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QUANTUM LAURENT POSITIVITY FOR MUTATION-ACYCLIC CLUSTER ALGEBRAS

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Abstract. We prove quantum Laurent positivity for mutation-acyclic skew-symmetrizable cluster algebras of arbitrary finite rank. For every integral compatible quantization with invertible frozen variables, each cluster variable has a Laurent expansion with coefficients in $\mathbb{N}[v^{\pm 1}]$ in every seed. For an indecomposable rigid representation of an acyclic valued quiver, we identify its Hall shuffle character with the character of a self-dual simple quiver Hecke module. These modules are real and admit affinizations; modules corresponding to the summands of a rigid direct sum strongly commute. A Feigin localization and a comparison of graded composition multiplicities establish positivity at an arbitrary target seed.

1. Introduction

Quantum cluster algebras are generated by variables obtained through iterated mutation. The quantum Laurent phenomenon states that every cluster variable is a finite Laurent polynomial in the variables of any seed [2, Corollary 5.2]. Quantum Laurent positivity asks whether, when this expansion is written in normalized quantum monomials, each coefficient is a Laurent polynomial in the quantum parameter with nonnegative integer coefficients.

For classical cluster algebras, Lee and Schiffler proved positivity for skew-symmetric cluster algebras of arbitrary rank [9, Theorem 1.1]. Davison established quantum Laurent positivity for skew-symmetric quantum cluster algebras by cohomological methods [3, Theorem 2.4].

We write n for the number of mutable directions.

Let $\widetilde{B} \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ have skew-symmetrizable principal part B, and let Λ be an integral skew-symmetric matrix such that

$$\widetilde{B}^T\Lambda = (D \;\; 0), \qquad D = \mathrm{diag}(d_1,\dots,d_n), \quad d_i > 0. \tag{1.1}$$

Let v be an indeterminate, and work over $\mathbb{Z}[v^{\pm 1}]$. When using [2, 15], we identify their formal parameter $q^{1/2}$ with v; this is separate from the finite-field parameter q used in Section 3.1. We use the based quantum torus convention

$$M(a)M(b) = v^{a^T\Lambda b}M(a+b). \tag{1.2}$$

Frozen variables are formal invertible generators of the based quantum torus.

A principal matrix is *mutation-acyclic* if some matrix in its mutation class has an acyclic oriented graph.

Theorem 1.1. *In every mutation-acyclic quantum cluster pattern of finite rank, each cluster variable has a Laurent expansion with coefficients in $\mathbb{N}[v^{\pm 1}]$ in every seed.*

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The representation-theoretic input for Theorem 1.1 is the following character identity. We use the finite-field valued-quiver realization and the quiver Hecke parameters over an algebraically closed field of characteristic zero fixed in Section 3.1. Here Ω is the Hall-to-shuffle map; its value on $[V]^*$ is understood generically, using the counting polynomials of rigid representations, as explained in that section.

Theorem 1.2. *For an indecomposable rigid representation V of an acyclic valued quiver, there is a self-dual simple quiver Hecke module L_V such that*

$$\Omega([V]^*) = \mathrm{ch}_{t^{-1}} L_V.$$

The module L_V is real and admits an affinization. If $V_1, \dots, V_s$ are indecomposable representations and $V_1 \oplus \cdots \oplus V_s$ is rigid, the modules L_{V_i} strongly commute and their ordered convolution is simple. Repeated isomorphism classes among the V_i are allowed.

For an acyclic valued quiver over a finite field, the quantum cluster character identifies isomorphism classes of indecomposable rigid representations with non-initial quantum cluster variables [11, Theorem 1.1]; the corresponding Feigin realization is given in [1, Main Theorem 2.9(b)]. The character identity in Theorem 1.2 is the indecomposable part of Rupel's Conjecture 8.2 in [12]. Rupel proved this conjecture for equal vertex weights [12, Theorem 8.4]. Here we treat general valuations and obtain the affinization and commutation properties needed for the positivity argument. The categorical passage to Laurent expansions uses the localizations of [4, 5]; simplicity of convolutions is obtained from the normal-sequence criterion [6, Lemma 2.10].

The proof proceeds through two categorical constructions. In Section 3, stability places the Hall character of a rigid indecomposable in a single semicuspidal direction. A support lemma and an extremal word identify the corresponding simple character. Reflection then supplies affinizations, which allow us to turn Hall commutation identities into simple convolutions. Section 4 realizes these objects in a localized category and identifies its Grothendieck ring with a subalgebra of the Feigin torus.

In infinite irreducible type, for a variable X and a target cluster T, a Coxeter belt supplies an acyclic reference seed relative to which X and every member of T are noninitial. The other types reduce to this case by adjoining and then freezing one vertex. The cluster-character correspondence now lifts X to a simple object and the mutable variables of T to strongly commuting simples. Frozen variables lift to invertible objects. After clearing mutable Laurent denominators, each target monomial is a simple class, while the left-hand product is an object of finite length. Comparing its composition multiplicities proves positivity. Coefficient transport returns the result to the prescribed quantization. Section 5 carries out this argument.

2. Quantum seeds and coefficient transport

2.1. **Quantum seeds and normalization.** A toric frame has based monomials $M(a)$, $a \in \mathbb{Z}^m$, satisfying (1.2). Mutable labels are $1, \dots, n$. For a column $\widetilde{B}_k$, write $[\widetilde{B}_k]_+$ for its componentwise positive part. Mutation at k replaces $X_k = M(\mathbf{e}_k)$ by

$$X_k' = M(-\mathbf{e}_k + [\widetilde{B}_k]_+) + M(-\mathbf{e}_k + [-\widetilde{B}_k]_+). \tag{2.1}$$

The other generators are retained, and the matrix and form mutate according to the usual BZ rules [2].

The initial torus carries the $\mathbb{Z}$-linear bar anti-involution defined by

$$\overline{v^r M(a)} = v^{-r} M(a), \qquad \overline{AB} = \overline{B}\,\overline{A}.$$

For the pairwise quasi-commuting generators of any one seed, the normalized monomial is

$$M(a) = v^{-\sum_{i<j} a_i a_j \lambda_{ij}} X_1^{a_1} \cdots X_m^{a_m}.$$

If these generators are fixed by bar, reversing their order cancels the change in the normalization factor, so $M(a)$ is also fixed, including when some a_i are negative. Equation (2.1) therefore propagates bar invariance along every mutation word. All cluster variables are fixed by the bar associated with the initial seed; see also [2, Proposition 6.2 and Remark 6.4].

2.2. **Transport between compatible realizations.**

Lemma 2.1. *Fix B, D as in (1.1), a finite mutation word, and a mutable output label $j \in \{1, \dots, n\}$. All realizations below start with principal part B and compatible diagonal D, and use this same word and label. Set*

$$P = \begin{pmatrix} B \\ I_n \end{pmatrix}, \qquad \Lambda_P = \begin{pmatrix} 0 & -D \\ D & -DB \end{pmatrix}.$$

Let M_P be the initial toric frame of this principal realization, and let $g \in \mathbb{Z}^n$ be the classical g-vector of the chosen output variable relative to the initial seed. Write

$$X^P = \sum_{e \in \mathbb{N}^n} C_e(v) M_P((g, 0) + Pe).$$

Here $C_e(v) \in \mathbb{Z}[v^{\pm 1}]$, and only finitely many coefficients are nonzero. For every compatible realization with the same B, D, the corresponding variable is exactly

$$X = \sum_{e} C_e(v) M(\widetilde{g} + \widetilde{B}e), \tag{2.2}$$

Here M is the initial toric frame of the chosen realization, and $\widetilde{g} \in \mathbb{Z}^m$ is the extended classical g-vector of the corresponding output variable, as defined in [15, Definition 3.7]. *Its first n coordinates are g; its frozen coordinates are the exponents of the inverse tropical denominator in the classical separation formula. Both exponent maps are injective.*

Proof. Since $B^T D = -DB$, the matrix Λ_P is skew-symmetric and $P^T \Lambda_P = (D \;\; 0)$. Let $\mathcal{T}_{DB}$ be the based quantum torus over $\mathbb{Z}[v^{\pm 1}]$ with basis $\{Z(e) : e \in \mathbb{Z}^n\}$ and multiplication

$$Z(e)Z(f) = v^{e^T DBf} Z(e + f).$$

Let $\mathcal{T}_P$ and $\mathcal{T}_M$ denote the initial based quantum tori with bases $\{M_P(a) : a \in \mathbb{Z}^{2n}\}$ and $\{M(a) : a \in \mathbb{Z}^m\}$, respectively. Define $\mathbb{Z}[v^{\pm 1}]$-linear maps

$$\iota_P : \mathcal{T}_{DB} \longrightarrow \mathcal{T}_P, \qquad \iota_{\widetilde{B}} : \mathcal{T}_{DB} \longrightarrow \mathcal{T}_M$$

by their values on basis elements:

$$\iota_P(Z(e)) = M_P(Pe), \qquad \iota_{\widetilde{B}}(Z(e)) = M(\widetilde{B}e) \quad (e \in \mathbb{Z}^n).$$

Compatibility gives

$$P^T \Lambda_P P = DB = \widetilde{B}^T \Lambda \widetilde{B}.$$

Thus, for all $e, f \in \mathbb{Z}^n$,

$$\begin{aligned} \iota_{\widetilde{B}}(Z(e))\iota_{\widetilde{B}}(Z(f)) &= M(\widetilde{B}e)M(\widetilde{B}f) \\ &= v^{e^T DBf} M(\widetilde{B}(e + f)) \\ &= \iota_{\widetilde{B}}(Z(e)Z(f)). \end{aligned}$$

The same calculation applies to ι_P, and both maps send $Z(0)$ to 1. Hence they are unital $\mathbb{Z}[v^{\pm 1}]$-algebra homomorphisms.

By [15, Theorem 5.3(1),(2)], with its parameter $q^{1/2}$ identified with v, there is a common left quantum F-polynomial

$$F = \sum_{e \in \mathbb{N}^n} f_e(v) Z(e), \qquad f_e(v) \in \mathbb{Z}[v^{\pm 1}],$$

with finite support, such that

$$X^P = \iota_P(F)M_P((g,0)), \qquad X = v^s \iota_{\widetilde{B}}(F)M(\widetilde{g})$$

for some $s \in \mathbb{Z}$. The first n coordinates of $\widetilde{g}$ are g, so

$$(\widetilde{B}e)^T \Lambda \widetilde{g} = e^T Dg, \qquad (Pe)^T \Lambda_P(g,0) = e^T Dg.$$

Consequently,

$$C_e(v) = v^{e^T Dg} f_e(v), \qquad X = \sum_e v^s C_e(v) M(\widetilde{g} + \widetilde{B}e).$$

Here coefficient comparison in the principal realization is valid because $(g,0) + Pe = (g + Be, e)$ determines e.

