

HILBERT-TYPE INEQUALITY WITH PRIMES

KANNAN SOUNDARARAJAN

ABSTRACT. Let P be a finite set of primes, and let K_P be the matrix whose off-diagonal entries are $|p - q|^{-1}$ and whose diagonal entries vanish. It is shown that the operator norm of K_P on $\ell^2(P)$ is bounded by an absolute constant, uniformly in P .

1. INTRODUCTION

Answering Problem 22 in Montgomery's list [1], this note establishes the following Theorem.

Theorem 1. *Let P be a finite set of primes. Define an operator K_P on the Hilbert space $\ell^2(P)$ by setting*

$$(K_P a)(p) = \sum_{\substack{q \in P \\ q \neq p}} \frac{a(q)}{|p - q|}.$$

Then there is an absolute constant C such that $\|K_P\| \leq C$. In other words, if $a(p)$ (for $p \in P$) are arbitrary complex numbers then

$$(1) \quad \left| \sum_{\substack{p, q \in P \\ p \neq q}} \frac{a(p)\overline{a(q)}}{|p - q|} \right| \leq \sum_{\substack{p, q \in P \\ p \neq q}} \frac{|a(p)| |a(q)|}{|p - q|} \leq C \sum_{p \in P} |a(p)|^2.$$

The proof given here is a rewriting of an argument supplied by ChatGPT 6. My initial interest had been Problem 21 in Montgomery's list, which asked for the optimal constant in a weighted version of Hilbert's inequality. This was a well-known open problem, with a guessed optimal constant of π . The LLM drew my attention to a very recent preprint of Brad Rodgers [3], which established that in fact the optimal answer cannot be π and must be a little larger (although its true value is not yet determined). On a whim, I then prompted the LLM to consider Problem 22, and it quickly produced the solution explained here.

For each integer $j \geq 0$, set $L_j = 2^{2^j}$, and define the operator T_j on $\ell^2(P)$ by setting

$$(T_j a)(p) = \sum_{\substack{q \in P \\ L_j \leq |p - q| < L_{j+1}}} \frac{a(q)}{|p - q|}.$$

The author was partially supported by NSF grant DMS-2100933.

Note that only finitely many of the T_j are non-zero, and that we may write

$$K_P = E + \sum_{j=0}^{\infty} T_j,$$

where the “error” E accounts for the possibility that 2 and 3 both belong to P . Explicitly $E = 0$ unless 2 and 3 are both in P , in which case E sends $a(2)$ to $a(3)$ and vice versa, and sends all other $a(p)$ to 0.

Since E has operator norm ≤ 1 , to establish the boundedness of K_P it is enough to show the boundedness of $\sum_{j=0}^{\infty} T_j$. Using Schur’s inequality (see Lemma 1) together with the Brun-Titchmarsh inequality for the number of primes in an interval, one can show that $\|T_j\| \leq A$ for an absolute constant A and all j . Further, by applying Schur’s inequality with an upper bound sieve for counting pairs of primes $n, n+h$ with n lying in an interval, one can establish that $\|T_j T_k\| \leq B 2^{-|j-k|/2}$ for an absolute constant B . This estimate implies that the operators T_j and T_k are nearly orthogonal when j and k are not close. The Cotlar-Stein Lemma (see Lemma 2 below) gives a bound for the operator norm of a sum of nearly orthogonal operators, and enables one to deduce the boundedness of K_P .

The immediate upper bound for $\|K_P\|$ is $\ll \log \log(3 + |P|)$. In his notes to Problem 22, Montgomery mentions a construction of Ruzsa which indicates that the Brun-Titchmarsh bound alone is insufficient to go beyond this first bound. The new ingredient used in the proof here is a sieve bound for pairs of primes (see (5)) combined with the Cotlar-Stein Lemma. Indeed the proof would apply to any set of integers satisfying the sieve bounds recorded in Lemma 3.

Once K_P is shown to be bounded, the inequality (1) follows by Cauchy-Schwarz: setting $b(p) = |a(p)|$ we have

$$|\langle a, K_P a \rangle| \leq \langle b, K_P b \rangle \leq \|b\| \|K_P b\| \leq \|K_P\| \|b\|^2 = \|K_P\| \|a\|^2.$$

2. PRELIMINARY LEMMAS

This section collects together three lemmas used in our proof. The first lemma records a simple bound for the operator norm of a matrix in terms of its absolute row and column sums.

