

A counterexample to ordering-independence for chromatic operators

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Abstract

We give a 10-vertex counterexample to a conjecture of Pawłowski asserting that the characteristic polynomial of a chromatic operator associated with a forest is independent of the ordering of its edges. The two operators already have different traces of fifth powers.

1 The question

Let n be a nonnegative integer, and write $[n] = \{1, 2, \dots, n\}$. We encode a simple graph F on the vertex set $[n]$ by its edge set, so that every edge is a 2-element subset of $[n]$. If

$$\pi = (e_1, e_2, \dots, e_m)$$

is a linear ordering of the edges of F , define¹

$$\mathbf{a}_{F,\pi} = \prod_{r=1}^m (1 - t_{e_r}) \in \mathbb{Z}[S_n], \quad (1)$$

where $t_{\{i,j\}} = (i \ j) \in S_n$, and where the factors in (1) are multiplied from left to right in the order prescribed by π .

The *characteristic polynomial* of an element $x \in \mathbb{Q}[S_n]$ means the characteristic polynomial of left multiplication by x on the regular module $\mathbb{Q}[S_n]$. Pawłowski's conjecture [Paw22, Conjecture 1(a)] asks the following.

If F is a forest, is the characteristic polynomial of $\mathbf{a}_{F,\pi}$ independent of the edge ordering π ?

We answer this question in the negative.

2 The counterexample

Let F be the tree whose spine is

$$10 - 9 - 5 - 4 - 3 - 1,$$

with the additional leaves 2 and 7 attached to 1, the leaf 6 attached to 3, and the leaf 8 attached to 4. Thus,

$$E(F) = \{\{5, 9\}, \{9, 10\}, \{1, 3\}, \{4, 5\}, \{3, 6\}, \{1, 2\}, \{3, 4\}, \{4, 8\}, \{1, 7\}\}. \quad (2)$$

¹Pawłowski includes an overall factor of $n!$ in the definition of the chromatic operator. Since multiplication by this nonzero scalar does not affect whether the characteristic polynomials associated with two edge orderings are equal, we omit it here.

For compactness, write ij for the edge $\{i, j\}$. Consider the two edge orderings

$$\pi_1 = (59, 9\ 10, 13, 45, 36, 12, 34, 48, 17), \quad (3)$$

$$\pi_2 = (59, 9\ 10, 17, 36, 34, 13, 12, 45, 48). \quad (4)$$

Put $\mathbf{a}_i = \mathbf{a}_{F, \pi_i}$ for $i \in \{1, 2\}$.

Theorem 1. *The elements $\mathbf{a}_1$ and $\mathbf{a}_2$ have different characteristic polynomials. Concretely,*

$$[\text{id}]\mathbf{a}_1^5 = 323917419 \quad \text{and} \quad [\text{id}]\mathbf{a}_2^5 = 323917499, \quad (5)$$

where $[\text{id}]x$ denotes the coefficient of the identity permutation in $x \in \mathbb{Z}[S_{10}]$. (This entails that the traces of $\mathbf{a}_1^5$ and $\mathbf{a}_2^5$ differ, because the trace of any element $\mathbf{a} \in \mathbb{Z}[S_n]$ acting on the regular representation of S_n is $n![\text{id}]\mathbf{a}$. And thus, the characteristic polynomials of $\mathbf{a}_1$ and $\mathbf{a}_2$ must differ as well.)

The first five identity coefficients are as follows.

k	$[\text{id}]\mathbf{a}_1^k$	$[\text{id}]\mathbf{a}_2^k$
1	1	1
2	65	65
3	7922	7922
4	1417045	1417045
5	323917419	323917499

Before giving the verification, we record two elementary reductions.

Lemma 2 (Subword interpretation). *Let $w = (t_1, t_2, \dots, t_m)$ be any word of transpositions, and put*

$$a_w = (1 - t_1)(1 - t_2) \cdots (1 - t_m).$$

For every nonnegative integer k , the coefficient $[\text{id}]a_w^k$ is the number of subwords of the concatenated word w^k whose product is the identity. In particular,

$$0 \leq [\text{id}]a_w^k \leq 2^{mk}. \quad (6)$$

Proof. Expand the mk factors in a_w^k . Choosing either 1 or the transposition term from each factor is the same as choosing a subword of w^k . A chosen subword of length r has coefficient $(-1)^r$. If its product is the identity, then r is even, since every transposition is odd. Hence every identity subword contributes $+1$, and there is no cancellation. This proves both assertions. $\square$

Lemma 3 (Fourier inversion at the identity). *For each partition $\lambda \vdash n$, let ρ^λ be the corresponding irreducible representation of S_n , and let f^λ be its dimension. Then every $x \in \mathbb{Q}[S_n]$ satisfies*

$$n![\text{id}]x = \sum_{\lambda \vdash n} f^\lambda \text{tr}(\rho^\lambda(x)). \quad (7)$$

Proof. The left-hand side is the trace of left multiplication by x on the regular module $\mathbb{Q}[S_n]$. The regular module decomposes as

$$\mathbb{Q}[S_n] \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}.$$

Taking traces gives (7). $\square$

Applying (7) avoids the $10!$ -dimensional regular representation. There are only 42 partitions of 10, and the largest possible dimension of an irreducible representation of S_{10} is $\max_{\lambda \vdash 10} f^\lambda = 768$. Thus, when computing the identity coefficient $[\text{id}]x$ of an element $x \in \mathbb{Q}[S_n]$ using Lemma 3, we only need to deal with matrices of size at most 768×768 (namely, the matrices $\rho^\lambda(x)$). Moreover, if x is a k -th power a^k of some $a \in \mathbb{Q}[S_n]$, then $\rho^\lambda(x) = \rho^\lambda(a^k) = (\rho^\lambda(a))^k$, so that we need not compute x itself but can get by with computing the k -th power of the matrix $\rho^\lambda(a)$. This makes the coefficients $[\text{id}]\mathbf{a}_1^5$ and $[\text{id}]\mathbf{a}_2^5$ realistically computable.

