

A Counterexample to a Conjecture on Fused Specht Polynomials

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Abstract

Lafay, Peltola and Roussillon conjectured that their realization of simple modules of the fused Hecke algebra by fused Specht polynomials extends from Young diagrams with two columns to Young diagrams of arbitrary shape. We give a counterexample for

$$n = 8, \quad \lambda = (3, 2, 2, 1), \quad \varsigma = (2, 2, 2, 2).$$

More precisely, we exhibit an explicit linear dependence among three fused Specht polynomials indexed by row-strict Young tableaux. The same example also yields an infinite family of counterexamples.

We use the notation of [1]. In particular, p_ς denotes the ς -antisymmetrizer idempotent, P_N denotes the Specht polynomial associated with a numbering N , and $\mathcal{F}_T$ denotes the fused Specht polynomial associated with a filling T .

1 The counterexample

Let

$$\lambda = (3, 2, 2, 1) \quad \text{and} \quad \varsigma = (2, 2, 2, 2).$$

Consider the following three elements of $\text{RSYT}_\varsigma^\lambda$:

$$A = \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 4 & \\ 2 & 4 & \\ 3 & & \end{array}, \quad B = \begin{array}{ccc} 1 & 2 & 4 \\ 1 & 3 & \\ 2 & 3 & \\ 4 & & \end{array}, \quad C = \begin{array}{ccc} 1 & 3 & 4 \\ 1 & 3 & \\ 2 & 4 & \\ 2 & & \end{array}. \quad (1)$$

Their numberings from [1, Definition 2.9] are

$$\tilde{A} = \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 7 & \\ 3 & 8 & \\ 5 & & \end{array}, \quad \tilde{B} = \begin{array}{ccc} 1 & 4 & 8 \\ 2 & 5 & \\ 3 & 6 & \\ 7 & & \end{array}, \quad \tilde{C} = \begin{array}{ccc} 1 & 5 & 8 \\ 2 & 6 & \\ 3 & 7 & \\ 4 & & \end{array}. \quad (2)$$

To avoid using the same variables before and after fusion, we temporarily write $z_1, \dots, z_8$ for the variables of the Specht polynomials and $x_1, \dots, x_4$ for the variables of the fused Specht polynomials. For a subset $J \subseteq [8]$, set

$$\Delta_J := \prod_{\substack{i, j \in J \\ i < j}} (z_j - z_i).$$

It follows immediately from the columns of the numberings in (2) that

$$P_{\tilde{A}} = \Delta_{\{1,2,3,5\}} \Delta_{\{4,7,8\}}, \quad (3)$$

$$P_{\tilde{B}} = \Delta_{\{1,2,3,7\}} \Delta_{\{4,5,6\}}, \quad (4)$$

$$P_{\tilde{C}} = \Delta_{\{1,2,3,4\}} \Delta_{\{5,6,7\}}. \quad (5)$$

The following elementary observation makes the fusion calculation short.

Lemma 1. *For $Q \in \mathbb{C}[z_1, \dots, z_8]$, let*

$$D := \prod_{k=1}^4 \left(\frac{\partial}{\partial z_{2k}} - \frac{\partial}{\partial z_{2k-1}} \right).$$

Then

$$\psi(p_\varsigma.Q) = \frac{1}{16} [DQ]_{z_{2k-1}=z_{2k}=x_k \text{ for all } k \in [4]}. \quad (6)$$

Proof. For two variables z_a, z_b , Taylor expansion at $z_a = z_b = x$ gives

$$\left[\frac{Q - (a \ b).Q}{z_b - z_a} \right]_{z_a=z_b=x} = \left[\left(\frac{\partial}{\partial z_b} - \frac{\partial}{\partial z_a} \right) Q \right]_{z_a=z_b=x}. \quad (7)$$

Since

$$p_\varsigma = \frac{1}{16} \prod_{k=1}^4 (1 - (2k-1 \ 2k)),$$

applying (7) successively to the four pairs proves (6). $\square$

We can now state the counterexample.

Proposition 1. *The three fused Specht polynomials associated with the tableaux in (1) satisfy*

$$2\mathcal{F}_A - 2\mathcal{F}_B + \mathcal{F}_C = 0. \quad (8)$$

Consequently, the map ψ in [1, (2.16)] is not injective on $p_\varsigma(P^\lambda)$. In particular, Conjecture 2.17 of [1] is false.