For the other exponent map, if $\widetilde{B}h = 0$, then

$$0 = (\widetilde{B}h)^T \Lambda = h^T (D \;\; 0),$$

so $h = 0$ since D is invertible. Thus both expansions have distinct based monomials. Bar invariance of the output variables, applied in their respective initial tori, now gives

$$C_e(v^{-1}) = C_e(v), \qquad v^s C_e(v) = v^{-s} C_e(v^{-1}).$$

Since $X^P \neq 0$, some $C_e(v)$ is nonzero. For this e,

$$(v^s - v^{-s}) C_e(v) = 0.$$

The ring $\mathbb{Z}[v^{\pm 1}]$ is an integral domain and v is indeterminate, hence $s = 0$. This proves (2.2). □

To compare Laurent expansions at a specified target, we root the pattern at that target and apply Lemma 2.1 to a word leading to the variable to be expanded. Frozen coordinates remain part of the exponent vector. We shall also need to double the compatible diagonal when passing to the Feigin realization. The next lemma removes this change of parameter; "canonical" refers to the pair (P, Λ_P) in Lemma 2.1.

Lemma 2.2. *Fix B, D as in (1.1), a finite mutation word, and a mutable output label. Let X^D and X^{2D} be the corresponding cluster variables in the canonical realizations for D and $2D$, with indeterminates v and u, respectively. Write their Laurent expansions in the respective initial based tori as*

$$X^D = \sum_{a \in \mathbb{Z}^{2n}} c_a^D(v) M_D(a), \qquad X^{2D} = \sum_{a \in \mathbb{Z}^{2n}} c_a^{2D}(u) M_{2D}(a),$$

where M_D and M_{2D} are the initial toric frames. Then

$$c_a^{2D}(u) = c_a^D(u^2) \qquad (a \in \mathbb{Z}^{2n}).$$

In particular, one expansion has nonnegative Laurent coefficients if and only if the other does.

Proof. The two realizations have the same exchange matrix P, and their forms satisfy $\Lambda_P(2D) = 2\Lambda_P(D)$. Thus the assignment

$$f(v) M_D(a) \longmapsto f(u^2) M_{2D}(a) \qquad (f(v) \in \mathbb{Z}[v^{\pm 1}])$$

extends additively to an injective homomorphism of based quantum tori: multiplication agrees by (1.2), and injectivity follows from the independence of the based monomials and the injectivity of $f(v) \mapsto f(u^2)$. It extends to their fraction skew fields and intertwines mutation by (2.1). Induction along the fixed word therefore sends X^D to X^{2D}. Comparing coefficients in the initial based torus gives the stated identity. Finally, substitution $v = u^2$ doubles each exponent of a coefficient polynomial without changing its integer coefficient, which proves the equivalence of positivity. □

3. Rigid representations and simple characters

The proof of Theorem 1.2 has three parts. First, stability and semicuspidal decomposition show that a rigid Hall character is a scalar multiple of one simple character. An extremal word determines that scalar. We then construct an affinization and use Hall multiplication to treat rigid direct sums.

3.1. **Hall and shuffle conventions.** Let Q be a finite acyclic valued quiver with vertex set I, realized over the vertex fields $\mathbb{F}_{q^{d_i}}$. Write S_i for the simple representation at vertex i and $\alpha_i = \dim S_i$. Dimension vectors record dimensions over the vertex fields and belong to $Q_+ = \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i$; we also use $Q_{\mathbb{Q}} = \bigoplus_i \mathbb{Q}\alpha_i$. A representation is *rigid* if $\operatorname{Ext}^1(V,V) = 0$. The Euler form uses dimensions over $\mathbb{F}_q$:

$$E(\dim V, \dim W) = \dim_{\mathbb{F}_q} \operatorname{Hom}(V,W) - \dim_{\mathbb{F}_q} \operatorname{Ext}^1(V,W).$$

For the exchange matrix B of Q, our conventions are

$$A = 2I - |B|, \qquad E(\alpha_i,\alpha_j) = \begin{cases} d_i, & i = j, \\ -[d_i b_{ij}]_+, & i \neq j. \end{cases}$$

The symmetric form satisfies

$$(\alpha_i,\alpha_j) = E(\alpha_i,\alpha_j) + E(\alpha_j,\alpha_i) = d_i a_{ij}.$$

Put $t = q^{1/2}$ for finite-field specializations. In generic expressions t is formal; we set $t = u^2$ and work over $\mathbb{Q}(u)$ to allow half-integral powers of t.

We use quiver Hecke algebras over an algebraically closed field k of characteristic zero, with grading parameter $\mathsf{q} = t^{-1}$. For $\nu \in Q_+$, write $R(\nu)$ for the algebra, $e_{\mathbf{j}}$ for a word idempotent, and $L \circ N$ for induction. We label $R(\nu)$-modules by ν, so $\operatorname{wt} L = \nu$ here; their weight in [4, 5] is $-\nu$. Write $R\text{-gmod} = \bigoplus_{\nu \in Q_+} R(\nu)\text{-gmod}$ for the category of finite-dimensional graded modules. The finitely generated modules used for affinizations need not be finite-dimensional. The parameters are those of [12, Section 8]:

$$Q_{ij}(x,y) = \begin{cases} 0, & i = j, \\ 1, & i \neq j,\ a_{ij} = 0, \\ \varepsilon_{ij}(x^{-a_{ij}} - y^{-a_{ji}}), & a_{ij} < 0, \end{cases} \qquad \varepsilon_{ij} = 1 = -\varepsilon_{ji} \quad \text{if } \operatorname{Ext}^1(S_i,S_j) \neq 0.$$

These polynomials satisfy $Q_{ij}(x,y) = Q_{ji}(y,x)$ and are homogeneous of degree $-2(\alpha_i,\alpha_j)$ for $\deg x = 2d_i$, $\deg y = 2d_j$, with nonzero pure power coefficients on every edge. The character and crystal results of [8] hold for this choice of parameters; see [14, Remark 1.37].

The parameter and endpoint conventions used below are summarized here:

BZ torus parameter	v
finite-field size	q
Hall parameter	$t = q^{1/2}$ at finite-field specializations
formal scalar extension	$t = u^2$
KLR grading parameter	$\mathsf{q} = t^{-1} = u^{-2}$
KLR weight	$\operatorname{wt} L = \nu$

Hall flags are descending, so left word prefixes record quotients. We use left KLR restriction; standard graded duality preserves word order.

A word $\mathbf{j} = (j_1,\dots,j_h)$ has weight $\sum_a \alpha_{j_a}$. For a finite-dimensional graded $R(\nu)$-module L, its character is

$$\operatorname{ch}_{\mathsf{q}} L = \sum_{\mathbf{j}} \left(\sum_{r \in \mathbb{Z}} \dim_k (e_{\mathbf{j}} L)_r \, \mathsf{q}^r \right) \mathbf{j}.$$

Its word support consists exactly of the nonzero word spaces. A grading shift multiplies the character by a power of q. Self-duality means duality under the graded k-dual with the action induced by the anti-involution fixing the KLR generators [4, Section 1.2]; this duality does not reverse the word.

For a representation V of dimension $\nu = \sum_i \nu_i \alpha_i$, let $\delta_{[V]}$ be the dual Hall basis element corresponding to its isomorphism class. We use Rupel's normalization [12, Section 5.1]:

$$[V]^* = t^{-\frac{1}{2}E(\nu,\nu)+\frac{1}{2}\sum_i d_i\nu_i}\delta_{[V]}.$$

The word $\mathbf{j}$ records the order of the simple quotients in a flag. The Hall shuffle character packages the counts of all such flags, while the KLR character records the graded dimensions of the corresponding word spaces. The Euler factors below fix the normalization needed to compare these two types of coefficients. Let $\mathcal{F}_{\mathbf{j}}(V)$ denote the descending flags

$$V = V^0 \supset V^1 \supset \cdots \supset V^h = 0, \qquad V^{a-1}/V^a \simeq S_{j_a}.$$

The Hall shuffle character [12, (7.2) and Theorem 7.1] is

$$F_V = \Omega([V]^*) = \sum_{\mathbf{j}} t^{-\sum_{a<b} E(\alpha_{j_b},\alpha_{j_a})} |\mathcal{F}_{\mathbf{j}}(V)|\,\mathbf{j}. \tag{3.1}$$

For example, S_i has just one such flag, with word (i), so $F_{S_i} = (i)$. This is also the character of the one-dimensional $R(\alpha_i)$-module concentrated in degree zero, on which the polynomial generator acts by zero. For larger weights the two character maps use different product normalizations, as follows.

The symmetric shuffle product of [12, Section 6] differs from the usual KLR character product [8, Section 2.2]. For homogeneous elements f, g of weights ν, μ,

$$f * g = t^{-(\nu,\mu)/2}(f *_{\mathrm{KLR}} g).$$

Indeed, a shuffle has KLR exponent equal to the sum of the pairings of its inverted cross pairs, whereas the symmetric exponent subtracts half the sum over all cross pairs. Set

$$\kappa(\nu) = -\frac{(\nu,\nu)}{4} + \frac{1}{2}\sum_i d_i\nu_i, \qquad \mathrm{sch}(L) = t^{\kappa(\nu)}\,\mathrm{ch}_{t^{-1}}\,L. \tag{3.2}$$

Since $\kappa(\nu+\mu) - \kappa(\nu) - \kappa(\mu) = -(\nu,\mu)/2$, the map sch is multiplicative for the symmetric shuffle product. Also $2\kappa(\nu) \in \mathbb{Z}$ and $\mathsf{q} = u^{-2}$. We use this scalar extension in Section 5.

For indecomposable rigid V of dimension ν, the proof of [12, Corollary 7.8] constructs one generic quantum coordinate algebra element, depending on the root datum and ν. Its shuffle image, denoted by F_V, has Laurent polynomial word coefficients in t; at $t = \sqrt{q}$ it specializes to (3.1) for the corresponding rigid representation over $\mathbb{F}_q$. Equivalently, each flag count in that formula is replaced by its counting polynomial at $q = t^2$. Thus a nonzero generic word coefficient remains nonzero for all sufficiently large prime powers q. These specializations are made on the Laurent polynomial coefficients of F_V.

To compare this generic element with KLR characters, extend scalars by $t = u^2$ and $\mathsf{q} = u^{-2}$ to $\mathbb{Q}(u)$. By KLR categorification [4, Theorem 1.2] and character injectivity [8, Theorem 3.1], the character image in weight ν has a basis given by the characters of self-dual simple modules, one representative from each grading-shift class. Over $\mathbb{Q}(u)$, the symmetric character image is generated by the one-letter words: the quantum coordinate algebra is generated by its simple-root elements, and $\kappa(\alpha_i) = 0$. This is also the image of the shuffle embedding in [12, Theorem 6.10]. Thus the generic F_V belongs to that image. In fixed weight ν, multiplication by $t^{\kappa(\nu)}$ does not change the character span. Consequently F_V is a finite $\mathbb{Q}(u)$-linear combination of simple characters. We call such a combination a *virtual character*; its coefficients need not be nonnegative.

