E_+) = 1$, which follows from (4.4) and the Whitney sum formula, give

$$(4.5) \quad a + b = 0, \quad ab = 0, \quad \text{and hence} \quad a^2 = -ab = 0.$$

Consequently, by (3.1) and (4.5),

$$(4.6) \quad \text{ch}^{(4)}(E_-) = \frac{1}{2}(c_1(E_-)^2 - 2c_2(E_-)) = 0.$$

By the construction of Y and (4.6), all hypotheses of Lemma 3.2 hold for $B(T) = T$. Hence the map u of (4.2) satisfies $[u] = 0$, contradicting (4.3). The contradiction proves Theorem 1.2, and with it Corollary 2.3.

5. PROOF OF THE MAIN THEOREM

Corollary 2.3 gives the lower bounds for bases and for rank-one POVMs. Jaming's four-basis construction [5], as presented in [4, Proposition 4], matches the first bound, so $b_4 = 4$, which settles the remaining case of (1.1). To complete Theorem 1.1, we transfer the POVM bound to phase-retrieval frames without changing their cardinality, and then apply Vinzant's construction.

Lemma 5.1 (Phase-retrieval frames and rank-one POVMs). *In every finite dimension, the minimum cardinality of a complex phase-retrieval frame equals the minimum number of nonzero effects of a rank-one POVM distinguishing all pure states.*

Proof. Discarding zero vectors from a frame, or zero effects from a POVM, affects neither phase retrieval nor the property of distinguishing all pure states, so we assume that all vectors and effects are nonzero. Suppose $a_1, \dots, a_N \in \mathbb{C}^d$ recover every vector from its intensities up to a global phase. They span $\mathbb{C}^d$, because phase retrieval distinguishes every nonzero vector from 0, whereas all intensities vanish on the orthogonal complement of their span. Their frame operator $S = \sum_j a_j a_j^*$ is therefore positive definite, and the canonical Parseval frame associated with $\{a_j\}$ [12, Section 2] is

$$(5.1) \quad \tilde{a}_j = S^{-1/2} a_j, \quad \sum_j \tilde{a}_j \tilde{a}_j^* = I_d.$$

Equality of the $\tilde{a}_j$ -intensities for x, y is equality of the a_j -intensities for $S^{-1/2}x, S^{-1/2}y$. Since $S^{-1/2}$ is invertible, the normalized vectors retain phase retrieval, and their rank-one effects form a POVM distinguishing all pure states.

Conversely, let $F_j = \tilde{a}_j \tilde{a}_j^*$ be the effects of a rank-one POVM distinguishing all pure states, and suppose $x, y \in \mathbb{C}^d$ give equal intensities. Summing the intensities gives $\|x\|^2 = \|y\|^2$. If this common norm is nonzero, the unit vectors $x/\|x\|$ and $y/\|y\|$ give the same statistics, so they define the same pure state, and $x = e^{i\theta}y$ because $\|x\| = \|y\|$; the zero case is immediate. Thus the vectors $\tilde{a}_j$ perform phase retrieval on all of $\mathbb{C}^d$. $\square$

That passing to the canonical Parseval frame preserves phase retrieval goes back to [12, Proposition 2.1 and Section 3]; in the quantum setting, the same normalization by $S^{-1/2}$ appears in [10, Section 2] and in the rank-one conversion of [15, Theorem 1].

By Corollary 2.3 and Lemma 5.1, no ten or fewer vectors do phase retrieval in $\mathbb{C}^4$. Vinzant's eleven-vector frame [14, Equation (2) and Theorem 1] attains this bound, and its canonical Parseval frame (5.1) attains the POVM bound. With $b_4 = 4$ established above, this completes the proof of Theorem 1.1.

6. DISCUSSION

We have shown that in $\mathbb{C}^4$ a measured span of dimension at most ten that contains the identity and a single rank-one operator cannot distinguish all pure states (Proposition 2.2). Every rank-one Hermitian matrix is a real multiple of a rank-one positive operator. Hence, if a POVM on $\mathbb{C}^4$ with ten outcomes distinguishes all pure states, the real span of its effects contains no rank-one matrix; in particular, every nonzero effect has rank at least two. Whether such a POVM exists is open. If the effects may have any rank, all that is known in $\mathbb{C}^4$ is that ten outcomes are necessary and eleven suffice. Heinosaari, Mazzarella, and Wolf count self-adjoint operators besides the identity, and in their terms the lower and upper bounds are nine and ten [7, Theorem 6, and Theorem 3 as corrected in arXiv v3]; an n -outcome POVM corresponds to $n - 1$ such operators [7, Proposition 3]. The canonical Parseval frame associated with Vinzant's frame also gives a rank-one POVM with eleven outcomes.

The dimension in Theorem 1.2 is optimal.

