

UNDER MEASURABLES, $\text{HOD}_{\text{Ord}^\omega}$ IS LOCALLY ARBITRARY

ABSTRACT. For the inner model $\text{HOD}_{\text{Ord}^\omega}$, we derive an analogue for Roguski’s classical result that without nontrivial assumptions on V , HOD is an arbitrary model of ZFC . For any sentence σ , letting CM denote “class of measurables,” the following theories are equiconsistent:

- (1) $\text{ZFC} + \text{CM} + [\text{HOD}_{\text{Ord}^\omega} \models (\exists \kappa V_\kappa \models \sigma)]$;
- (2) $\text{ZFC} + \text{GCH} + \text{CM} + [\text{HOD}_{\text{Ord}^\omega} \models (\text{SVC} \wedge \text{CM} \wedge \exists \kappa V_\kappa \models \sigma)]$;
- (3) $\text{ZF} + \text{DC} + \text{CM} + \exists \kappa V_\kappa \models \sigma$.

Thus, under measurables the only local constraint on $\text{HOD}_{\text{Ord}^\omega}$ is the requisite choice principle.

We also derive an analogue for Woodins. Letting CW denote “class of Woodins,” the following are equiconsistent:

- (1) $\text{ZFC} + \text{CW} + [\text{HOD}_{\text{Ord}^\omega} \models (\exists \kappa V_\kappa \models \sigma)]$;
- (2) $\text{ZFC} + \text{GCH} + \text{CW} + [\text{HOD}_{\text{Ord}^\omega} \models (\exists \kappa V_\kappa \models \sigma)] + \text{HOD}_{\text{Ord}^\omega}$ is a ground of V ;
- (3) $\text{ZFC} + \text{CW} + \text{some symmetric extension } V(G) \models \text{DC} + \exists \kappa V_\kappa \models \sigma$.

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1. EXTERNAL HOD AND LOCAL ARBITRARINESS

Roguski's work on the theory of HOD is the starting point for this paper [1]. In the countable transitive formulation of his realization theorem, every countable transitive model M of ZFC has a larger model N of ZFC such that

$$M = \text{HOD}^N.$$

Ordinary HOD can thus realize a prescribed ZFC universe. We seek an analogue when definitions may use countable sequences of ordinals as parameters. Our first result concerns the theories of rank segments of the resulting inner model under a class of measurable cardinals. A second result relates its rank theory under a proper class of Woodins to that of symmetric extensions of Woodin grounds.

For $U \models \text{ZFC}$, write

$$\mathcal{H}^U = \text{HOD}_{\text{Ord}^\omega}^U.$$

An allowed parameter is one function from ω to the ordinals, together with finitely many ordinal parameters; the hereditary requirement is retained. This inner model contains every ordinal ω -sequence of U and satisfies $\text{ZF} + \text{DC}$ (Lemma 10.1). The local analogue of Roguski's theorem will therefore compare its rank theory with that of choiceless models satisfying dependent choice.

Write **CM** for the assertion that measurable cardinals are unbounded in the ordinals, abbreviated as “class of measurables.” A measurable cardinal is uncountable and carries a nonprincipal, cardinal-complete ultrafilter. We require normality only in choice models. Put

$$T_0 = \text{ZF} + \text{DC} + \text{CM}.$$

For each first-order sentence σ , consider the theories

$$\begin{aligned} E_1(\sigma) &= \text{ZFC} + \text{CM} + [\mathcal{H} \models \exists \kappa (V_\kappa \models \sigma)], \\ E_2(\sigma) &= \text{ZFC} + \text{GCH} + \text{CM} + [\mathcal{H} \models \text{SVC} \wedge \text{CM} \wedge \exists \kappa (V_\kappa \models \sigma)], \\ E_3(\sigma) &= T_0 + \exists \kappa (V_\kappa \models \sigma). \end{aligned}$$

In the first two theories, the witnessing rank is computed inside $\mathcal{H}$. Theory E_2 requires GCH in the ambient universe and SVC together with a class of measurables inside $\mathcal{H}$. Our first main result is the following.

Theorem 1.1 (External rank equiconsistency). *For every σ ,*

$$\text{Con}(E_1(\sigma)) \longleftrightarrow \text{Con}(E_2(\sigma)) \longleftrightarrow \text{Con}(E_3(\sigma)).$$

This is the sense in which $\text{HOD}_{\text{Ord}^\omega}$ is locally arbitrary under measurables. Every rank-sentence assertion consistent with $\text{ZF} + \text{DC} + \text{CM}$ can hold inside the sequence-parameter HOD of a $\text{ZFC} + \text{GCH} + \text{CM}$ universe, even when that HOD is itself required to satisfy $\text{SVC} + \text{CM}$. GCH is imposed on the ambient universe, so it does not restrict the rank theory being realized inside $\mathcal{H}$. The comparison concerns consistency of rank-sentence theories. Exact realization of an entire starting model is available under the prepared-ground hypotheses stated below.

1.1. The Woodin equiconsistency. Write CW for the assertion that Woodin cardinals are unbounded in the ordinals. For a fixed first-order sentence σ , put

$$R_\sigma \equiv \exists \alpha (V_\alpha \models \sigma).$$

Consider the following three theories:

$$\begin{aligned} W_1(\sigma) &= \text{ZFC} + \text{CW} + [\mathcal{H} \models R_\sigma], \\ W_2(\sigma) &= \text{ZFC} + \text{CW} + [\text{some set-symmetric extension satisfies } \text{DC} + R_\sigma], \\ W_3(\sigma) &= \text{ZFC} + \text{GCH} + \text{CW} + [\mathcal{H} \models R_\sigma] \\ &\quad + [\mathcal{H} \text{ is a symmetric ground of } V]. \end{aligned}$$

The symmetric extension in W_2 is a model of ZF. The assertion CW is imposed on its ZFC ground, and need not hold in the symmetric extension. To read W_2 syntactically, assert the existence of a set symmetric system and a condition forcing $\text{DC} + R_\sigma$ under the hereditary-symmetric interpretation. The symmetric forcing theorem supplies ZF in that interpretation. In W_3 , both GCH and CW are assertions about the ambient universe. We impose no Woodin-cardinal assertion on $\mathcal{H}$. The term symmetric ground has its set-forcing meaning. In a ZFC ambient universe, it is equivalent to being an ordinary set-forcing ground, which may fail choice [7, Corollary 5.9].