3.2. **Stability and virtual semicuspidal characters.** A *charge* is a real-linear map $c = \theta + \mathrm{i}h$ on the real root space, with $h(\alpha_i) > 0$ for every simple root. Thus every nonzero positive weight has phase in $(0, \pi)$. A word of total weight β is *semicuspidal* if each nonempty proper left prefix of weight γ satisfies

$$\arg c(\gamma) \le \arg c(\beta) \quad \Longleftrightarrow \quad \theta(\gamma)h(\beta) - h(\gamma)\theta(\beta) \ge 0.$$

A module is semicuspidal if every word in its character is semicuspidal. When $\theta(\beta) = 0$, this condition is simply $\theta(\gamma) \ge 0$; we then also say θ semicuspidal. This is the left-prefix convention of [14, Definitions 2.1 and 2.3]. Equivalently, every nonzero restriction $\mathrm{Res}_{\gamma,\beta-\gamma} L$, with $0 < \gamma < \beta$, has $\arg c(\gamma) \le \arg c(\beta)$, since this restriction selects exactly the words with left prefix weight γ. We use the same phase order and the same left-to-right induction order as [14, Theorem 2.4]. That paper works with finite-dimensional nilpotent modules [14, Definitions 1.41–1.42]. A finite-dimensional graded module in our setting is nilpotent for the polynomial generators, since their degrees are positive. Its grading does not affect word support, and grading shifts do not affect semicuspidality.

For Hall characters, left prefixes in descending flags are quotients. The following Euler-form calculation will make their inequalities strict.

Lemma 3.1. *Let V be indecomposable and rigid, with $\nu = \dim V$. Define*

$$\theta_\nu(x) = E(\nu, x) - E(x, \nu).$$

Every nonzero proper subrepresentation U satisfies $\theta_\nu(\dim U) < 0$. Every nonzero proper quotient has positive θ_ν weight.

Proof. By [11, Theorem 3.4(1)], every nonzero endomorphism of V is injective or surjective, hence invertible. Put $T = V/U$. Applying $\mathrm{Hom}(-, V)$ gives

$$\mathrm{Ext}^1(V, V) \longrightarrow \mathrm{Ext}^1(U, V) \longrightarrow \mathrm{Ext}^2(T, V).$$

Rigidity and heredity give $\mathrm{Ext}^1(U, V) = 0$. A nonzero map $V \to U$ would yield a nonzero noninvertible endomorphism of V, so $\mathrm{Hom}(V, U) = 0$. Therefore, with dimensions taken over $\mathbb{F}_q$,

$$\theta_\nu(\dim U) = -\dim \mathrm{Ext}^1(V, U) - \dim \mathrm{Hom}(U, V) < 0.$$

The inclusion $U \hookrightarrow V$ gives strictness. Additivity and $\theta_\nu(\nu) = 0$ give the quotient assertion. □

Every proper nonempty prefix in the support of F_V therefore has positive θ_ν weight. This also holds generically: a nonzero counting polynomial is nonzero at all sufficiently large finite field parameters, where Lemma 3.1 applies.

Perturb θ_ν among real linear functionals vanishing on ν, preserving these finitely many strict inequalities, to obtain

$$\ker \theta \cap Q_{\mathbb{Q}} = \mathbb{Q}\nu. \tag{3.3}$$

Such a choice exists because the excluded loci are countably many proper hyperplanes. Choose a real linear h positive on the simple roots and put $c = \theta + \mathrm{i}h$. Since $h(\nu) > 0$, equation (3.3) gives $\ker c \cap Q_{\mathbb{Q}} = 0$. Thus c is injective on the rational root space, and

$$\arg c(\gamma) < \arg c(\nu) = \pi/2$$

for every proper nonempty prefix weight γ in F_V.

The following support lemma passes from virtual characters to semicuspidal simple characters.

Lemma 3.2. *Let c be injective on the rational root space. A finite virtual KLR character of fixed weight whose nonzero words are all c semicuspidal lies in the linear span of c semicuspidal simple characters.*

Proof. Write the virtual character as

$$F = \sum_L a_L \operatorname{ch}_{t^{-1}} L, \qquad a_L \in \mathbb{Q}(u),$$

using distinct simple modules up to grading shift and omitting zero coefficients. There is nothing to prove if $F = 0$.

The polygon of a simple module. By [14, Theorem 2.4], each L is, up to shift, the simple head of $L_1 \circ \cdots \circ L_s$, where the L_j are semicuspidal and

$$\beta_j = \operatorname{wt} L_j, \qquad \arg c(\beta_1) > \cdots > \arg c(\beta_s).$$

In coordinates $(h, -\theta)$, define P_L by the consecutive vertices

$$p_r = \left(h\left(\sum_{j=1}^{r} \beta_j\right), -\theta\left(\sum_{j=1}^{r} \beta_j\right)\right), \qquad 0 \le r \le s.$$

Its slopes $-\theta(\beta_j)/h(\beta_j)$ strictly decrease, so P_L is concave. A word path joins the images of its consecutive prefix weights. Semicuspidality puts each factor word below its chord. For a prefix of a shuffle, let x_j be the h-length taken from the jth factor. Its height in these coordinates is at most

$$\sum_{j=1}^{s} \frac{-\theta(\beta_j)}{h(\beta_j)} x_j, \qquad 0 \le x_j \le h(\beta_j), \qquad \sum_j x_j = x.$$

At fixed x, this sum is maximized by filling the factors in decreasing slope order: transferring available length from a smaller slope to a larger one increases the sum. The maximum is the height of P_L at x. Thus every word path of the convolution, and hence of its quotient L, lies below P_L.

The unmixing property [14, Lemma 2.9 and the proof of Lemma 2.6] also gives

$$\operatorname{Res}_{\beta_1,\dots,\beta_s} L \simeq \mathsf{q}^{r_L}(L_1 \boxtimes \cdots \boxtimes L_s), \qquad r_L \in \mathbb{Z}.$$

On characters this restriction cuts words into blocks of the specified weights; extend it linearly to virtual characters.

Eliminating a maximal polygon. Suppose some term has $s > 1$. Choose a pointwise maximal polygon P among such terms, and let $(\beta_1, \dots, \beta_s)$ be its edge weights. A semicuspidal term has the single chord as its polygon, strictly below P in the interior, so P is maximal among all terms. Since $\arg c(\beta_1)$ exceeds the total phase, the support hypothesis gives

$$\operatorname{Res}_{\beta_1,\dots,\beta_s} F = 0.$$

Any contributing simple term has a word path through every vertex of P. Its polygon P' dominates those vertices, and concavity then gives $P' \ge P$ on every edge interval: a concave function lies above the chord joining two of its points. Maximality forces $P' = P$.

Injectivity of c recovers the block weights from the polygon's edges. The uniqueness in Theorem 2.4 gives distinct ordered tuples of simple factors for distinct terms with this polygon. Their tensor characters are linearly independent, even after extending scalars to $\mathbb{Q}(u)$. The displayed restriction formula therefore prevents their nonzero coefficients from cancelling, a contradiction. □

Proposition 3.3. *Let V be indecomposable and rigid, with $\nu = \dim V$, and choose θ as in (3.3). Then ν is a positive real root, and there is a unique θ semicuspidal simple L_V of weight ν up to shift. Its self-dual representative satisfies $F_V = \lambda_V(t) \operatorname{ch}_{t^{-1}} L_V$ for some $\lambda_V(t) \in \mathbb{Q}(t)^\times$.*

Proof. We first show that ν is a multiple of a positive real root, then exclude nontrivial multiples. Rigidity gives

$$(\nu,\nu) = 2E(\nu,\nu) = 2\dim_{\mathbb{F}_q} \mathrm{End}(V) > 0.$$

Since $F_V \neq 0$, Lemma 3.2 and [14, Corollary 2.12] express ν as a sum of positive roots of phase $\pi/2$. By (3.3) these roots are positive rational multiples of ν, so each has positive norm. Positive imaginary roots have nonpositive norm [7, Proposition 5.3.1(ii)–(iii)]: each is Weyl-conjugate to a positive root γ with $(\gamma,\alpha_i) \leq 0$ for all i, whence $(\gamma,\gamma) = \sum_i \gamma_i(\gamma,\alpha_i) \leq 0$. Thus the roots in the sum are real. Each real root is a Weyl translate of a simple root and hence is primitive, since the Weyl group acts by integral automorphisms of the root lattice. Thus all are the same positive real root α, and $\nu = k\alpha$ for some $k \geq 1$.

Corollary 2.12 gives exactly one semicuspidal simple, up to shift, at each weight $k\alpha$: the only positive root on this ray is α, of multiplicity one. Hence F_V is a nonzero scalar multiple of the simple character at $k\alpha$ and has exactly its word support.

Let L_α be the semicuspidal simple of weight α. Every shuffle of its words has nonnegative θ weight at each prefix, so all constituents of $L_\alpha^{\circ k}$ are semicuspidal. If $k > 1$, concatenating k words of L_α gives a word of this convolution with a proper prefix of weight α and $\theta(\alpha) = 0$. Since every constituent is a shift of the unique semicuspidal simple at $k\alpha$, that word occurs in this simple, hence in F_V. This contradicts the strict prefix inequality. Therefore $k = 1$. Choosing the self-dual representative L_V, we obtain

$$F_V = \lambda_V(t)\,\mathrm{ch}_{t^{-1}} L_V. \tag{3.4}$$

Here $\lambda_V(t) \in \mathbb{Q}(t)^\times$: compare any nonzero word coefficient on both sides, which belong to $\mathbb{Q}(t)$. □

3.3. **The extremal word coefficient.** To determine the scalar in (3.4), we compare one word coefficient, writing $[\mathbf{w}]F$ for the coefficient of the word $\mathbf{w}$ in F. We use the factorial conventions

$$(r)!_z = \prod_{a=1}^{r}(1 + z + \cdots + z^{a-1}), \qquad [r]_z! = \prod_{a=1}^{r} \frac{z^a - z^{-a}}{z - z^{-1}},$$

with both empty products equal to 1.

Proposition 3.4. *For every indecomposable rigid V, the self-dual simple module L_V constructed above satisfies*

$$\Omega([V]^*) = \mathrm{ch}_{t^{-1}} L_V. \tag{3.5}$$

Equivalently, $\lambda_V(t) = 1$ in (3.4).

Proof. The Hall coefficient. Choose a source topological order $i_1, \ldots, i_n$ for Q, and consider

$$\mathbf{w} = i_1^{\nu_{i_1}} \cdots i_n^{\nu_{i_n}}.$$

For $0 \leq r \leq n$, let $V^{(r)}$ have zero space at $i_1, \ldots, i_r$ and the full space V_j at every other vertex, with the induced arrow maps. Every arrow goes from an earlier to a later vertex in the chosen order, so $V^{(r)}$ is a subrepresentation. The dimensions force these vertex spaces at the end of each color block; hence any flag of word $\mathbf{w}$ has this same coarse flag.