Lemma 1 (Schur’s inequality). *Let $M = (m_{uv})$ be a finite rectangular complex matrix. Suppose*

$$\max_u \sum_v |m_{uv}| \leq R, \quad \max_v \sum_u |m_{uv}| \leq C,$$

so that the absolute row sums are bounded by R , and the absolute column sums by C . Then, the norm of M as an operator from ℓ^2 to ℓ^2 satisfies

$$\|M\| \leq \sqrt{RC}.$$

Proof. For every vector x , Cauchy–Schwarz gives

$$|(Mx)_u|^2 = \left| \sum_v m_{uv} x_v \right|^2 \leq \left(\sum_v |m_{uv}| \right) \left(\sum_v |m_{uv}| |x_v|^2 \right) \leq R \sum_v |m_{uv}| |x_v|^2.$$

Summing over u and using the column bound yields

$$\|Mx\|^2 \leq R \sum_v |x_v|^2 \sum_u |m_{uv}| \leq RC \|x\|^2.$$

□

Our next lemma is the key Cotlar-Stein estimate for the norm of nearly orthogonal operators (see Chapter VII §2 of [4]).

Lemma 2 (Cotlar–Stein). *For $1 \leq j \leq N$, let $S_j : \mathcal{H} \rightarrow \mathcal{K}$ be bounded linear operators between two complex Hilbert spaces. Put*

$$A = \max_j \sum_k \|S_j^* S_k\|^{\frac{1}{2}}, \quad B = \max_j \sum_k \|S_j S_k^*\|^{\frac{1}{2}}.$$

Then

$$\left\| \sum_{j=1}^N S_j \right\| \leq \sqrt{AB}.$$

Proof. Write $S = \sum_j S_j$, and note that for any integer $m \geq 1$

$$(2) \quad \|S\|^{2m} = \|(S^* S)^m\| \leq \sum_{i_1, \dots, i_{2m}=1}^N \|S_{i_1}^* S_{i_2} S_{i_3}^* S_{i_4} \cdots S_{i_{2m-1}}^* S_{i_{2m}}\|,$$

upon expanding out the power and using the triangle inequality. Observe that by submultiplicativity

$$\|S_{i_1}^* S_{i_2} S_{i_3}^* S_{i_4} \cdots S_{i_{2m-1}}^* S_{i_{2m}}\| = \left\| \prod_{r=1}^m S_{i_{2r-1}}^* S_{i_{2r}} \right\| \leq \prod_{r=1}^m \|S_{i_{2r-1}}^* S_{i_{2r}}\|.$$

On the other hand, by pairing terms differently, we also have

$$\|S_{i_1}^* S_{i_2} S_{i_3}^* S_{i_4} \cdots S_{i_{2m-1}}^* S_{i_{2m}}\| = \left\| S_{i_1}^* \left(\prod_{r=1}^{m-1} S_{i_{2r}} S_{i_{2r+1}}^* \right) S_{i_{2m}} \right\| \leq B^2 \prod_{r=1}^{m-1} \|S_{i_{2r}} S_{i_{2r+1}}^*\|,$$

where we used that $\|S_{i_1}^*\| = \|S_{i_1} S_{i_1}^*\|^{\frac{1}{2}} \leq B$ and similarly $\|S_{i_{2m}}\| = \|S_{i_{2m}} S_{i_{2m}}^*\|^{\frac{1}{2}} \leq B$. Using the geometric mean of both bounds, we conclude that

$$(3) \quad \|S_{i_1}^* S_{i_2} S_{i_3}^* S_{i_4} \cdots S_{i_{2m-1}}^* S_{i_{2m}}\| \leq B \left(\prod_{r=1}^m \|S_{i_{2r-1}}^* S_{i_{2r}}\|^{\frac{1}{2}} \right) \left(\prod_{r=1}^{m-1} \|S_{i_{2r}} S_{i_{2r+1}}^*\|^{\frac{1}{2}} \right).$$

Substitute (3) into (2) and sum first over i_{2m} . This index features only in one term in the products in (3), namely $\|S_{i_{2m}}^* S_{i_{2m}}\|^{\frac{1}{2}}$, so that we may sum over all choices for i_{2m} and

bound that contribution by the maximum row sum A . Sum now over i_{2m-1} , which features now only in the product $\|S_{i_{2m-2}}S_{i_{2m-1}}^*\|^{\frac{1}{2}}$ so that this sum may be bounded by B . Proceed in this manner, summing in order over the indices $i_{2m-2}, \dots, i_2$, obtaining m factors of A and $m-1$ factors of B . We thus obtain

$$\|S\|^{2m} \leq BA^m B^{m-1} \sum_{i_1=1}^N 1 = A^m B^m N.$$

Taking $2m$ -th roots on both sides, and letting $m \rightarrow \infty$, the lemma follows. $\square$

Finally our third lemma records classical sieve estimates for primes and pairs of primes in intervals.