Theorem 1.2 (Woodin rank equiconsistency). *For every fixed first-order sentence σ ,*

$$\text{Con}(W_1(\sigma)) \longleftrightarrow \text{Con}(W_2(\sigma)) \longleftrightarrow \text{Con}(W_3(\sigma)).$$

The proof preserves the relevant membership structure V_α . It uses a relative inner model containing a given real and a proper class of Woodins, together with a set-forcing presentation of that inner model over its own HOD. The relative construction is invariant under Turing-equivalent recodings in the same ambient universe. These properties allow the collapse argument to descend to a model with the required rank and dependent choice.

For the converse, retain the Woodin ground of the given symmetric extension. Its old normal measures supply the local Prikry construction. After the sequence generator has been coded, a bounded homogeneous code makes enough of the original ground ordinal-definable to recover the prescribed rank. This uses only set forcing over the Woodin ground. This rank-realization argument works over the original W_2 ground without an eventual-GCH hypothesis.

To obtain W_3 , we instead use the particular ground supplied by the reverse construction. It satisfies eventual GCH, and each of its sets is internally definable from ordinals and one fixed set parameter. A bounded $\diamond^*$ code makes that parameter and a ground-defining parameter ordinal-definable while preserving GCH. The spectrum then recovers the whole sequence-generated model as $\mathcal{H}$, with the ambient universe a set-forcing extension of it. This proves the consistency strengthening without a class preparation preserving Woodin cardinals.

1.2. Realization and rank preservation. The construction first preserves a specified rank while passing to a model generated by a set of ordinal sequences. It then realizes that rank inside external HOD. Two presentation principles enter the first step. Write $\text{SVC}(S)$ when every nonempty set is a surjective image of $\eta \times S$ for some ordinal η ; SVC asserts that such a set S exists. The assertion $\text{KW}_1^{*,\omega}$ says that every nonempty set is a surjective image of μ^ω for some ordinal μ , with the sequence power computed in the model under discussion. Write $T_s = T_0 + \text{SVC}$.

The external realization step takes the following form.

Theorem 1.3 (External realization). *Suppose $M \models T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$. For every ordinal α there is an outer model $U \models \text{ZFC} + \text{GCH} + \text{CM}$ of M such that*

$$\mathcal{H}^U \models \text{SVC} + \text{CM}, \quad V_\alpha^{\mathcal{H}^U} = V_\alpha^M.$$

If M has a presentation as in Proposition 13.1 whose choice envelope satisfies eventual GCH, one may require $\mathcal{H}^U = M$.

The secure-ground case gives equality with the whole generated model by further set forcing over its fresh choice envelope. The general preparation uses class forcing, so the outer model in Theorem 1.3 is not asserted to be a set-forcing extension of M .

The next theorem supplies the required presentation while retaining any chosen rank and its full power set. In particular, SVC is not an assumption on the original model. Here $D(A)$ denotes the model generated over a transitive base. It contains D and the transitive closure of $\{A\}$.

Theorem 1.4 (Rank transfer without SVC). *Suppose $V \models T_0$ and $\alpha \in \text{Ord}$. In a set-generic extension of V there is a model M such that*

$$M \models T_0 + \text{SVC} + \text{KW}_1^{*,\omega}, \quad V_\alpha^M = V_\alpha^V, \quad \mathcal{P}(V_\alpha^V)^M = \mathcal{P}(V_\alpha^V)^V.$$

Moreover, $M = D(A)$ for a choice symmetric ground $D \models \text{ZFC} + \text{CM}$ and a set $A \subseteq \mu^\omega$.

The model M contains a rank-preserving SVC inner model of V , but need not contain all of V . Once the starting model satisfies SVC, the local construction also gives containment.

Theorem 1.5 (Local rank-to-sequence transfer). *Suppose $N \models T_s$ and $\alpha \in \text{Ord}$. In a set-generic extension of N there is an inner model M containing N such that*

$$M \models T_s + \text{KW}_1^{*,\omega}, \quad V_\alpha^M = V_\alpha^N.$$

More precisely, one can choose $\lambda > \alpha$ and $X = V_\lambda^N$ so that

$$\mathcal{P}(X)^M = \mathcal{P}(X)^N, \quad V_\lambda^M = X.$$

There are a choice symmetric ground $D \models \text{ZFC} + \text{CM}$, an ordinal μ , and $A \subseteq \mu^\omega$ in M such that $M = D(A)$.

The two transfer statements also give the following equiconsistency.

Corollary 1.6 (Rank-sentence equiconsistency). *For every first-order sentence σ , the theories*

$$T_0 + \exists \alpha (V_\alpha \models \sigma) \quad \text{and} \quad T_0 + \text{SVC} + \text{KW}_1^{*,\omega} + \exists \alpha (V_\alpha \models \sigma)$$

are equiconsistent.

Proof. Apply Theorem 1.4 to a witnessing rank. The converse is immediate. $\square$

Combining rank transfer with external realization gives the implication from $E_3(\sigma)$ to $E_2(\sigma)$ at the level of consistency. The implication from $E_2(\sigma)$ to $E_1(\sigma)$ is immediate. For the remaining implication, Proposition 10.3 uses core-model preparation and homogeneous descent to obtain an inner model of the given sequence-parameter HOD while retaining its witnessing rank. This step does not assume that the original HOD already has a class of measurables.