Inside the i_r block, earlier vertex spaces are zero and later ones are full. Every $\mathbb{F}_{q^{d_{i_r}}}$-subspace $U \subseteq V_{i_r}$ is therefore allowed: incoming maps have zero source, and outgoing maps land in full target spaces. The same zero-source and full-target conditions apply to every species structure map. Successive quotients in the block are S_{i_r} precisely when these subspaces form a complete descending flag over $\mathbb{F}_{q^{d_{i_r}}}$. Thus refinement gives a bijection

$$\mathcal{F}_{\mathbf{w}}(V) \simeq \prod_{i \in I} \mathrm{Flag}_{\mathbb{F}_{q^{d_i}}}(V_i),$$

where a zero-dimensional space contributes one flag. An m-dimensional space over $\mathbb{F}_{q^{d_i}}$ has $(m)!_{q^{d_i}}$ complete flags: choose a hyperplane in $(q^{d_i m}-1)/(q^{d_i}-1)$ ways and continue inside it. Consequently

$$|\mathcal{F}_{\mathbf{w}}(V)| = \prod_i (\nu_i)!_{t^{2d_i}}.$$

For $r < s$, there is no arrow from i_s to i_r, and hence $E(\alpha_{i_s}, \alpha_{i_r}) = 0$. The reverse Euler terms between distinct blocks therefore vanish. Within block i, $E(\alpha_i, \alpha_i) = d_i$, and there are $\binom{\nu_i}{2}$ pairs. Using

$$t^{-d_i\binom{\nu_i}{2}}(\nu_i)!_{t^{2d_i}} = [\nu_i]_{t^{d_i}}!,$$

we obtain

$$[\mathbf{w}]F_V = \prod_i [\nu_i]_{t^{d_i}}!. \tag{3.6}$$

The simple-character coefficient. For a simple module H of weight β, let m be the largest number of initial i's in a supported word. The maximal left-block restriction is

$$\mathrm{Res}_{m\alpha_i, \beta - m\alpha_i} H \simeq \mathsf{q}^a L(i^m) \boxtimes H_0$$

for a simple H_0 and an integer a. Here $L(i^m)$ denotes the self-dual single-color simple. This is the left-handed form of [8, Section 2.5.1(2)(b),(6)]: remove the entire maximal string by the crystal restriction, then recover H by the inverse crystal induction. The left functors are (2.18)–(2.21) there, obtained from the right functors by the word-reversing involution of Section 1.1.4. The results cover symmetrizable Cartan data; Remark 1.3 of that paper also allows the graded parameter generalization used here.

By (3.4) and (3.6), $\mathbf{w}$ occurs in L_V. Its first nonempty block contains all occurrences of that color, so its length is a maximal left string. The restriction above leaves a simple module whose support contains the remaining blocks. Repeating, and omitting empty blocks, gives a single tensor product of the $L(i^{\nu_i})$, up to one overall grading shift.

For color i, the nilHecke polynomial generators have degree $2d_i$. The graded dimension of its simple module, in the self-dual shift, is

$$\mathsf{q}^{-d_i\binom{m}{2}}(m)!_{\mathsf{q}^{2d_i}} = [m]_{\mathsf{q}^{d_i}}!;$$

see [8, Section 2.4]. At $\mathsf{q} = t^{-1}$ this equals $[m]_{t^{d_i}}!$. Whole-block restriction extracts this simple factor once; forgetting that factor by repeated one-strand restriction would instead retain its quantum-factorial graded dimension.

Restriction is compatible with the standard duality and retains the order of the blocks. The complete restriction is simple as a module over the tensor product algebra. If the shift is r, self-duality of L_V and of the normalized tensor S gives $\mathsf{q}^r S \simeq \mathsf{q}^{-r} S$. A nonzero finite-dimensional graded module has no nonzero grading period, so $r = 0$. Therefore

$$[\mathbf{w}]\,\mathrm{ch}_{t^{-1}} L_V = \prod_i [\nu_i]_{t^{d_i}}!.$$

Comparison with (3.6) proves the assertion. □

3.4. **Reflection and affinization.** A simple module L is *real* if $L \circ L$ is simple. We use the affinization condition of [4, Definition 3.1]. It consists of a graded module Δ and a positive degree injective endomorphism z, with Δ finite free over $k[z]$, $\Delta/z\Delta \simeq L$, and

$$p_i = \sum_{\mathbf{j}} \prod_{a : j_a = i} x_a e_{\mathbf{j}}$$

acting nontrivially for every color occurring in its weight. A real simple with an affinization is called affreal.

We construct the affinization needed for normalized R-matrices by reflecting to a simple root and transporting the polynomial family $k[z]$ back along the reflection equivalences.

We use the graded monoidal equivalences in [10, Theorems 1.4.1 and 3.2.10]. The relevant categories consist of finitely generated graded modules with no word beginning in i, respectively no word ending in i. Reflection exchanges these categories, changes weight by s_i, and preserves finite-dimensional self-dual simple modules [10, Proposition 4.1.3]. We use version 4 throughout.

Here the admissible side is determined by the sign of $\theta(\alpha_i)$: negative sign excludes words beginning in i, and positive sign excludes words ending in i.

Lemma 3.5. *Let L be a θ semicuspidal simple of weight β, where $\theta(\beta) = 0$ and $\theta(\alpha_i) \neq 0$. Its image under the reflection equivalence on the admissible side is $\theta \circ s_i$ semicuspidal.*

Proof. Case $\theta(\alpha_i) < 0$. No word of L begins in i. Its reflection N has no word ending in i, and $\theta' = \theta \circ s_i$ satisfies $\theta'(\alpha_i) > 0$.

If N is not semicuspidal, write its ordered semicuspidal factors as $L_1, \dots, L_s$, with $s > 1$, and put

$$A = L_1, \qquad U = L_2 \circ \cdots \circ L_s, \qquad B = \operatorname{hd} U.$$

Then $\theta'(\operatorname{wt} A) < 0$. We claim that

$$\operatorname{Res}_{\operatorname{wt} A, \operatorname{wt} B} N \simeq A \boxtimes B, \tag{3.7}$$

up to grading.

Consider the Mackey filtration of $\operatorname{Res}_{\operatorname{wt} A, \operatorname{wt} U}(A \circ U)$. In a mixed term, a nonzero suffix of A would have the same weight as a sum of prefixes from the factors of U. Semicuspidality puts the suffix at a phase at least that of A, whereas every nonzero such prefix, and hence their sum, has strictly smaller phase. No mixed term can occur. The restriction is therefore $A \boxtimes U$.

The same separation applies to $A \circ B$: every word of B, being a word of the quotient of U, has prefixes of phase strictly smaller than that of A. Thus $\operatorname{Res}_{\operatorname{wt} A, \operatorname{wt} B}(A \circ B) \simeq A \boxtimes B$. The surjection $A \circ U \twoheadrightarrow A \circ B$ and the unique simple head of $A \circ U$ imply $\operatorname{hd}(A \circ B) \simeq N$. Exact restriction makes $\operatorname{Res}_{\operatorname{wt} A, \operatorname{wt} B} N$ a quotient of the simple module $A \boxtimes B$. This quotient is nonzero by Frobenius reciprocity applied to $A \circ B \twoheadrightarrow N$, proving (3.7).

The phase of A is greater than $\pi/2$, while that of color i is less. Semicuspidality forbids a word of A ending in i. The restriction and the corresponding prohibition for N also forbid a word of B ending in i. The inverse monoidal equivalence therefore applies to A, B and the epimorphism $A \circ B \twoheadrightarrow N$. Frobenius reciprocity gives a nonzero prefix restriction of L with weight $s_i \operatorname{wt} A$. Its θ value is $\theta'(\operatorname{wt} A) < 0$, a contradiction.

Case $\theta(\alpha_i) > 0$. The words of L cannot end in i and the words of its image cannot begin in i. Take the last semicuspidal factor B, whose θ' value is positive, and let A be the simple head of the preceding factors. The corresponding Mackey filtration has no mixed term: a nonzero prefix of B has phase at most that of B, while a sum of nonzero suffixes from preceding factors has strictly larger phase. The same head and adjunction argument gives $\operatorname{Res}_{\operatorname{wt} A, \operatorname{wt} B} N \simeq A \boxtimes B$, up to grading. Since $\theta'(\alpha_i) < 0$, B cannot begin in i; the restriction implies the same for A. Applying the inverse equivalence to $A \circ B \twoheadrightarrow N$ produces a suffix of L with positive θ value, contradicting semicuspidality. □

For a graded module $M = \bigoplus_j M_j$, write

$$\operatorname{END}_R(M) = \bigoplus_{r \in \mathbb{Z}} \{ f \in \operatorname{End}_R(M) : f(M_j) \subseteq M_{j+r} \text{ for all } j \}$$

for its total graded endomorphism ring.

Proposition 3.6. *The module L_V is real. It has a finitely generated graded family (Δ, z) with $\deg z > 0$, finite free over $k[z]$, such that*

$$\Delta/z\Delta \simeq L_V, \qquad \mathrm{END}_{R(\dim V)}\,\Delta = k[z], \qquad \operatorname{supp}\Delta = \operatorname{supp} L_V.$$

Proof. Descending to a simple root. Start with $L = L_V$, $\beta = \dim V$, and the functional θ used in (3.3). The preceding argument shows that β is a positive real root and L is θ semicuspidal. If β is simple, no reflection is needed. Otherwise retain this functional and proceed as follows. Write h_i for the simple coroot, so that $\langle h_i, \beta\rangle = 2(\alpha_i, \beta)/(\alpha_i, \alpha_i)$. Since

$$(\beta, \beta) = \sum_i \beta_i(\beta, \alpha_i) > 0,$$

there is an i with $\langle h_i, \beta\rangle > 0$. By [7, Lemma 3.2.2], $s_i\beta$ is positive because $\beta \neq \alpha_i$, and

$$\mathrm{ht}(s_i\beta) = \mathrm{ht}(\beta) - \langle h_i, \beta\rangle < \mathrm{ht}(\beta).$$

Also $\theta(\alpha_i) \neq 0$. Repeated application of Lemma 3.5 sends the actual simple module to a simple root module. At the next step one may choose a new height function positive on the simple roots, since semicuspidality at total θ weight zero does not depend on that choice. Moreover

$$\ker(\theta \circ s_i) \cap Q_{\mathbb{Q}} = \mathbb{Q}s_i\beta,$$

so the genericity condition is preserved. The descent terminates because root height is a positive integer.