Remark 6.1 (Sharpness of the dimension). Let $\tilde{a}_1, \dots, \tilde{a}_{11}$ be the canonical Parseval frame (5.1) associated with Vinzant's frame, and let K be the orthogonal complement of the span of the effects $\tilde{a}_j \tilde{a}_j^*$. Then K is traceless, has dimension at least $16 - 11 = 5$, and has every $\tilde{a}_j$ as a common isotropic vector. By [14, Theorem 1] and the proof of Lemma 5.1, these effects form a POVM distinguishing all pure states, so Lemma 2.1 shows that K contains no nonzero matrix of rank at most two. By Theorem 1.2, K cannot contain a six-dimensional subspace, so $\dim K = 5$. Thus Theorem 1.2 fails for five-dimensional subspaces.

The existence of a ten-outcome POVM distinguishing all pure states in $\mathbb{C}^4$ has an exact matrix formulation. If $V \subset \text{Herm}_0(4)$ is a six-dimensional subspace, then $\mathcal{M} = V^\perp$ is a ten-dimensional subspace containing I_4 . It is spanned by ten positive effects that sum to the identity. Indeed, if $I_4, A_1, \dots, A_9$ is a basis of $\mathcal{M}$, we may take $F_j = \frac{1}{10}(I_4 + \varepsilon A_j)$ for $j \leq 9$ and $F_{10} = \frac{1}{10}(I_4 - \varepsilon \sum_{j=1}^9 A_j)$ with $\varepsilon > 0$ small. Conversely, the effects of a ten-outcome POVM span a subspace of dimension at most ten that contains I_4 ,

and its orthogonal complement contains a six-dimensional traceless subspace. Together with [Lemma 2.1](#), these two constructions show that a ten-outcome POVM distinguishing all pure states in $\mathbb{C}^4$ exists if and only if $\text{Herm}_0(4)$ contains a six-dimensional subspace in which every nonzero matrix has rank at least three. By [Theorem 1.2](#), such a subspace has no common isotropic vector.

We close with two open problems.

- (1) Does $\text{Herm}_0(4)$ contain a six-dimensional real subspace in which every nonzero matrix has rank at least three? Equivalently, can a ten-outcome POVM distinguish all pure states in $\mathbb{C}^4$? For such a subspace, our argument breaks down only in its last step. [Sections 4.1](#) and [4.2](#) use only the rank condition, so the odd unitary map u still has nonzero class, and [Lemma 3.2](#) then forces $\text{ch}^{(4)}(E_-) \neq 0$ on Y , as happens for the Dirac map ([Remark 3.3](#)). A negative answer would therefore follow from any argument showing that $\text{ch}^{(4)}(E_-) = 0$ for such a subspace without a common isotropic vector.
- (2) A rank-one measurement with N outcomes supplies N common isotropic vectors of the orthogonal complement of its effects, one for each effect, whereas [Theorem 1.2](#) uses only one of them. Can several common isotropic vectors be combined to improve lower bounds on the size of phase-retrieval frames in dimensions where the minimum is still unknown? Our argument uses dimension four in two places: in [Lemma 3.1](#), through $4([L] - [\mathbb{C}]) \neq 0$ in $\widetilde{K}^0(\mathbb{RP}^6)$, and in [Section 4](#), where a traceless 4×4 matrix of rank at least three with $\text{tr} T^3 = 0$ is invertible. In higher dimensions the absence of matrices of rank at most two gives no comparable control of the spectrum, so the spectral bundles would have to be constructed differently.

CONCURRENT WORK

While this manuscript was in preparation, Huang posted a preprint (30 July 2026) with the same lower bounds: no ten vectors in $\mathbb{C}^4$ do phase retrieval, and consequently neither a rank-one POVM with ten outcomes nor three bases distinguish all pure states [\[20\]](#). Later versions of that preprint, with the same proof, are coauthored with Kim. The two proofs were found independently, and they take place on different spaces. Huang works on the pure-state manifold. There, if ten vectors did phase retrieval in $\mathbb{C}^4$, the outcome probabilities of the rank-one POVM given by their canonical Parseval frame would embed $\mathbb{CP}^3$ in the affine hyperplane of $\mathbb{R}^{10}$ on which they sum to one, a copy of $\mathbb{R}^9$. Each of these probabilities vanishes on the projective plane of states orthogonal to the corresponding vector, so its differential vanishes there, and this yields a nonvanishing normal vector field over that plane. Such a field is ruled out by the first Pontryagin class of the normal bundle, which is determined by the tangent bundle of $\mathbb{CP}^3$ [\[20, Theorem 1.2 and Section 3.3\]](#). We work instead on the unit sphere of a six-dimensional space of traceless Hermitian matrices orthogonal to the measured effects, where antipodal symmetry and complex K -theory take the place of the normal bundle and its Pontryagin class. In both proofs, the rank-one effect enters through a nowhere-vanishing section of a bundle.

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