1.3. Structural conservativity. The rank-preserving construction also yields a structural conservativity theorem. This result concerns full second-order truth in arbitrary set-sized structures and allows a starting theory without Foundation. Put

$$T_- = (\text{ZF} - \text{Foundation}) + \text{DC} + \text{CM}.$$

Definition 1.7 (Structural Π_2 sentences). A *structural Π_2 sentence* is a set-theoretic assertion of the form

$$\forall \mathcal{M} (\mathcal{M} \text{ is a set-sized } \mathcal{L}\text{-structure} \implies \mathcal{M} \models_{\text{full}} \varphi),$$

where φ is a fixed second-order sentence in a finite signature $\mathcal{L}$. Satisfaction uses full second-order semantics: if A is the domain of $\mathcal{M}$, each n -ary second-order relation variable ranges over all of $\mathcal{P}(A^n)$ in the ambient universe. This assertion has a Π_2 formulation in the Lévy hierarchy. The second-order quantifier complexity of φ is unrestricted.

Theorem 1.8 (Structural conservativity). *The theory $T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$ is conservative over T_- for structural Π_2 assertions. That is, for every fixed full second-order sentence φ , the assertion that φ holds in every set-sized structure has the same provability in the two theories.*

Foundation removal preserves structural truth. Theorem 1.8 does not identify the non-well-founded membership structure of an arbitrary starting universe with a rank of a well-founded model. The argument represents a full second-order counterexample by a structure on atoms. After the local forcing construction, fibers of ordinal-sequence surjections give a pure copy of that structure and of an SVC witness. The full power sorts retain the truth of the original sentence.

Remark 1.9 (Role of the core-model preparation). The core-model argument gives a passage to SVC that preserves each specified rank. We use this rank-preserving result directly. The proof constructs a uniform choice model over the coding real inside relative HOD, using the core model in the negative branch. Canonical refinement preserves its pure universe under finite recodings and gives a set-forcing presentation over its own HOD. The proof does not require $V = \text{HOD}(r)$. The earlier $\text{SVC} + \text{KW}_1^{*,1}$ conservativity result is compatible with this preparation and is not an additional hypothesis of the structural conservativity theorem.

1.4. Limits of the conclusions. The large-cardinal assumption cannot simply be deleted from Theorem 1.1. Put $\mathbb{R} = \mathcal{P}(\omega)$ and

$$A_L \equiv \mathbb{R} = \mathbb{R}^L \wedge \omega_2 = \omega_2^L.$$

Over ZF, this agreement implies that every countable ordinal sequence belongs to L . To see this, fix such a sequence s and apply Jensen covering in the choice model $L[s]$. A constructible cover of its range has size at most ω_1 there. Agreement at ω_2 makes its constructible size at most ω_1 as well; agreement on reals then returns s to L . Thus

$$\text{ZF} + A_L + \text{KW}_1^{*,\omega} \vdash \text{AC}.$$

Likewise, if $\mathcal{H}^U \models A_L$ for $U \models \text{ZFC}$, all its allowed sequence parameters are constructible, and $\mathcal{H}^U = \text{HOD}^U$. A symmetric extension of L using countable conditions and countable supports satisfies $\text{ZF} + \text{DC} + A_L$ while $\mathcal{P}(\omega_1)$ is not wellorderable and DC_{ω_1} fails. Appendix B gives the construction and a single guarded rank sentence σ that can hold in this model but in no rank of any external $\mathcal{H}^U$. Hence the rank-equiconsistency conclusion itself fails when CM is removed.

There is also an obstruction below measurables to the structural conservativity theorem. Let CR assert a class of Ramsey cardinals, with Ramseyness understood as $\kappa \rightarrow (\kappa)_2^{<\omega}$. There is a parameter-free Π_3^2 sentence Φ of third-order arithmetic such that, for $T_R = \text{ZF} + \text{DC} + \text{CR}$,

$$T_R + \text{KW}_1^{*,\omega} \vdash \Phi, \quad \text{Con}(\text{ZFC} + \text{CR}) \implies \text{Con}(T_R + \neg\Phi).$$

Its semantic form over this base is

$$[\neg\exists 0^\dagger \wedge \mathbb{R} = \mathbb{R}^K \wedge \omega_2 = \omega_2^K] \implies \text{Fodor}(\omega_1),$$

where K is the classical Dodd–Jensen core computed in HOD. The Ramsey class and the absence of $0^\dagger$ exclude an inner model with a measurable. Strong covering then returns every countable ordinal sequence to K , so $\text{KW}_1^{*,\omega}$ implies Choice under the displayed agreement. A countably supported symmetric extension of the classical core retains the agreement, DC, and a class of Ramseys, but has a stationary regressive map with no stationary fiber. Appendix C proves the claim and gives its third-order formulation. The Π_3^2 bound counts quantifiers over sets of reals; it is not a claim of minimal complexity. The same sentence has a structural formulation as in Definition 1.7. Thus CR cannot replace CM in Theorem 1.8.

The choice of inner model matters as well. Let $C = L[\text{Ord}^\omega]$ denote the Chang model, the least transitive inner model of ZF containing every countable ordinal sequence. Under the same ambient measurables, its failure of SVC prevents the strengthened conclusion of Theorem 1.1.

Proposition 1.10 (The Chang obstruction). *If $U \models \text{ZFC} + \text{CM}$, then $C^U \models \neg \text{SVC}$.*

Proof. Every member of C is uniquely definable over C from a countable ordinal sequence, with finite ordinal parameters absorbed into that sequence [24, Section 1.2]. For a normal ultrapower $j : U \rightarrow M$, countable closure gives $C^M = C^U$; hence $j \restriction C : C \rightarrow C$ is elementary. We also use Kunen’s finite-movement lemma: after choosing normal measures at distinct measurable cardinals, each ordinal is moved by only finitely many of the corresponding ultrapowers [23, Lemma 4.10]. The stated form follows from the countable-sequence version by selecting infinitely many exceptions if they exist.