Transporting the family. A finitely generated graded module of fixed KLR weight has degrees bounded below: the PBW spanning set has finitely many permutation factors with bounded negative degrees, and positive polynomial degrees. At each inverse step, write L for the current simple and Δ for its polynomial family; at the end $L = L_V$ and $\beta = \dim V$. At the endpoint take the regular module $R(\alpha_j) = k[z]$. Transport it back along the equivalences, together with

$$0 \longrightarrow \mathsf{q}^{\deg z}\Delta \xrightarrow{z} \Delta \longrightarrow L \longrightarrow 0. \tag{3.8}$$

At each inverse step, the whole family lies in the required domain. Let e be a forbidden endpoint idempotent. The filtration of $\Delta/z^N\Delta$ by shifts of L implies $e(\Delta/z^N\Delta) = 0$, since the functor $M \mapsto eM$ is exact. Consequently

$$e\Delta \subseteq \bigcap_{N\geq 1} z^N\Delta = 0.$$

The intersection is zero because degrees are bounded below and $\deg z > 0$. Thus the inverse equivalence applies to (3.8).

Choose homogeneous lifts $b_1, \ldots, b_d$ of a basis of L. Reduction modulo z expresses a homogeneous element of Δ as a linear combination of these lifts plus z times an element of smaller degree. Induction from the lower degree bound proves that the b_j generate over $k[z]$. If

$$\sum_{j=1}^{d} f_j(z)b_j = 0, \qquad f_j(z) \in k[z],$$

then reduction modulo z gives $f_j(0) = 0$ for every j. Injectivity of z permits division of the relation by z. Iteration makes each f_j divisible by every power of z, so each is zero. Hence $\Delta = \bigoplus_{j=1}^{d} k[z]b_j$.

Full faithfulness gives

$$\mathrm{END}_{R(\beta)}\,\Delta = k[z]. \tag{3.9}$$

Finite generation ensures that every ungraded endomorphism is a finite sum of homogeneous endomorphisms, so the same equality holds for the ungraded endomorphism ring. The word support of Δ equals that of L, by reduction modulo z and the lower degree bound.

The same monoidal reflection chain sends every $L^{\circ a}$ to a simple single-color nilHecke module up to grading. Hence every $L^{\circ a}$ is simple and L is real; see also [14, Proposition 2.21]. □

The family in Proposition 3.6 satisfies the remaining affinization condition as follows.

Lemma 3.7. *The module Δ just constructed satisfies the central nonvanishing condition for an affinization.*

Proof. Central elements on the generic fiber. Put $K = k(z)$, $R_K = R(\beta)\otimes_k K$ and $\Delta_K = \Delta\otimes_{k[z]} K$. Choose a finite $k[z]$-basis of Δ. If $f \in \mathrm{End}_{R_K}\Delta_K$, clearing the denominators of its matrix gives a nonzero $p(z) \in k[z]$ such that $p(z)f$ preserves Δ. It still commutes with $R(\beta)$, and hence (3.9) implies $p(z)f = a(z)\,\mathrm{id}$ for some $a(z) \in k[z]$. Therefore

$$\mathrm{End}_{R_K}\Delta_K = K\,\mathrm{id}\,.$$

The commutant of the matrices of a finite set of KLR generators is the kernel of a finite system of K-linear equations. Extending scalars to an algebraic closure of K preserves this kernel and its dimension, so the endomorphisms there are still scalar. In particular, every central element acts by a scalar.

Excluding mixed spectra. Consider a joint generalized eigenvalue tuple of the commuting x_a on a nonzero word space. Suppose it contains both zero and nonzero entries. An adjacent zero and nonzero entry can be interchanged invertibly. For distinct colors this follows from $\psi_a^2 = Q_{ij}(x_a, x_{a+1})$ and $Q_{ij}(0, b) \neq 0$ for $b \neq 0$. For equal colors use the nilHecke intertwiner with square one. Thus the zero entries can be moved entirely to the left or entirely to the right.

Let γ be the weight of the zero entries. Semicuspidal support gives both $\theta(\gamma) \geq 0$ and $\theta(\beta - \gamma) \geq 0$. Therefore $\theta(\gamma) = 0$. By (3.3), $\gamma = r\beta$ for a rational $0 < r < 1$. Since β is a primitive integral root, any rational multiple of β that is integral has integer multiplier. This is a contradiction. Every joint spectrum is consequently either entirely zero or entirely nonzero.

Excluding the zero spectrum. If an entirely zero spectrum occurs, each positive degree symmetric polynomial in the variables of each color has eigenvalue zero there. These polynomials are central and hence act by scalars on Δ_K, so they act by zero everywhere. They also kill Δ, which is torsion free over $k[z]$.

For each word, the polynomial ring in its strands is finite over the ring generated by the elementary symmetric polynomials in the strands of each color: every strand of color i satisfies the monic polynomial whose coefficients are those elementary symmetric functions. The sums over word idempotents of these functions form a central subalgebra Z_0 of $R(\beta)$. There are finitely many words and permutation factors in the PBW spanning set, so $R(\beta)$ is finite as a Z_0-module. Modulo the ideal $Z_{0,+}$ of positive degree elements, this gives a finite-dimensional algebra over k. Finite generation would then make Δ finite-dimensional, contradicting its nonzero finite free $k[z]$ structure.

All joint spectra are therefore entirely nonzero. Each p_i is invertible on the generic fiber, and in particular acts nontrivially on Δ. □

Thus each L_V in (3.5) is real and admits an affinization.

3.5. **Rigid sums and simple convolutions.** Two simple modules *strongly commute* if their convolution is simple.

Lemma 3.8. *Let V, W be indecomposable rigid representations such that $V \oplus W$ is rigid. Then L_V and L_W strongly commute.*

Proof. Write $\nu = \dim V$, $\mu = \dim W$. The split extension term in the normalized dual Hall product is

$$[V]^*[W]^* = t^{\{E(\mu,\nu)-E(\nu,\mu)\}/2}[V \oplus W]^*. \tag{3.10}$$

Rigidity of $V \oplus W$ excludes all other extension terms. In the Hall number, the automorphism ratio is

$$F_{VW}^{V\oplus W} \frac{|\operatorname{Aut} V|\,|\operatorname{Aut} W|}{|\operatorname{Aut}(V\oplus W)|} = |\operatorname{Hom}(V,W)|^{-1}.$$

Combining this with the Euler twist and the difference of the κ factors gives (3.10). Reversing V, W, then using (3.2), (3.5) and injectivity of the KLR character map, gives

$$[L_V][L_W] = \mathsf{q}^{E(\nu,\mu)-E(\mu,\nu)}[L_W][L_V].$$

The exponent is integral and both modules are real and admit affinizations. Thus [4, Lemma 3.20] implies strong commutation. □

For a rigid sum $V_1 \oplus \cdots \oplus V_s$, the corresponding modules are pairwise strongly commuting, including repeated factors. Every normalized pairwise R matrix is an isomorphism. Their ordered composite is nonzero and an isomorphism. The simple image statement for normal sequences [6, Lemma 2.10] proves that the whole convolution is simple.

For a decomposable rigid object, iterating (3.10) and the symmetric shuffle twist gives

$$\Omega([\textstyle\bigoplus_i V_i]^*) = t^{-\sum_{i<j} E(\dim V_j, \dim V_i)} \operatorname{ch}_{t^{-1}}(L_{V_1} \circ \cdots \circ L_{V_s}). \tag{3.11}$$

The exponent is integral. Thus this is the character of a graded shift of a simple module. Equation (3.11) gives the rigid-sum form of Rupel's Conjecture 8.2 for the parameters of Section 3.1. This completes the proof of Theorem 1.2.

4. Feigin realization and localization

We construct an injective map from the localized Grothendieck ring to the Feigin torus and invertible simple lifts of frozen Laurent monomials.

4.1. **The kernel and localization.** Let $\mathbf{i} = (i_1, \ldots, i_\ell)$ be a reduced word for a Weyl group element w, with every simple reflection occurring. Let $\mathcal{T}_{\mathbf{i}}$ be the Laurent Feigin torus over $\mathbb{Z}[u^{\pm 1}]$, with $t = u^2$. Set $(\Lambda_{\mathbf{i}})_{rs} = -(\alpha_{i_r}, \alpha_{i_s})$ for $r < s$ and extend skew-symmetrically. The generators satisfy $t_r t_s = t^{(\Lambda_{\mathbf{i}})_{rs}} t_s t_r$. We use normalized monomials

$$t^{\mathbf{a}} = t^{-\frac{1}{2}\sum_{r<s}(\Lambda_{\mathbf{i}})_{rs} a_r a_s} t_1^{a_1} \cdots t_\ell^{a_\ell}, \qquad \mathbf{a} \in \mathbb{Z}^\ell.$$

Thus $t^{\mathbf{a}} t^{\mathbf{b}} = t^{\mathbf{a}^T \Lambda_{\mathbf{i}} \mathbf{b}/2} t^{\mathbf{a}+\mathbf{b}}$, and $[t^{\mathbf{a}}]$ extracts a coefficient in this basis. For $\mathbf{a} \in \mathbb{N}^\ell$, the notation $e(i_1^{a_1}, \ldots, i_\ell^{a_\ell})L$ denotes the corresponding block restriction, a module over the tensor product of the single-color KLR algebras. Work over the scalar extension

$$K_0(R\text{-gmod})_u := K_0(R\text{-gmod}) \otimes_{\mathbb{Z}[\mathsf{q}^{\pm 1}]} \mathbb{Z}[u^{\pm 1}], \qquad \mathsf{q} \longmapsto u^{-2}, \qquad t = u^2.$$

A *positive formal unit* means u^r, $r \in \mathbb{Z}$. Only the even powers correspond to genuine integer grading shifts. We represent lifts by classes of actual objects multiplied by these scalar units. Define

$$\Xi_{\mathbf{i}} : K_0(R\text{-gmod})_u \longrightarrow \mathcal{T}_{\mathbf{i}}, \qquad [L] \longmapsto \overline{\Psi}_{\mathbf{i}}(\operatorname{sch}(L)), \tag{4.1}$$

where $\overline{\Psi}_{\mathbf{i}}$ is the map from the symmetric shuffle algebra to the Feigin torus, with scalar parameter t. Its block-word formula is [12, Theorem 6.12]. The commuting Hall–shuffle–Feigin triangle [12, Theorem 7.6], where $\widetilde{\Psi}_{\mathbf{i}}$ is the Hall-to-Feigin map, gives

$$\overline{\Psi}_{\mathbf{i}}(F_V) = \widetilde{\Psi}_{\mathbf{i}}([V]^*).$$

For the source-adapted doubled word in the representation dictionary [1, Theorem 2.5 and Main Theorem 2.9(b)], denote this image by X_V^{Feigin}. For indecomposable rigid V, equations (3.2) and (3.5) then give explicitly

$$\Xi_{\mathbf{i}}([L_V]) = t^{\kappa(\dim V)} X_V^{\mathrm{Feigin}}.$$

Up to a positive formal unit, the coefficient at $t_1^{a_1}\cdots t_\ell^{a_\ell}$ is

$$\frac{\dim_{\mathsf{q}} e(i_1^{a_1},\dots,i_\ell^{a_\ell})L}{\prod_k [a_k]_{\mathsf{q}^{d_{i_k}}}!}. \tag{4.2}$$

The numerator is the graded dimension of a restriction to a tensor product of single-color nilHecke algebras. The denominator is the graded dimension of its unique simple module S, normalized to be self-dual. A graded composition series of the restriction H gives

$$\dim_{\mathsf{q}} H = \sum_{r\in\mathbb{Z}} m_r \mathsf{q}^r \dim_{\mathsf{q}} S, \qquad m_r \in \mathbb{N}.$$

Consequently the ratio is $\sum_r m_r \mathsf{q}^r \in \mathbb{N}[\mathsf{q}^{\pm 1}]$.