Here is the complete SageMath code that performs this computation. It uses Sage's built-in graph, partition, permutation, matrix, and Specht-representation functionality.

```
"""
```

```
Exact SageMath verification of a counterexample to Pawlowski's
ordering-independence conjecture for chromatic operators.
```

```
Run with:
```

```
sage pawlowski_counterexample.sage
```

```
The computation uses all 42 Specht representations of S_10, not the
10!-dimensional regular representation. Its largest matrix is 768 by 768.
```

```
"""
```

```
from sage.all import *
from sage.combinat.symmetric_group_representations import \
    SymmetricGroupRepresentations
```

```
n = 10
```

```
# The tree is the path 10-9-5-4-3-1, with leaves 2 and 7 at 1,
# leaf 6 at 3, and leaf 8 at 4.
```

```
order_1 = [
    (5, 9), (9, 10), (1, 3), (4, 5), (3, 6),
    (1, 2), (3, 4), (4, 8), (1, 7),
]
```

```
order_2 = [
    (5, 9), (9, 10), (1, 7), (3, 6), (3, 4),
    (1, 3), (1, 2), (4, 5), (4, 8),
]
```

```
edges = set(order_1)
```

```
# Elementary checks on the input.
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```
assert set(order_2) == edges
```

```
G = Graph(list(edges))
```

```
assert G.vertices(sort=True) == list(range(1, n + 1))
```

```
assert G.is_tree()
```

```
K = QQ
```

```

def transposition(i, j):
    """Return (i j) as a Sage permutation in one-line notation."""
    w = list(range(1, n + 1))
    w[i - 1], w[j - 1] = w[j - 1], w[i - 1]
    return Permutation(w)

# Load Sage's integral Specht models. The identity
#
#  $n! [1]x = \sum_{\lambda \vdash n} f^\lambda \text{tr}(\rho^\lambda(x))$ 
#
# is the decomposition of the regular character.
representations = SymmetricGroupRepresentations(
    n, "specht", ring=K, cache_matrices=False
)
# regular_traces will be a list of traces of  $A_1^k$  and  $A_2^k$ 
# on the regular representation of  $S_n$ , for all
#  $k \in \{1, 2, 3, 4, 5\}$ , where  $A_1$  and  $A_2$  are the actions
# of the two products of  $1 - t$ 's in the two orders.
regular_traces = [[K.zero() for _ in range(6)] for _ in range(2)]
dimensions = [] # will be the list of the dimensions  $f^\lambda$ 

for lam in Partitions(n):
    rho = representations(lam) # the Specht module for partition lam

    # Compute each transposition matrix only once for this partition.
    T = {e: rho(transposition(*e)) for e in edges}
    d = next(iter(T.values())).nrows() # the dimension of rho
    dimensions.append(d)
    I = identity_matrix(K, d)

    def chromatic_operator(order): # action of order-operator on rho
        A = I
        for e in order:
            A = A * (I - T[e])
        return A

    A = [chromatic_operator(order_1), chromatic_operator(order_2)]
    for which in range(2):
        power = A[which]
        for k in range(1, 6):
            regular_traces[which][k] += K(d) * power.trace()
            if k < 5:
                power = power * A[which]

# Sanity checks on the irreducible dimensions.
assert len(dimensions) == Partitions(n).cardinality() == 42
assert max(dimensions) == 768

```

```

assert sum(d^2 for d in dimensions) == factorial(n)

# Divide the regular trace by 10! to obtain the identity coefficient.
# All computations are exact over QQ.
coefficients = []
for which in range(2):
    row = []
    for k in range(1, 6):
        residue = regular_traces[which][k] / K(factorial(n))
        row.append(QQ(residue))
    coefficients.append(row)

expected = [
    [1, 65, 7922, 1417045, 323917419],
    [1, 65, 7922, 1417045, 323917499],
]
assert coefficients == expected

print("identity coefficients for order 1:", coefficients[0])
print("identity coefficients for order 2:", coefficients[1])
print("difference of fifth coefficients:",
      coefficients[1][4] - coefficients[0][4])

```

The output is

```

identity coefficients for order 1: [1, 65, 7922, 1417045, 323917419]
identity coefficients for order 2: [1, 65, 7922, 1417045, 323917499]
difference of fifth coefficients: 80

```

This proves (5).

Finally, for any $x \in \mathbb{Q}[S_{10}]$, the trace of its left-regular action is $10![\text{id}]x$. Hence

$$\text{tr}(L_{\mathbf{a}_2}^5) - \text{tr}(L_{\mathbf{a}_1}^5) = 10! \cdot 80 = 290304000 \neq 0.$$

Two endomorphisms with the same characteristic polynomial have equal traces of all powers. Therefore $L_{\mathbf{a}_1}$ and $L_{\mathbf{a}_2}$ have different characteristic polynomials, as claimed.

Remark 4. *For a faster certificate that only disproves equality, one may work over finite fields whose characteristic is larger than 10. However, in the SageMath version used for this computation, the Specht-representation implementation did not work over the desired finite fields, so we used exact rational arithmetic instead.*

References

[Paw22] Brendan Pawlowski, *Chromatic symmetric functions via the group algebra of S_n* , Algebraic Combinatorics 5 (2022), no. 1, 1–20; [arXiv:1802.05470](https://arxiv.org/abs/1802.05470).