Suppose $S \in C$ witnesses SVC. Choose an uncountable set I of ambient measurables above $\text{rank}(S)$, with a normal measure at each, and let $\lambda > \sup I$. There is in C a surjection $f : \eta \times S \rightarrow \lambda^\omega$; Chang contains the full ambient sequence space. Let $p \in \text{Ord}^\omega$ define f uniquely over C . Only countably many of the selected ultrapowers move a coordinate of p . Choose distinct remaining measurables $\langle \kappa_n : n < \omega \rangle$ and put $z = \langle \kappa_n : n < \omega \rangle$. Take one preimage $f(\xi, s) = z$. Some selected ultrapower j with critical point κ_n fixes ξ . It fixes s by rank and fixes f by its definition from p over the unchanged Chang model. Therefore

$$j(z) = j(f)(j(\xi), j(s)) = f(\xi, s) = z, \quad j(z)(n) = j(\kappa_n) > \kappa_n = z(n),$$

a contradiction. All choices here are made in U . $\square$

Replacing $\mathcal{H}$ by C in $E_2(\sigma)$ therefore makes that theory inconsistent for every σ . This already rules out the three-way Chang replacement. The proposition uses internal definability over Chang and its invariance under ambient normal ultrapowers. It does not assert invariance of the ambiently computed $\text{HOD}_{\text{Ord}^\omega}$, nor does global failure of SVC exclude SVC from each individual Chang rank.

Even the comparison of theories (1) and (3) can fail after replacing $\mathcal{H}$ by C . Relative to ZFC with a proper class of strong cardinals and an inaccessible limit of strong cardinals, there is a sentence σ for which theory (3) is consistent but the Chang version of theory (1) is inconsistent. Appendix D gives the statement and proof. This explains why the choice of $\text{HOD}_{\text{Ord}^\omega}$ matters even without the SVC requirement in theory (2). Whether an anti-inner-model hypothesis restores the two-way comparison remains open; Question D.5 considers excluding inner models with a measurable of Mitchell order at least two.

1.5. Proof outline and formalization. Sections 2–7 prove the local rank-to-sequence theorem. A high Prikry sequence and a countably closed labeling forcing produce $Q \subseteq \kappa^\omega$ and $F : Q \rightarrow X$. Paired comparison removes the labeling coordinate from low-rank names while keeping their values fixed. External countable domination gives DC in $N(Q, F)$. Section 8 supplies the initial passage to SVC without changing the protected rank. The core-model and atomization reductions originate in the research archive [2]; Section 9 gives the atomization and fiber purification needed for structural conservativity.

Sections 10–14 complete the external-HOD argument. Sparse Cohen rows with evaluated terminal caps code the sequence generator over a recognizable choice envelope. The code makes the generator ordinal-definable, while homogeneous descent bounds the resulting HOD by the intended model. Repeated $\diamond^*$ codes provide a secure ground with eventual GCH. A countably closed collapse gives GCH in the choice envelope, and the final coding preserves it. The measurable consistency implications combine this realization with the rank-preserving preparation.

Sections 15–19 prove the Woodin theorem. The relative fully backgrounded construction and its top-iteration alternative give a canonical choice inner model with a proper class of Woodins, presented by set forcing over its own HOD. Homogeneous descent produces the required symmetric model. In the other direction, an old-ground form of the local Prikry construction and bounded high coding realize its protected rank. Section 18 uses the special reverse ground to obtain GCH

and exact equality with a symmetric ground. The proof first codes a fixed approximation-and-cover parameter, then places the spectrum above that code. This identifies the full choice envelope in the final universe. Every forcing used in the Woodin argument is set forcing. Appendix D gives the separate strong-cardinal obstruction to the Chang replacement.

We use transitive-model notation for the forcing and inner-model interpretations. This notation does not assume that consistency supplies a transitive model. The set-forcing arguments use the forcing theorem. Generated models use the definable transitive-base construction, and the Foundation-free construction uses the interpreted cumulative hierarchy over a set of atoms. These constructions can equivalently be carried out over countable models, using their internal notions of well-foundedness and satisfaction. The KW and Foundation reductions in Sections 2–9 use only set forcing. The measurable exact-realization preparation uses the progressively closed $\diamond^*$ coding of Section 12. Each of its set arguments takes place in a set-sized initial segment.

2. SMALL FORCING AND GENERATED-MODEL PRESENTATIONS

Lemma 2.1 (Small forcing in ZF). *Let $V \models \text{ZF}$, let $\mathbb{B} \in V$ be a complete Boolean algebra, and let $\mathcal{U}$ be a nonprincipal κ -complete ultrafilter on κ , where $\kappa > h(\mathbb{B})$. For every V -generic $G \subseteq \mathbb{B}$, the filter*

$$\mathcal{U}^G = \{a \in \mathcal{P}(\kappa)^{V[G]} : \exists b \in \mathcal{U} (b \subseteq a)\}$$

is a nonprincipal κ -complete ultrafilter in $V[G]$. In particular, arbitrary set forcing over ZF preserves a proper-class tail of measurable cardinals.

Proof. For a name $\dot{a} \subseteq \check{\kappa}$, the range of

$$\zeta \mapsto \|\dot{\zeta} \in \dot{a}\|_{\mathbb{B}}$$

is canonically well-orderable by least preimages. Its cardinality is below $h(\mathbb{B})$, hence below κ . The partition into Boolean-value fibers therefore has a unique $\mathcal{U}$ -positive fiber $b_{\dot{a}}$, with associated Boolean value $v_{\dot{a}}$. In the generic extension, this old fiber is contained in a if $v_{\dot{a}} \in G$, and is contained in $\kappa \setminus a$ otherwise. Thus the generated filter decides every subset of κ .

To prove completeness, take a name for a sequence $\langle a_i : i < \delta \rangle$, where $\delta < \kappa$. Localize and give the name a fixed default value outside the localization. For every i , form the unique positive Boolean-value fiber b_i in V . Uniqueness makes this construction uniform, so the sequence $\langle b_i : i < \delta \rangle$ belongs to V without a choice of antichain witnesses. If each evaluated $a_i \in \mathcal{U}^G$, its associated Boolean value belongs to G . Hence

$$b = \bigcap_{i < \delta} b_i \in \mathcal{U}, \quad b \subseteq \bigcap_{i < \delta} a_i.$$

Nonprincipality is immediate. Completeness also prevents κ from acquiring cardinality below κ . A short enumeration would express κ as a union of fewer than κ null singletons. $\square$

For a normal measure in a ZFC ground and forcing of size below its measurable cardinal, we also use the usual normal generated lift and the downward Lévy–Solovay theorem; see [6]. If $D \subseteq W \subseteq D[G]$ and the forcing is small relative to a normal D -measure $\mathcal{U}$ on κ , its restriction to W is the set

$$\mathcal{U}^W = \{a \in \mathcal{P}(\kappa)^W : \exists b \in \mathcal{U} (b \subseteq a)\}. \quad (2.1)$$

This definition belongs to W when $D \subseteq W$ and $W \models \text{ZF}$. Restricting the ambient lift gives normality and completeness. The argument requires this restriction construction; measurability need not be downward absolute to an arbitrary inner model.