Detecting the kernel. Choose the self-dual representative in each grading-shift orbit of simple R-modules. In the next paragraph, distinct simples mean distinct such representatives. The kernel of $\Xi_{\mathbf{i}}$ is taken in $K_0(R\text{-gmod})_u$. A grading shift changes coefficients, not word support or the maximal-string tuple.

The maximal-string tuple $a(L)$ is obtained by removing as many initial i_1's as possible, then as many i_2's as possible from the remaining simple, and so on. By [5, Proposition 3.1], a simple module has a nonzero block restriction along $\mathbf{i}$ precisely when this process reaches the unit module. In that case, $a(L)$ is the lexicographically largest point of its Feigin support. Successive maximal restriction shows that the coefficient there is a positive formal unit. Inverse crystal operations recover L from the unit and $a(L)$, so different surviving simples have different largest tuples. Their images are linearly independent over $\mathbb{Z}[u^{\pm 1}]$: in a finite nonzero linear combination, choose the largest of the distinct leading tuples. Its monomial occurs in only that summand, with a nonzero coefficient times a unit, so cannot cancel. Define the submodule of the original Grothendieck ring

$$\mathcal{I}_w = \bigoplus_{L:\,\Xi_{\mathbf{i}}([L])=0} \mathbb{Z}[\mathsf{q}^{\pm 1}][L].$$

The direct sum is over the chosen self-dual representatives. Thus

$$\ker \Xi_{\mathbf{i}} = \mathcal{I}_w \otimes_{\mathbb{Z}[\mathsf{q}^{\pm 1}]} \mathbb{Z}[u^{\pm 1}]. \tag{4.3}$$

Inverting determinantial modules. Let ϖ_j be the jth fundamental weight and let $C_j = \mathsf{M}(w\varpi_j, \varpi_j)$ be the corresponding determinantial simple module. Denote by $\mathcal{L}_w$ the braider localization of R-gmod obtained by inverting these modules, and by $Q_w : R\text{-gmod} \longrightarrow \mathcal{L}_w$ its localization functor. We use the localization of the whole category in the general-type diagram of [5, Remark 3.6], constructed in [4]. Proposition 3.3 of [5], with its full support hypothesis, identifies the objects killed by the localization with those having all the block restrictions above equal to zero. Hence its Grothendieck kernel is $\mathcal{I}_w$, a two-sided ideal of $K_0(R\text{-gmod})$; equation (4.3) describes this kernel after scalar extension. Proposition 4.8 and Theorem 5.9 of [4], together with the general type diagram in [5, Remark 3.6], identify the localized Grothendieck ring as

$$K_0(\mathcal{L}_w) \simeq (K_0(R\text{-gmod})/\mathcal{I}_w)\,[[C_j]^{-1} : j \in I]. \tag{4.4}$$

Localization is exact and sends a simple to a simple or zero. Every localized object is the image of a finite-dimensional module tensored with inverses of the C_j. It has finite length, and its class is a nonnegative graded sum of simple classes.

4.2. **Frozen variables.** For a dominant weight λ, block restrictions of the determinantial module and coefficients of the quantum minor are both detected by raising operators on the extremal weight line. The symbol t_λ below is the torus monomial attached to λ in [1, Section 6]; in particular it is not a power of the scalar t. Here $\Psi_{\mathbf{i}}$ denotes the Feigin map on the quantum coordinate algebra used in that paper, applied below to quantum minors.

Lemma 4.1. *Let $\mathbf{i} = (i_1, \dots, i_\ell)$ be a reduced expression of w, and let λ be dominant. Put $C_\lambda = \mathsf{M}(w\lambda, \lambda)$ and $\nu = \lambda - w\lambda$. For every $\mathbf{a} \in \mathbb{N}^\ell$ with $\sum_k a_k \alpha_{i_k} = \nu$,*

$$e(i_1^{a_1}, \dots, i_\ell^{a_\ell}) C_\lambda \neq 0 \quad \Longleftrightarrow \quad [t^{\mathbf{a}}] \Psi_{\mathbf{i}}(\Delta_{w\lambda}) \neq 0.$$

Consequently $\Xi_{\mathbf{i}}([C_\lambda])$ is a positive formal unit times t_λ^{-1}.

Proof. A common coefficient test. We compute both supports in the integrable highest weight module. Write $V_z(\lambda)$ for this module at a formal parameter z, with highest vector v_λ, and choose a nonzero vector $v_{w\lambda}$ on its one-dimensional extremal weight space. Define $m_{\mathbf{a}}(z)$ by

$$E_{i_\ell}^{(a_\ell)} \cdots E_{i_1}^{(a_1)} v_{w\lambda} = m_{\mathbf{a}}(z) v_\lambda. \tag{4.5}$$

Changing the normalization of $v_{w\lambda}$ multiplies every $m_{\mathbf{a}}$ by the same nonzero scalar and does not change its zero set.

The categorical coefficient. Use the cyclotomic quotient with relations on the rightmost strand, as in [4, Section 1.3]. Its functor E_i^λ removes a leftmost i, so the order of functors associated with the word $i_1^{a_1} \cdots i_\ell^{a_\ell}$ is exactly the order in (4.5). The module C_λ is the cyclotomic module constructed in [4, Section 3.3], inflated to $R(\nu)$. By [4, Theorem 1.3], the cyclotomic Grothendieck group, extended to $\mathbb{Q}(\mathsf{q})$, is the integrable highest weight module. Over this field the integral and dual integral forms have the same span. The nonzero class of C_λ lies on the extremal weight line $w\lambda$.

Divided restriction extracts the graded multiplicity space for the unique simple module over each single-color nilHecke algebra. Thus

$$(E_{i_\ell}^\lambda)^{(a_\ell)} \cdots (E_{i_1}^\lambda)^{(a_1)} C_\lambda$$

is zero precisely when the block restriction in the statement is zero. The grading normalizations in the categorical action multiply each operator on a fixed weight space by a nonzero power of q; they cannot change this condition. The resulting object is a finite-dimensional graded vector space, whose class vanishes precisely when the object vanishes. It follows that

$$e(i_1^{a_1}, \dots, i_\ell^{a_\ell}) C_\lambda = 0 \quad \Longleftrightarrow \quad m_{\mathbf{a}}(t^{-1}) = 0. \tag{4.6}$$

The bar isomorphism $V_t(\lambda) \to V_{t^{-1}}(\lambda)$ fixes E_i, F_i and the highest vector, and preserves each weight space. Choose extremal vectors compatibly with this isomorphism. Then $m_{\mathbf{a}}(t^{-1}) = 0$ if and only if $m_{\mathbf{a}}(t) = 0$. The operator order is preserved.

The quantum-minor coefficient. We now compute the other side in the realization of [1, Section 6.1], using t for its quantum parameter. For a vector y of weight μ in $V_t(\lambda) \subset U_t(\mathfrak{b}_+)^*$, Lemma 6.1 of that paper gives

$$E_i(y v_{-\lambda}) = t^{-(\alpha_i, \lambda)} E_i(y) v_{-\lambda}.$$

The weight after applying E_i is $\mu + \alpha_i - \lambda$. Hence, with $\partial_i = K_i^{-1/2} E_i$,

$$\partial_i(y v_{-\lambda}) = t^{-\frac{1}{2}(\alpha_i, \mu + \lambda + \alpha_i)} E_i(y) v_{-\lambda}. \tag{4.7}$$

This identity holds on the whole weight space, including vectors killed by E_i.

On each weight space the factor in (4.7) is an invertible power of t. In the ambient algebra of [1, (6.1)], $v_\lambda v_{-\lambda} = v_0 = 1$. Thus the highest weight vector after multiplication by $v_{-\lambda}$ is identified

with a scalar. Iterating in the order of (4.5) and dividing by the same quantum factorials therefore gives

$$\partial_{i_\ell}^{(a_\ell)} \cdots \partial_{i_1}^{(a_1)}(v_{w\lambda} v_{-\lambda}) = t^{\eta(\mathbf{a})} m_{\mathbf{a}}(t)$$

for some $\eta(\mathbf{a}) \in \frac{1}{2}\mathbb{Z}$. By [1, Proposition 6.14], the coefficient $[t^{\mathbf{a}}]\Psi_{\mathbf{i}}(\Delta_{w\lambda})$ differs from this value by a nonzero normalization factor. Hence

$$[t^{\mathbf{a}}]\Psi_{\mathbf{i}}(\Delta_{w\lambda}) = 0 \quad \Longleftrightarrow \quad m_{\mathbf{a}}(t) = 0. \tag{4.8}$$

Together with (4.6) and parameter inversion, this proves the support equivalence for every $\mathbf{a}$.

The monomial image. By [1, Proposition 6.16], $\Psi_{\mathbf{i}}(\Delta_{w\lambda}) = t_\lambda^{-1}$. Therefore the block support of C_λ is the single exponent of this monomial. The factor $t^{\kappa(\nu)}$ in sch and the nonzero factors in the shuffle-to-Feigin map preserve this support. Its unique coefficient is a positive formal unit by the maximal-string calculation preceding the lemma. Substitution $t = u^2$ gives the assertion over $\mathbb{Z}[u^{\pm 1}]$. □

Apply Lemma 4.1 to $\lambda = \varpi_j$. We obtain

$$\Xi_{\mathbf{i}}([C_j]) = u^{s_j} t_{\varpi_j}^{-1}, \qquad s_j \in \mathbb{Z}. \tag{4.9}$$

We check that the seed frame of the representation dictionary preserves the frozen Laurent lattice of the standard Feigin seed.