Proof. Applying Lemma 1 to the factorizations (3)–(5) gives

$$\begin{aligned} \mathcal{F}_A = & -\frac{1}{2}(x_1 - x_2)(x_1 - x_3)(x_2 - x_4) \\ & \cdot (x_1^2 + x_1x_2 - 3x_1x_3 + x_2^2 - 3x_2x_4 + 3x_3x_4), \end{aligned} \quad (9)$$

$$\begin{aligned} \mathcal{F}_B = & -\frac{1}{2}(x_1 - x_2)(x_1 - x_4)(x_2 - x_3) \\ & \cdot (x_1^2 + x_1x_2 - 3x_1x_4 + x_2^2 - 3x_2x_3 + 3x_3x_4), \end{aligned} \quad (10)$$

$$\mathcal{F}_C = (x_1 - x_2)^4(x_3 - x_4). \quad (11)$$

A direct subtraction of (9) from (10) yields

$$\mathcal{F}_B - \mathcal{F}_A = \frac{1}{2}(x_1 - x_2)^4(x_3 - x_4) = \frac{1}{2}\mathcal{F}_C. \quad (12)$$

This proves (8).

It remains to check that this dependence comes from a nonzero element of $p_\varsigma(P^\lambda)$. By [1, Proposition 2.11] and the isomorphism $\phi : V^\lambda \rightarrow P^\lambda$ from [1, Lemma 2.3], the collection

$$\left\{ p_\varsigma \cdot P_{\tilde{T}} \mid T \in \text{RSYT}_\varsigma^\lambda \right\}$$

is a basis of $p_\varsigma(P^\lambda)$. In particular,

$$2p_\varsigma \cdot P_{\tilde{A}} - 2p_\varsigma \cdot P_{\tilde{B}} + p_\varsigma \cdot P_{\tilde{C}} \neq 0,$$

whereas its image under ψ is zero by (8). Thus ψ is not injective. $\square$

2 Further remarks

Remark 1 (An infinite family). *For each $r \geq 0$, set*

$$\lambda^{(r)} = (r+3, 2, 2, 1) \quad \text{and} \quad \varsigma^{(r)} = (2, 2, 2, 2, \underbrace{1, \dots, 1}_{r \text{ times}}).$$

Append the entries $5, 6, \dots, 4+r$, in this order, to the first row of each of A, B, C . This gives three elements

$$A^{(r)}, B^{(r)}, C^{(r)} \in \text{RSYT}_{\varsigma^{(r)}}^{\lambda^{(r)}}.$$

The added columns have one box each, so they contribute no factors to the associated Specht polynomials. The added blocks of the antisymmetrizer also have size one. Therefore

$$\mathcal{F}_{A^{(r)}} = \mathcal{F}_A, \quad \mathcal{F}_{B^{(r)}} = \mathcal{F}_B, \quad \mathcal{F}_{C^{(r)}} = \mathcal{F}_C,$$

where the right hand sides are regarded as polynomials independent of the additional variables. Thus

$$2\mathcal{F}_{A^{(r)}} - 2\mathcal{F}_{B^{(r)}} + \mathcal{F}_{C^{(r)}} = 0$$

for every $r \geq 0$, giving counterexamples in every size $n \geq 8$.

Remark 2 (Small sizes). *There are seven tableaux in $\text{RSYT}_{(2,2,2,2)}^{(3,2,2,1)}$. Exact row reduction gives*

$$\dim p_{(2,2,2,2)}(P^{(3,2,2,1)}) = 7 \quad \text{and} \quad \dim \text{span}_{\mathbb{C}} \left\{ \mathcal{F}_T \mid T \in \text{RSYT}_{(2,2,2,2)}^{(3,2,2,1)} \right\} = 6.$$

Thus the kernel in Proposition 1 is one-dimensional. An exhaustive computation for $n \leq 8$, up to permuting the parts of ς , found no counterexample for $n \leq 7$ and found this as the only counterexample for $n = 8$. Thus the example above is computationally minimal.

Remark 3 (The collision in Remark 2.33 of [1]). *The two tableaux in [1, Remark 2.33], which have shape $(3, 2, 2, 1)$ and content $(3, 1, 2, 2)$, do not themselves give a dependence. Indeed, their fused Specht polynomials are, up to nonzero scalar factors,*

$$(x_3 - x_1)^2(x_4 - x_2)^2 \quad \text{and} \quad (x_4 - x_1)^2(x_3 - x_2)^2,$$

respectively, and are therefore linearly independent. The dependence in Proposition 1 occurs for the more symmetric content $(2, 2, 2, 2)$.

References

- [1] Augustin Lafay, Eveliina Peltola and Julien Roussillon, *Fused Specht Polynomials and $c = 1$ Degenerate Conformal Blocks*, arXiv:2410.09798v3, 2025. <https://arxiv.org/abs/2410.09798v3>.