Lemma 2.2 (Bounded presentations). *Suppose $N \models \text{ZF}$ and $M = N(Q, F)$, where $Q \subseteq \kappa^\omega$ and $F : Q \rightarrow X \in N$. Each set in M is the value of a bounded, set-sized evaluator whose inputs are the whole sets Q, F , finitely many members of Q , parameters in N , and ordinal codes. The evaluator*

is absolute between transitive outer models containing the same inputs. For every nonempty $Y \in M$, there are an ordinal η , a nonempty set $Z \in N$, and a surjection in M

$$\pi : \eta \times Z \times Q^{<\omega} \longrightarrow Y.$$

The whole sets Q, F remain fixed parameters of this surjection; they are not replaced by finitely many of their members.

Proof. We use transitive-base constructibility. For a sufficiently large ordinal θ , put

$$B_\theta = \text{TC}(\{V_\theta^N, Q, F\}), \quad J_0(B_\theta) = B_\theta, \quad J_{\xi+1}(B_\theta) = \text{Def}(J_\xi(B_\theta)),$$

and take unions at limits. The transitive base makes these levels transitive and increasing. Taking the union of these hierarchies over ground ranks and construction stages gives $N(Q, F)$. Thus each $Y \in M$ is contained in a set-sized level $J_\beta(B_\theta)$. This transitive-base construction includes the members of the adjoined sets, which predicate constructibility may omit; see [4, 7].

At a fixed level, code each element by a finite expression. A base element is a leaf. An element first produced at a successor stage is specified by a formula over the preceding set-sized level and finitely many parameters from that level. Substitute descriptions of those parameters. Transfinite induction gives a finite expression for each element, with leaves in B_θ and ordinal stage labels. At a limit stage use an earlier stage. At a successor stage combine the finitely many earlier descriptions. A description of a member of $J_\beta(B_\theta)$ uses stages bounded by β , allowing a finite increase in β if needed.

To evaluate an expression, use satisfaction for the indicated set structures $J_\xi(B_\theta)$. Set satisfaction, the definable-subset operation, and the recursion through a bounded list of levels are absolute between transitive outer models with the same base and ordinals. This proves the evaluator's absoluteness without a satisfaction predicate for the proper class M or N .

We next reduce its leaves. Apart from the whole sets Q, F , an element of $\text{TC}(\{Q, F\})$ is obtained by finitely many pairing and projection operations from individual members of Q and ground elements. Indeed, values of F and their hereditary members belong to N , and the members occurring in a sequence $q \in Q$ are built from natural numbers and ordinals below κ . The same statement holds for the ordered pairs coding the graph of F . Leaves from $\text{TC}(\{V_\theta^N\})$ are already ground elements. Consequently every expression can be converted to one using the fixed whole parameters Q, F , a finite tuple from Q , and a finite tuple from one set V_ρ^N , where ρ may be chosen uniformly from θ, X, κ .

Let $Z = (V_\rho^N)^{<\omega}$, increasing ρ so that Z is nonempty and contains all the required fixed ground data. Finite syntactic expressions, finite ordinal labels below the chosen bound, and finite type information form a wellorderable set. Index it by an ordinal η . Evaluate the codes with all parameters in Z and all finite tuples from Q . Keep an output exactly when the code is valid and its value belongs to Y . This defines a set-sized partial function in M with range Y . Fix one $y_0 \in Y$ and send every unused argument to y_0 , obtaining the total map π . Using all codes avoids choosing a definition for each $y \in Y$ or a wellordering of the ground parameter set Z . $\square$

3. THE CHOICE SYMMETRIC GROUND FROM SVC

Proposition 3.1 (A fixed small presentation with a measurable tail). *If $N \models \text{ZF} + \text{SVC} + \text{CM}$, there is a choice inner model D with*

$$D \models \text{ZFC} + \text{CM}, \quad N = D(z)$$

for some $z \in N$. There are fixed set forcings $P \in N$ and $T \in D$ and a common extension

$$E = N[H] = D[G]$$

such that N is a symmetric submodel of $D[G]$ for a set-sized symmetric system in D .

Proof. By [7, Proposition 5.4], SVC gives a choice symmetric ground D and a set generator z with $N = D(z)$. Use [7, Lemma 4.2] to fix a common-cover presentation in which N is a symmetric submodel. An arbitrary pair of forcings witnessing a common extension need not give this symmetric system; see [7, Note 4.6]. A sufficiently large pair of collapses gives the required presentation.

It remains to show that this same D has a proper class of measurables. Put

$$\theta = h^N(\text{RO}^N(P)), \quad \tau = |T|^D.$$

These are fixed ordinals. For every N -measurable $\kappa > \max\{\theta, \tau, \omega\}$, Lemma 2.1 lifts its measure to $N[H]$. Thus κ is measurable in the choice model $D[G]$. Downward Lévy–Solovay for the D -forcing T , whose D -size is $\tau < \kappa$, puts a measure on κ in D . The bounds and the ground were fixed independently of κ , so $D \models \text{CM}$. No absoluteness of the Hartogs calculation between N and D is used. $\square$

Now assume also $N \models \text{DC}$. Fix an SVC witness $S \in N$, the generator z , the protected rank, and the ground symmetric-system data. Choose an uncountable regular D -cardinal ρ above the size of T and the rank bounds for this presentation. Then choose a sufficiently high beth fixed point λ of D with $\text{cf}^D(\lambda) = \rho$. One obtains such a point as the supremum of a sufficiently high, strictly increasing continuous sequence of beth fixed points of length ρ . The forcing T preserves this cofinality. Since $D \subseteq N \subseteq D[G]$, it follows that $\text{cf}^N(\lambda) = \rho > \omega$.