Lemma 6.15 of [1] identifies t_{ϖ_j} with the frozen variable at the last occurrence of j in the standard Feigin seed. For the rest of this subsection specialize to $\mathbf{i} = (\mathbf{i}_0, \mathbf{i}_0)$, as in the representation dictionary. This dictionary uses the seed $\Sigma_{\mathbf{i}}^-$. Relabel $\mathbf{i}_0 = (1, \dots, n)$ and let $f_k = \mathbf{e}_{n+k}$ be the basis vector at the last occurrence of k. In the notation of [1, (2.10), (4.15), Lemma 4.10], the additional map $\rho_{\mathbf{i}}^-$ restricts to

$$\rho_{\mathbf{i}}^-(f_k) = f_k - \sum_{\ell \le k} a_{\ell k} f_\ell = -f_k - \sum_{\ell < k} a_{\ell k} f_\ell, \qquad 1 \le k \le n, \tag{4.10}$$

where $(a_{\ell k})$ is the Cartan matrix. In particular, the frozen sublattice $F = \bigoplus_k \mathbb{Z} f_k$ is preserved. The restriction matrix is integral triangular with diagonal entries -1, hence unimodular. The new frozen exponent vectors in the Feigin coordinates are $\varphi_{\mathbf{i}}^{-1}\rho_{\mathbf{i}}^-(f_k)$, whereas the standard frozen variables have exponents $\varphi_{\mathbf{i}}^{-1}(f_k)$; see [1, Section 7.1]. Applying $\varphi_{\mathbf{i}}^{-1}$ to (4.10) expresses each new frozen variable as a normalized Laurent monomial in the standard frozen variables, and the integral inverse gives the converse. Together with (4.9), this expresses every positive and negative frozen power in the required seed as the image of an invertible simple class times a positive formal unit.

Proposition 4.2. *For a reduced word* $\mathbf{i}$ *with full support,* $\Xi_{\mathbf{i}}$ *extends to an injective homomorphism*

$$K_0(\mathcal{L}_w) \otimes_{\mathbb{Z}[\mathsf{q}^{\pm 1}]} \mathbb{Z}[u^{\pm 1}] \hookrightarrow \mathcal{T}_{\mathbf{i}}, \qquad \mathsf{q} \longmapsto u^{-2}.$$

For the doubled word, every frozen Laurent monomial of $\Sigma_{\mathbf{i}}^-$ *is the image of an invertible simple class times* u^r*. Nonzero localized simples have no nonzero grading period.*

Proof. By (4.3), the induced map on the quotient by $\mathcal{I}_w$ is injective. Equation (4.9) sends each localizing class to an invertible monomial, so the localization remains injective. The frozen assertion follows from (4.10). A grading period $a \neq 0$ would give $u^{-2a} f = f$ for the nonzero image f of a simple class, which is impossible in the Laurent torus. □

4.3. **A valued rank-two example.** Take

$$B=\begin{pmatrix}0&2\\-1&0\end{pmatrix},\qquad D=\operatorname{diag}(1,2),\qquad A=\begin{pmatrix}2&-2\\-1&2\end{pmatrix},\qquad \mathbf{i}=(1,2,1,2).$$

The word $\mathbf{i}$ is reduced. Let V have dimension $(2,1)$, with arrow map an $\mathbb{F}_q$-linear isomorphism $\mathbb{F}_q^2\to\mathbb{F}_{q^2}$. Then $\operatorname{End}(V)\simeq\mathbb{F}_{q^2}$ and $E(\dim V,\dim V)=2$, so V is indecomposable and rigid. Its only flag word is 112, with $q+1$ flags. Thus Proposition 3.4 gives

$$F_V=(t+t^{-1})\,112=\operatorname{ch}_{t^{-1}}L_V,\qquad \kappa(\dim V)=1.$$

Dividing by the single-color factorials in the block formula gives

$$X_V^{\mathrm{Feigin}}=t^{(2,1,0,0)}+t^{(2,0,0,1)}+(t+t^{-1})t^{(1,0,1,1)}+t^{(0,0,2,1)},\qquad \Xi_{\mathbf{i}}([L_V])=tX_V^{\mathrm{Feigin}}. \tag{4.11}$$

Here $t^{\mathbf{a}}$ denotes a normalized torus monomial. For the negative principal seed, $M(a)=t^{\Phi a}$ with

$$\Phi=\varphi_{\mathbf{i}}^{-1}\rho_{\mathbf{i}}^{-}=\begin{pmatrix}-1&-2&1&-2\\0&-1&1&-1\\0&0&1&0\\0&0&0&1\end{pmatrix},\qquad \Phi^T\Lambda_{\mathbf{i}}\Phi=\begin{pmatrix}0&2&0&0\\-2&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Consequently (4.11) becomes

$$X_V=M(0,-1,0,0)+M(-2,-1,0,1)+(t+t^{-1})M(-2,0,1,1)+M(-2,1,2,1).$$

If Z_1,Z_2 are the standard frozen variables, the frozen variables of this seed are Z_1^{-1} and $Z_1^2Z_2^{-1}$; they commute here. Finally,

$$\begin{pmatrix}B\\-I\end{pmatrix}^T\Phi^T\Lambda_{\mathbf{i}}\Phi=(2D\ \ 0),\qquad t+t^{-1}=u^2+u^{-2}.$$

This exhibits both the frozen lattice change and the parameter conversion used in the general argument.

5. Mutation-acyclic positivity

We first work in irreducible infinite type; the final subsection reduces the remaining acyclic cases to this setting.

5.1. **A reference seed avoiding finitely many variables.** A *Coxeter belt* is obtained by repeating a sequence of mutations at sources in a topological order. A completed period means that every mutable direction has been mutated once in that order.

Lemma 5.1. *Let B be acyclic and suppose that its associated Cartan matrix is irreducible and of infinite type. There is a Coxeter belt of acyclic seeds whose clusters at completed periods have pairwise disjoint sets of coefficient-free classical mutable variables. In particular, for any prescribed finite set of cluster variables, one can choose a completed period whose initial mutable variables avoid all their coefficient-free classical specializations. Along the same mutation words, the prescribed variables are then noninitial in every compatible realization of the principal pattern.*

Proof. Relabel so that $B_{ij}\ge 0$ for $i<j$. Write $|B|=U+L$ with strictly upper and lower triangular parts, and put

$$\Gamma=(I-U)^{-1},\qquad c=s_1\cdots s_n=-\Gamma(I-L).$$

Mutate successively at the sources $1,\dots,n$. Every completed period restores the same principal matrix.

Valuation vectors along the belt. To distinguish variables, specialize to the classical coefficient-free pattern of this principal matrix. Let δ_k be the column whose jth entry is minus the valuation

of the current kth variable in the original variable x_j (the other original variables being coefficients). The rational expressions contain no subtraction, so their valuations satisfy exactly

$$\delta'_k = -\delta_k + \max\left(0, \sum_i |B_{ik}|\delta_i\right), \tag{5.1}$$

with componentwise maximum. Initially the column matrix is $-I$. During the first period, after mutation at k the new column is $\mathbf{e}_k + \sum_{i<k} B_{ik}\delta_i$: the previous columns are supported on earlier coordinates, while the still negative unit columns occupy later coordinates and are cut off by the componentwise maximum. Thus the first completed period has column matrix Γ. In a later source period, provided the old and already updated columns are nonnegative, the maximum is the nonnegative branch. Writing $\mathcal{D} = (\delta_1 \ \cdots \ \delta_n)$, the column equations are

$$\mathcal{D}_{\text{new}}(I - U) = -\mathcal{D}_{\text{old}}(I - L).$$

Starting from $\mathcal{D}_{\text{old}} = \Gamma$, this gives $\mathcal{D}_{\text{new}} = c\Gamma$; starting from $c^r\Gamma$ gives $c^{r+1}\Gamma$. The positive-root argument below verifies the nonnegative assumption at each intermediate step.

Distinctness and avoidance. For the first Coxeter word put

$$\beta_k = s_1 \cdots s_{k-1}\alpha_k = \Gamma\mathbf{e}_k, \qquad 1 \le k \le n.$$

The equality follows by applying the reflections in reverse order: the earlier coordinates satisfy the same triangular recurrence as the columns of $(I-U)^{-1}$. In the $(r+1)$st period the new column at step k is $c^r\beta_k$. Indeed, substitution in the column recurrence gives this formula, starting with $r = 0$; the columns not yet changed retain their values from the preceding period. Speyer's theorem [13, Theorem 1] says that every power of this Coxeter word is reduced. Its successive inversion roots are exactly $c^r\beta_k$, and hence are positive and pairwise distinct. Inductively, after the first period both the unchanged and the already changed columns are positive roots, so the sum inside the maximum in (5.1) is nonnegative. The linear recurrence is therefore valid at each intermediate step. Distinct roots give distinct valuation vectors, so distinct completed periods share no mutable variable.

Each prescribed coefficient-free classical specialization can occur in at most one completed period. Exclude those finitely many periods and choose a remaining seed S. If a corresponding variable in another compatible realization were initial relative to S, its coefficient-free classical specialization would be one of the initial variables of S, contrary to the choice. This proves the final assertion. □

5.2. **Simultaneous transport to negative principal coefficients.** We transport the simultaneous cluster-character correspondence from positive principal coefficients to the negative principal coefficients of the Feigin seed. A representation is *basic* if its indecomposable summands are pairwise nonisomorphic. A basic rigid representation is called *locally tilting* if its number of indecomposable summands equals the number of vertices in its support, as in [11, Section 3]. It is full tilting when its support is all n mutable vertices.

Lemma 5.2. *Let B be acyclic, and use the same finite mutation word in realizations with exchange matrices $P^+ = \binom{B}{I}$ and $P^- = \binom{B}{-I}$. In the negative principal realization used by the Feigin dictionary, the representations corresponding to the noninitial members of the resulting cluster are the same summands of one locally tilting representation as in the positive principal realization. They are therefore pairwise Ext orthogonal. If all members are noninitial, their sum is a full rigid tilting representation.*

Proof. Step 1: compare the exponent vectors. We compare the complete cluster characters, including frozen coordinates. Write $\nu = \dim V$, let e be the dimension vector of a subrepresentation, and put $f = \nu - e$. For the extended valued quiver define $(x^*)_j = E(x, \alpha_j)/d_j$ and $({}^*x)_j =$

$E(\alpha_j, x)/d_j$. Thus the mutable vector g_V has entries $(g_V)_j = -E(\nu, \alpha_j)/d_j$. In the convention of [11, Definition 5.1], the cluster character has the following exponents for the two frozen arrow orientations:

$$a_V^+(f) = (g_V + Bf, f), \qquad a_V^-(f) = (g_V + Bf, \nu - f). \tag{5.2}$$

To check the frozen parts, use the exponent $-e^* - {}^*(\nu - e)$. In the positive principal quiver, the frozen arrow is $n + i \to i$, so its frozen coordinate is $(\nu - e)_i = f_i$. Reversing this arrow gives the coordinate $e_i = \nu_i - f_i$. Mutable Euler entries do not change. The coefficients in both based expressions are the same Grassmannian counts with the same Euler factor: at Hall parameter t, they are $t^{-E(e,\nu-e)} |\operatorname{Gr}_e V|$. Only the exponent changes. We work generically; the rigid counting polynomials give these identities at the formal parameter.