Lemma 3. *There are absolute constants c_1, c_2 such that, for every real interval I of length $R \geq 2$,*

$$(4) \quad \#\{n \in I : n \text{ prime}\} \leq c_1 \frac{R}{\log R}.$$

Moreover, for every positive integer h ,

$$(5) \quad \#\{n \in I : n, n+h \text{ both prime}\} \leq c_2 \frac{h}{\phi(h)} \frac{R}{(\log R)^2}.$$

Proof. The bound (4) follows from the Brun-Titchmarsh inequality, see Corollary 3.4 of [2]. The bound (5) also follows from a straightforward application of the Selberg sieve. It may be found as Corollary 3.14 of [2], where the right side is given as

$$\frac{8c(h)R}{(\log R)^2} \left(1 + O\left(\frac{\log \log R}{\log R}\right)\right),$$

uniformly in h , where $c(h) = c \prod_{p|h, p>2} \frac{p-1}{p-2}$ with $c = 2 \prod_{p>2} (1 - 1/(p-1)^2)$ being the twin-prime constant. In Corollary 3.14 it is assumed that h is even and $R \geq 4$, but these are plainly unimportant, and the stated version follows immediately. $\square$

3. PROOF OF THE MAIN THEOREM

All operators in this section act on $\ell^2(P)$, and $\mathbf{1}$ denotes the all-ones vector. We begin by showing that the operator norms of T_j are bounded (uniformly for all j).

Lemma 4. *There is an absolute constant $A_0 \geq 1$ such that for all $j \geq 0$ we have $T_j \mathbf{1} \leq A_0 \mathbf{1}$, and therefore $\|T_j\| \leq A_0$.*

Proof. If $p \in P$, then by definition

$$(T_j \mathbf{1})(p) = \sum_{\substack{q \in P \\ L_j \leq |p-q| < L_{j+1}}} \frac{1}{|p-q|} \leq \sum_{m=2^j}^{2^{j+1}-1} \frac{1}{2^m} \sum_{\substack{q \in P \\ 2^m \leq |p-q| < 2^{m+1}}} 1,$$

upon splitting the interval $[L_j, L_{j+1})$ into dyadic blocks. Using the Brun-Titchmarsh inequality (4), the above is

$$\ll \sum_{m=2^j}^{2^{j+1}-1} \frac{1}{2^m} \frac{2^m}{m} \ll 1.$$

This bound shows that the row sums of the matrix defining T_j (in the obvious basis for $\ell^2(P)$) are bounded. Since the operator is symmetric, the same holds for the column sums. Therefore, by Schur's inequality (Lemma 1) we conclude that $\|T_j\| \ll 1$. $\square$

The next lemma is the key step and bounds the operator norm of $T_j T_k$, showing that this norm tends rapidly to zero as $|j - k|$ increases.

Lemma 5. *There is an absolute constant B_0 such that, for $0 \leq j < k$,*

$$(6) \quad T_k T_j \mathbf{1} \leq B_0 2^{j-k} \mathbf{1}.$$

As a consequence, there exists an absolute constant D such that for all $j, k \geq 0$,

$$\|T_k T_j\| \leq D 2^{-|j-k|/2}.$$

Proof. We begin with the proof of (6). Suppose $0 \leq j < k$ and let p be an element of P , and consider $(T_k T_j \mathbf{1})(p)$. Expanding out the definition, this is

$$(T_k T_j \mathbf{1})(p) = \sum_{\substack{r \in P \\ L_k \leq |p-r| < L_{k+1}}} \frac{1}{|p-r|} \sum_{\substack{s \in P \\ L_j \leq |r-s| < L_{j+1}}} \frac{1}{|r-s|}.$$

Write $h = |r - s|$ so that $s = r + h$ or $r - h$, and rewrite the above as

$$\sum_{L_j \leq h < L_{j+1}} \frac{1}{h} \left(\sum_{\substack{r, r+h \in P \\ L_k \leq |p-r| < L_{k+1}}} \frac{1}{|p-r|} + \sum_{\substack{r, r-h \in P \\ L_k \leq |p-r| < L_{k+1}}} \frac{1}{|p-r|} \right).$$