Choose λ large enough that the transitive set $X = V_\lambda^N$ contains z , S , and the protected data. Then

$$N = D(X), \quad N \models \text{SVC}(X).$$

For the first equality, $D(X) \subseteq N$ and $z \in X \subseteq D(X)$; for the second, extend each given surjection on $\eta \times S$ by a default value on the rest of $\eta \times X$. Increase the bounds once more if necessary, and choose a measurable κ of D above them. The fixed forcing T is small below κ , so κ remains inaccessible in $D[G]$. All rank-bounded data below λ , including X and the old countable-condition posets introduced below, have size below κ there.

This construction preserves N and its protected ranks. We use the common choice cover to choose subsequent generics and make the size calculations.

4. THE PRIKRY AMBIENT UNIVERSE

For the next four sections fix a symmetric presentation

$$D \subseteq N \subseteq D[G], \quad D \models \text{ZFC} + \text{CM}, \quad N \models \text{ZF} + \text{DC} + \text{SVC}.$$

Write $\mathcal{S} = (\mathbb{A}, \mathcal{G}, \mathcal{F}) \in D$ for its symmetric system, and take G to be $\mathbb{A}$ -generic over D .

Choose a transitive rank $X = V_\lambda^N$ containing a generator for N over D , an SVC witness, and the structure to be protected. Choose λ sufficiently closed, with uncountable cofinality in the full choice cover; one may take a ground beth fixed point of cofinality greater than the size of the fixed small presentation. Then choose a measurable κ of D above $\lambda + \omega$ and all presentation bounds. In particular, arrange

$$N = D(X), \quad N \models \text{SVC}(X), \quad |X|^{D[G]} < \kappa, \quad \mathcal{S}, \text{RO}(\mathbb{A}) \in V_\kappa^D. \quad (4.1)$$

The forcing $\mathbb{A}$ and all subsequent choice-envelope forcing below κ have size less than κ . We can impose these bounds before choosing κ . The presentation is fixed, and a small forcing has fewer than κ bounded-rank names below a sufficiently large ground inaccessible. The small cover preserves the chosen ground cofinality of λ , so $\text{cf}^N(\lambda) > \omega$.

Let $\mathcal{U} \in D$ be a normal measure on κ , and let $\mathbb{P}_{\mathcal{U}}$ be its *old* Prikry forcing in D . Its ground wellordering belongs to N . The old measure base generates the normal lift in every small choice

cover under consideration; therefore $\mathbb{P}_{\mathcal{U}}$ is dense in the Prikry forcing computed there from that lift.

Put $\nu = \omega_1^N$ and let

$$\mathbb{C} = \{k : k \text{ is an } N\text{-countable partial function } \nu \longrightarrow X\}^N,$$

ordered by reverse inclusion. We keep the poset computed in N fixed. Every condition belongs to X , since its domain is bounded in $\nu < \lambda$ and its countable range has bounded rank below λ . We may use the dense initial-segment presentation $\bigcup_{\gamma < \nu} {}^\gamma X$ computed in N . In the full choice cover $|\mathbb{C}| < \kappa$, after increasing the bound on the small presentation if necessary.

Choose (c, K) generic for the actual product $\mathbb{P}_{\mathcal{U}} \times \mathbb{C}$ over the *full choice cover* $D[G]$, and set

$$W = N[c][K] = N[K][c].$$

The product uses these fixed old posets. Genericity over the full cover gives the displayed genericities over N and the stronger genericity needed in part (3) below. We do not assume that the old collapse remains countably closed after adjoining c .

In N , DC makes $\mathbb{C}$ countably closed. Its generic union enumerates X with domain ν . By $\text{SVC}(X)$, $N[K]$ can wellorder every old set, and the same is then true of every set of names and its evaluated range. Thus $N[K] \models \text{ZFC}$. The full cover $D[G][K]$ is a small forcing extension of D , so its normal lift of $\mathcal{U}$ restricts to the normal generated measure in $N[K]$. Product genericity makes c Prikry generic over this choice model. Consequently $W \models \text{ZFC}$.

Lemma 4.1. *In this setup:*

- (1) ${}^\omega X \cap W = {}^\omega X \cap N$;
- (2) every W -countable family of elements of N is contained in an N -wellorderable set;
- (3) $V_\kappa^{N[c]} = V_\kappa^N$.

Proof. For (1), first force with $\mathbb{C}$. The deciding-sequence argument uses DC in N and the countable closure of $\mathbb{C}$. It shows that $N[K]/N$ adds no countable sequence of elements of any fixed old set. In $N[K]$, X has size at most $\nu < \kappa$. The Prikry property for the lifted normal measure implies that the subsequent forcing adds no bounded subset of κ . Use the enumeration of X to code a countable sequence from X in W . The sequence therefore already lies in $N[K]$, and countable closure returns it to N . The same argument gives equality of the reals. We use the Prikry property and infinite-subsequence criterion as stated in [3, Section 3].

For (2), bound the ranks of the countably many ground elements by an ordinal and place them in one set $Z \in N$. Let $\langle z_n : n < \omega \rangle \in W$ be the resulting sequence. View it as the value of a $\mathbb{C}$ -name $\dot{z} \in N[c]$. In W , for each n choose $k_n \in K$ which forces

$$\dot{z}(n) = \check{z}_n$$

over $N[c]$. Such a condition exists by the truth lemma; a fixed localization can be included in all the k_n . Since $\mathbb{C} \subseteq X$, (1) gives $\langle k_n : n < \omega \rangle \in N$. Inside $N[c]$, that array, Z , and $\dot{z}$ reconstruct $\langle z_n \rangle$ by the forcing relation: z_n is the unique $z \in Z$ for which $k_n \Vdash_{\mathbb{C}} \dot{z}(n) = \check{z}$. Hence the whole sequence belongs to $N[c]$.