Step 2: determine the tropical denominator. To identify these characters with the mutation variables, use the positive principal pattern. The simultaneous seed theorem [11, Section 7, Theorem 7.3] gives the representations V along the prescribed word. The classical principal F polynomial has support contained in $0 \le f \le \nu$ and contains both $f = 0$ and $f = \nu$, since the whole and the zero subrepresentations exist. Both endpoint coefficients equal 1, in the classical F polynomial as well as in the based cluster character.

Write the classical polynomial as

$$F(y) = \sum_{0 \le f \le \nu} A_f y^f, \qquad A_\nu = 1,$$

with $A_f \in \mathbb{Z}$. Substitute $y_j = z_j^{-1}$, where the z_j are the frozen variables. Fix j and view this as a Laurent polynomial in z_j over the rational function field of the remaining variables. No z_j exponent is below $-\nu_j$, and its coefficient at that degree is

$$\sum_{f_j = \nu_j} A_f \prod_{k \ne j} z_k^{-f_k}.$$

This coefficient is nonzero: the term $f = \nu$ has coefficient one, and different vectors with $f_j = \nu_j$ give different remaining monomials. The z_j valuation is therefore exactly $-\nu_j$.

This valuation also computes the tropical evaluation of the subtraction-free rational expression defining the classical F polynomial. Specialize the other variables to generic positive real numbers for which the displayed lowest coefficient is nonzero. In a subtraction-free rational expression, leading coefficients of nonzero summands are positive; addition takes the minimum of valuations, multiplication adds them, and division subtracts them. Thus its tropical valuation equals its ordinary valuation at this specialization, namely $-\nu_j$. This works for every j, so the tropical denominator has frozen exponent $-\nu$. Definition 3.7 of [15] consequently gives the extended g vector (g_V, ν) in the negative realization. Lemma 2.1 now gives the identical based coefficients, with exponent

$$(g_V, \nu) + P^- f = a_V^-(f).$$

Equation (5.2) identifies this actual variable with the negative principal cluster character of the same V, which is the character used in [1, Theorems 2.5 and 6.11, Section 7.1].

In the Feigin application the compatible diagonal is $2D$ and $t = u^2$. One may first use the positive principal character theorem in its canonical diagonal-D realization with parameter t, substitute $t = u^2$, and then use Lemma 2.1 for diagonal $2D$. The comparison thus takes place in integral compatible realizations.

Step 3: retain the entire cluster. Initial members also agree under this correspondence: their F polynomial is 1 and their extended g vector is the corresponding mutable basis vector, so coefficient transport preserves them. Conversely, for nonzero V the two distinct endpoint exponents in (5.2) preclude an initial monomial. Apply the calculation to each noninitial member of the same target

cluster, using the same word. The positive principal simultaneous seed theorem supplies one locally tilting representation, and the calculation keeps each of its summands unchanged. Thus the negative principal representations are pairwise Ext orthogonal. When no initial member remains, there are n distinct summands. A basic rigid representation has at most as many summands as support vertices, so their support is all n vertices. □

5.3. **Composition multiplicities at an arbitrary target.** *Choosing a common realization.* Fix a cluster variable X and a target seed T with mutable cluster $Y_1, \dots, Y_n$ in an irreducible infinite acyclic pattern. By the final assertion of Lemma 5.1, choose a common reference seed S relative to which $X, Y_1, \dots, Y_n$ are all noninitial in the Feigin realization used below. Let $\mathbf{i}_0$ be its source topological word and $\mathbf{i} = (\mathbf{i}_0, \mathbf{i}_0)$. This double word is reduced by Speyer's theorem and has full support.

Theorem 2.5 and Main Theorem 2.9(b) of [1] identify these noninitial variables with the Feigin images of exceptional valued representations. Lemma 5.2 places the representations of the target variables in one rigid tilting object. We use these representations to lift target monomials as follows.

Lemma 5.3. *In this common realization, X and every based target monomial $M_T(b)$ with $b_i \geq 0$ for $1 \leq i \leq n$ lift to simple localized classes times powers of u. Distinct exponent vectors b give distinct grading-shift orbits of simple objects.*

Proof. Theorem 1.2 provides the simple lift of X and the simple convolutions for nonnegative mutable powers. Tensoring with the invertible frozen lifts of Proposition 4.2 preserves simplicity. Normalizing the products changes their classes by powers of u. If two lifts were grading shifts of one another, their torus images would be proportional, contradicting the linear independence of the based target monomials. □

Clearing denominators. By the BZ Laurent phenomenon, in the parameter u appropriate to this realization,

$$X = \sum_a c_a(u) M_T(a), \qquad c_a(u) \in \mathbb{Z}[u^{\pm 1}],$$

with finite nonempty support $A = \{a : c_a(u) \neq 0\}$. Choose $p \in \mathbb{Z}^m$ by

$$p_i = \max\left\{0, -\min_{a \in A} a_i\right\} \quad (1 \leq i \leq n), \qquad p_i = 0 \quad (i > n).$$

Then $p_i \geq 0$ and $a_i + p_i \geq 0$ for every $a \in A$ and $1 \leq i \leq n$. Thus the preceding lifting statement applies both to $M_T(p)$ and to each $M_T(a + p)$. Right multiplication yields

$$X M_T(p) = \sum_a c_a(u) u^{a^T \Lambda_T p} M_T(a + p). \tag{5.3}$$

Comparing simple classes. Perform this comparison in

$$K_0(\mathcal{L}_w) \otimes_{\mathbb{Z}[\mathsf{q}^{\pm 1}]} \mathbb{Z}[u^{\pm 1}], \qquad \mathsf{q} \longmapsto u^{-2},$$

where $\mathcal{L}_w$ is the localized finite-length category. Choose one representative U in each orbit of simple objects under genuine grading shift. There are no nonzero grading periods, as proved in Section 4. The Jordan–Hölder theorem therefore gives

$$K_0(\mathcal{L}_w) = \bigoplus_U \mathbb{Z}[\mathsf{q}^{\pm 1}][U].$$

The extension is free, hence flat: $\mathbb{Z}[u^{\pm 1}]$ has basis $1, u$ over $\mathbb{Z}[u^{\pm 2}]$. Consequently the scalar-extended group is $\bigoplus_U \mathbb{Z}[u^{\pm 1}][U]$. An actual object H has graded composition multiplicities

$m_{H,U}(\mathsf{q}) \in \mathbb{N}[\mathsf{q}^{\pm 1}]$. For every integer r,

$$u^r[H] = \sum_U u^r m_{H,U}(u^{-2})[U], \qquad u^r m_{H,U}(u^{-2}) \in \mathbb{N}[u^{\pm 1}].$$

Thus multiplication by every u^r preserves the scalar-extended positive cone.

Each monomial $M_T(a+p)$ on the right of (5.3) lifts to $u^{r_a}[U_a]$, where the U_a are pairwise distinct shift-orbit representatives. For the left side, choose simple localized objects L_X, L_p and integers r, s such that

$$X = \Xi_{\mathbf{i}}(u^r[L_X]), \qquad M_T(p) = \Xi_{\mathbf{i}}(u^s[L_p]).$$

Multiplicativity gives

$$XM_T(p) = \Xi_{\mathbf{i}}(u^{r+s}[L_X \circ L_p]).$$

Here $L_X \circ L_p$ is an actual object of the finite-length category $\mathcal{L}_w$; it need not be simple. Its composition multiplicities, multiplied by the positive formal unit u^{r+s}, are nonnegative as above. Injectivity of $\Xi_{\mathbf{i}}$ on the scalar-extended localized Grothendieck ring identifies this class with the lifted right side of (5.3). Set $H = L_X \circ L_p$. Comparison of the coefficient of $[U_a]$ gives

$$u^{r+s} m_{H,U_a}(u^{-2}) = c_a(u) u^{a^T \Lambda_T p + r_a}.$$

The left side belongs to $\mathbb{N}[u^{\pm 1}]$. Multiplication by a Laurent monomial only shifts degrees, so $c_a(u) \in \mathbb{N}[u^{\pm 1}]$.

Returning to the prescribed quantization. In the Feigin coordinates, based multiplication is $t^{\Lambda_{\mathbf{i}}/2}$. If the seed frame is $M_\Sigma(a) = t^{\Phi a}$, its form is

$$\Lambda_\Sigma = \Phi^T \Lambda_{\mathbf{i}} \Phi.$$

For the doubled source-adapted word $\mathbf{i} = (1, \dots, n, 1, \dots, n)$, Section 7.1 of [1] gives $\Phi = \varphi_{\mathbf{i}}^{-1} \rho_{\mathbf{i}}^-$ and $\widetilde{B}_\Sigma = \left(\begin{smallmatrix} B \\ -I_n \end{smallmatrix}\right)$. Its exchangeable positions are exactly $1, \dots, n$, as required in [1, Proposition 4.9]. Writing b_k^- for the kth column of $\widetilde{B}_\Sigma$, that proposition gives

$$\Lambda_{\mathbf{i}}(\Phi b_k^-, \Phi \mathbf{e}_\ell) = 2d_k \delta_{k\ell} \quad (1 \le k \le n,\ 1 \le \ell \le 2n), \qquad \widetilde{B}_\Sigma^T \Lambda_\Sigma = (2D \ \ 0).$$

After $t = u^2$, this is an ordinary integral BZ realization with compatible diagonal $2D$. Apply Lemma 2.1 at the actual target to obtain canonical $2D$ positivity, then Lemma 2.2 to obtain canonical D positivity, and finally Lemma 2.1 again to recover any prescribed realization.

5.4. **Finite type, reducible matrices and zero edges.** The case of no mutable directions is vacuous. Let B, D be acyclic compatible principal data with $n \ge 1$. Put $L = \operatorname{lcm}(d_1, \dots, d_n)$ and adjoin a new source vertex 0 with

$$\widehat{D} = \operatorname{diag}(L, d_1, \dots, d_n), \qquad \widehat{B}_{0i} = 2d_i, \quad \widehat{B}_{i0} = -2L.$$

Retain the old principal block B. The new data are integral, acyclic, connected and skew-symmetrizable. The Cartan product on every new edge is $4Ld_i \ge 4$, so the Cartan matrix is of infinite type. The preceding argument applies.

Take an integral compatible realization of the enlarged data. Freeze direction 0: delete its mutable column and retain its torus generator among the frozen variables. After reordering the remaining mutable directions first, compatibility restricts to

$$\widetilde{B}_{\mathrm{fr}}^T \widehat{\Lambda} = (D \ \ 0).$$

Every mutation word in the original directions gives the same seeds in this frozen pattern as in the enlarged pattern. The established positivity therefore applies at each of its targets. Lemma 2.1, rooted at that target, gives positivity for every realization of the original principal data.

A mutation-acyclic pattern can be rooted at an acyclic seed and every specified source and target retained. This proves Theorem 1.1. In particular, rank-two quantum cluster patterns satisfy the same positivity statement, with arbitrary additional coordinates treated as frozen.

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