Divide the range $[L_k, L_{k+1})$ for $|p - r|$ into dyadic blocks $2^m \leq |p - r| < 2^{m+1}$ with $2^k \leq m < 2^{k+1}$, and use (5) from Lemma 3. For the pair $r, r + h$ being prime apply (5) to each of the two intervals for r defined by $2^m \leq |p - r| < 2^{m+1}$. For the pair $r, r - h$ put $n = r - h$ and apply (5) to each of the corresponding intervals for n translated appropriately. It follows that

$$(T_k T_j \mathbf{1})(p) \ll \sum_{L_j \leq h < L_{j+1}} \frac{1}{h} \sum_{2^k \leq m < 2^{k+1}} \frac{1}{2^m} \frac{h}{\phi(h)} \frac{2^m}{m^2} \ll \sum_{L_j \leq h < L_{j+1}} \frac{1}{\phi(h)} \frac{1}{2^k}.$$

An elementary argument gives (for all $x \geq 2$)

$$\sum_{h \leq x} \frac{1}{\phi(h)} = \sum_{h \leq x} \frac{1}{h} \sum_{d|h} \frac{\mu(d)^2}{\phi(d)} = \sum_{d \leq x} \frac{\mu(d)^2}{d\phi(d)} \sum_{m \leq x/d} \frac{1}{m} \leq (1 + \log x) \sum_d \frac{\mu(d)^2}{d\phi(d)} \ll \log x,$$

so that

$$\sum_{L_j \leq h < L_{j+1}} \frac{1}{\phi(h)} \ll \log L_{j+1} \ll 2^j.$$

This establishes (6).

Now suppose that $j < k$ and put $M = T_k T_j$. In the usual basis for $\ell^2(P)$, this matrix has non-negative entries, and the bound (6) just established shows that its row sums are bounded by $B_0 2^{j-k}$ for a suitable constant B_0 . The column sums of M are the row sums of its transpose $M^\top = T_j T_k$. By Lemma 4 and entrywise nonnegativity,

$$T_j T_k \mathbf{1} \leq A_0 T_j \mathbf{1} \leq A_0^2 \mathbf{1}.$$

Therefore by Schur's inequality (Lemma 1), we obtain

$$\|T_k T_j\| \leq \sqrt{(B_0 2^{j-k}) A_0^2} = A_0 \sqrt{B_0} 2^{-(k-j)/2}.$$

This establishes the desired bound on $\|T_k T_j\|$ when $j < k$. Since the operators T_j are self-adjoint, taking adjoints handles the opposite case $j > k$: $\|T_k T_j\| = \|(T_k T_j)^*\| = \|T_j^* T_k^*\| = \|T_j T_k\|$. Finally the case $j = k$ is already covered by Lemma 4. This completes our proof, and moreover a permissible value for D is $\max(A_0^2, A_0 \sqrt{B_0})$. $\square$

The proof of our main theorem is now at hand. Since the operators T_j are self-adjoint, and only finitely many of them are non-zero, we may apply the Cotlar-Stein Lemma together with the bound of Lemma 5. It follows that

$$\max_j \sum_k \|T_j^* T_k\|^{\frac{1}{2}} \leq \max_j \sum_k \sqrt{D} 2^{-|j-k|/4} \leq 2\sqrt{D} \sum_{\ell=0}^{\infty} 2^{-\ell/4} \leq 20\sqrt{D},$$

and the same bound applies to $\max_j \sum_k \|T_j T_k^*\|^{\frac{1}{2}}$. We conclude that

$$\|K_P\| \leq \|E\| + \left\| \sum_j T_j \right\| \leq 1 + 20\sqrt{D},$$

so that $\|K_P\|$ is uniformly bounded by an absolute constant, as desired.

REFERENCES

- [1] H. L. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS Regional Conference Series in Mathematics, vol. 84, American Mathematical Society, Providence, RI, 1994.
- [2] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, 2007.
- [3] Brad Rodgers, *On the optimal constant in the Montgomery-Vaughan weighted Hilbert inequality*, preprint, [arXiv:2608.12315v1](#), (2026).
- [4] E. M. Stein, *Harmonic Analysis: Real-variable methods, Orthogonality, and Oscillatory Integrals*. Princeton Mathematical Series, vol. 43, Princeton University Press, 1993.

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA, USA
 Email address: ksound@stanford.edu