Take a $\mathbb{P}_{\mathcal{U}}$ -name $\dot{v} \in N$ for it and let

$$B = \{z \in Z : \exists n < \omega \exists p \in \mathbb{P}_{\mathcal{U}} (p \Vdash \dot{v}(n) = \check{z})\}.$$

This is an N -set containing every z_n . Fix the old wellordering of $\omega \times \mathbb{P}_{\mathcal{U}}$. Send each $z \in B$ to its least deciding pair. Distinct values have distinct pairs, so the map induces an N -wellordering of B without wellordering Z .

For (3), use the following generic-intersection fact, valid without Choice:

$$N_0 \subseteq E_0, \quad \mathbb{P} \in N_0, \quad h \text{ generic over } E_0 \implies N_0[h] \cap E_0 = N_0,$$

where the transitive models satisfy **ZF** and have the same ordinals, and the forcing is the same old poset in both models.

We give the proof. First, if h_0, h_1 are mutually $\mathbb{P}$ -generic over N_0 , then

$$N_0[h_0] \cap N_0[h_1] = N_0.$$

Induct on the rank of a common set x . Its members belong to N_0 by induction, so $x \subseteq V_\gamma^{N_0}$ for some ordinal γ . Choose names $\tau_0, \tau_1 \in N_0$ for x and a product condition (p_0, p_1) forcing their equality and that common value to be a subset of $V_\gamma^{N_0}$. For each $y \in V_\gamma^{N_0}$, p_0 must decide $y \in \tau_0$: otherwise two extensions giving opposite decisions, paired with one extension of p_1 deciding $y \in \tau_1$, contradict the forced equality. Hence

$$x = \{y \in V_\gamma^{N_0} : p_0 \Vdash y \in \tau_0\} \in N_0.$$

Separation and the forcing theorem suffice; no choice of deciding conditions for all y is needed.

Now let $x = \tau^h \in N_0[h] \cap E_0$, where $\tau \in N_0$ is a forcing name. The forcing theorem over E_0 gives $p \in h$ with

$$p \Vdash_{E_0, \mathbb{P}} \tau = \check{x}.$$

In a further auxiliary extension, choose h' generic over $E_0[h]$ with $p \in h'$. The filters h, h' are mutually generic over E_0 , hence over N_0 , and $\tau^{h'} = x$. The preceding intersection argument gives $x \in N_0$. This assertion concerns membership in the original ground, so the auxiliary extension does not change it. The forcing theorem for the further product gives an equivalent formulation that does not infer transitive models from consistency.

Apply this fact with $E_0 = D[G]$, $N_0 = N$, and $\mathbb{P} = \mathbb{P}_{\mathcal{U}}$. The old forcing is dense in normal Prikry forcing for the lifted measure over the choice model E_0 . Small forcing preserves the inaccessibility of κ , and the Prikry property gives

$$V_\kappa^{E_0[c]} = V_\kappa^{E_0}.$$

The absence of new bounded subsets suffices because E_0 satisfies Choice and each old rank below the inaccessible κ has size below κ . Induction on rank proves the displayed equality. If $x \in V_\kappa^{N[c]}$, the inclusion $N[c] \subseteq E_0[c]$ puts x in E_0 . Generic intersection then puts x in N . The converse inclusion is immediate. This proves (3) using the fixed old posets, without reducing to bounded symmetric names or recomputing the collapse after Prikry forcing. $\square$

5. TAGGED CAPS AND THE LABELING FORCING

All sequences, carriers, records, and diagram witnesses in this section belong to N . The word *old* means “in N ,” even when we work in an auxiliary extension. Changing finitely many initial blocks does not affect any asymptotic requirement. Fix $m_0 \geq 4$ and, for $m \geq m_0$, put

$$I_m = [2^m, 2^{m+1}), \quad w_m = 2^{\lfloor \sqrt{m} \rfloor}, \quad G_m = 2^{\lfloor m^{1/4} \rfloor}, \quad J_m = [2^m + w_m, 2^m + 2w_m).$$

For $m < m_0$ use $I_0 = \{0, 1\}$ and $I_m = [2^m, 2^{m+1})$ for $1 \leq m < m_0$, put $J_m = \{2^m\}$ and $G_m = w_m = 1$, and give every row the value 2^m there. Thus the blocks partition all of ω , including the index 0. Include $E_* = [0, m_0)$ in every cap and diagram exceptional set. The point 0 and all data in the finite initial interval $[0, 2^{m_0})$ contribute only finite carrier data. Every core is therefore a subset of its block. All separation assertions concern the eventual tail. An eventually positive sequence d_m is *super-G* if

$$\frac{d_m}{G_m^k} \longrightarrow \infty \quad \text{for every fixed } k < \omega.$$

By changing an initial segment, we may take every witness reserve below to be a positive integer sequence. The core width w_m is super- G . A *row* is an old function f with $f(m) \in J_m$ and super- G distances from the two core endpoints. Write

$$A_f = \{f(m) : m < \omega\}, \quad f \prec g \iff g(m) - f(m) \text{ is super-}G.$$

A *row chain* is a set of rows linearly ordered by $\prec$. Its pairwise order and separation requirements hold on every sufficiently late block, including exceptional blocks.

The ideal of label errors is

$$\mathcal{J} = \{a \subseteq \omega : |a \cap n| = o(\log n)\}^N. \quad (5.1)$$

If $a \in \mathcal{J}$, its occupied block set $\{m : a \cap I_m \neq \emptyset\}$ has asymptotic density zero, and $|a \cap I_m| = o(m)$. Indeed, the number of occupied blocks below m is bounded by $|a \cap 2^{m+1}| = o(m)$. This also bounds the load of the m th block. Since $m/G_m \rightarrow 0$, this load is $o(G_m)$.

Definition 5.1 (Cap records and diagrams). A *cap record* consists of an old carrier $a \subseteq \omega$, a countable row support L_a , an old density-zero set $Z_a \subseteq \omega$, and, for every $m \notin Z_a$, a labeling of the *entire* finite set $a \cap I_m$ by members of L_a . It satisfies:

- (1) $|a \cap I_m| = o(G_m)$ on all blocks, and $a \cap I_m \subseteq J_m$ whenever $m \notin Z_a$;
- (2) for each fixed $f \in L_a$, on all sufficiently late blocks outside Z_a , the canonical point $f(m)$ belongs to a and has tag f .

A *finite diagram* consists of finitely many *whole cap records* and finitely many specified canonical row points. A witness for the diagram is an old density-zero exceptional set, containing the exceptions of its cap records, and a super- G reserve d_m . Outside that exceptional set, equal points have equal tags; the convex envelopes of distinct represented tags occur in row order; the gaps between consecutive distinct-tag envelopes and the two core margins are at least d_m . A canonical point $f(m)$ has tag f .

A single tag may occupy several widely separated pieces, with no bound on its diameter. The finite-diagram requirement controls *all* tags in each selected cap block, including tags whose canonical-point requirements have not yet stabilized. Clause (2) stabilizes separately for each cap–row pair, not uniformly over all its tags. On a record’s exceptional blocks there is no tag or canonical-point requirement; only its all-block load bound remains.

Definition 5.2 (The forcing $\mathbb{R}$ and designated tops). A condition consists of a countable row chain L , a coloring $\ell : L \rightarrow X$, and a countable nonempty family of immutable cap records supported on L . Fix once and for all the distinguished empty cap with empty support, exception set E_* , and empty labelings, together with the default witness $(E_*, \langle w_m : m < \omega \rangle)$ for its one-cap diagram and for the empty diagram. Require every condition to contain this record and these same default witnesses. This family is *finitely directed modulo finite inclusion*: for any finitely many of its carriers $a_1, \dots, a_k$, it contains a carrier b with $a_i \subseteq^* b$ for every i . For every finite diagram of its caps and canonical rows, the condition includes a witness as in Definition 5.1. Extension retains the rows, their colors, all cap records, and all previously assigned diagram witnesses.

A listed cap b is a *designated top* of this condition if its support is the entire current row chain and

$$a \subseteq^* b \quad \text{for every listed cap } a, \quad A_f \subseteq^* b \quad \text{for every } f \in L.$$

These containments hold modulo a finite set of *points*, including on exceptional blocks. They are additional properties of a designated top, not requirements on every cap record.

With the fixed default witnesses, the condition with no rows and one empty cap is a largest condition. All countable families and their assigned witness functions are actual old sets included

in the condition. Conditions admit uniform codes in X . The countably many colors have ranks bounded below λ , since $\text{cf}^N(\lambda) > \omega$. The remaining countable data and the finite rank overhead of coding also stay below λ . A countable decreasing chain of conditions has its literal union as a lower bound. Every pair of rows and every finite diagram occurs at an earlier stage, so its requirements and witness persist. Finite directedness modulo finite inclusion persists for the same reason. We use DC only to show that the union of the old countable families is countable. The diagram witnesses are already assigned.

Lemma 5.3 (Dense tops and insertions). *The following extensions are dense in $\mathbb{R}$.*

- (1) *A condition can be extended by a designated top containing every previous cap and every current canonical row modulo finite sets.*
- (2) *A new row of any prescribed color can be inserted at any old cut in the countable current row chain.*
- (3) *Every old $a \in \mathcal{J}$ can be included in a cap, and every fixed integer translate of a listed cap can be included in a cap.*

Proof. *Tops.* Enumerate the caps as a_i and the rows as f_i ; when a list is finite, omit nonexistent entries in the formulas below. Let $\mathcal{D}_n$ be the diagram of the first n whole caps and first n canonical rows. Take increasing old density-zero sets U_n containing their assigned diagram exceptions and cap exceptions. Choose increasing thresholds M_n , $n \geq 1$, such that, for $m \geq M_n$,

$$\sum_{i < n} |a_i \cap I_m| + n \leq G_m/n,$$

and, off U_n , the diagram $\mathcal{D}_n$ has reserve at least $4G_m^{n+4}$. We may also require $w_m \geq 4G_m^{n+4}$ and

$$|U_n \cap t|/t < 1/n \quad (t \geq M_n).$$

Each step imposes only finitely many eventual requirements. On $M_n \leq m < M_{n+1}$ define

$$b \cap I_m = \bigcup_{i < n} (a_i \cap I_m) \cup \{f_i(m) : i < n\}.$$

Take arbitrary finite initial data, for instance empty blocks. Let E contain the initial block indices and consist thereafter of the m which belong to U_n on the n th band. If t is in that band, the exceptional indices from all earlier bands lie in U_n ; consequently

$$|E \cap t|/t \leq 1/n + O(1/t),$$

so E is old and density zero.

On a block outside E , use the labels of $\mathcal{D}_n$. This copies every point of each selected cap, including points with transient high tags. Equal points receive consistent labels. The load is at most G_m/n . Every fixed old cap and row is included on *all* sufficiently late blocks. Give b support L and exception set E . It is a valid record. A super- G reserve is given, for example, by G_m^{n+2} on the n th band.

For a finite diagram involving b and specified old caps and rows, take the band index large enough to include all those old items. Their points are then already in b , with the same tags outside E , and their union diagram is a subconfiguration of $\mathcal{D}_n$. Thus E , enlarged by a finite initial segment, and the same bandwise reserve give a *new* witness. No old witness is changed. The enlarged cap family is directed modulo finite inclusion because b dominates all its members. The same construction applies to any countable compatible family of records and diagram witnesses, even if it is not directed. Adjoining the constructed top makes it directed.

Insertion at a cut. First choose a designated top b . Enlarge the construction's exceptional set to include the exception of b and the witness exception for the diagram consisting of the whole

cap b . On a good block, partition its finitely many active tag packets according to the given cut. Choose the new point in the gap between the last left packet and the first right packet, using a core endpoint if a side is empty. If both sides are empty, use the core itself. The gap or margin has a super- G reserve. Its integer midpoint is separated from both bounding packets or endpoints by half that reserve minus one.

On exceptional blocks, use a separate slow interpolation of canonical rows. Enumerate the old rows and activate the first n only after their finite all-block order and margin bounds exceed, say, $4G_m^{n+4}$. Between the last active row on the left and first on the right, with core endpoints at empty sides, choose an integer midpoint. The activation thresholds tend to infinity, and every fixed row is eventually active. This defines an old row with the required cut and super- G margins on the exceptional blocks. On good blocks, every fixed old row eventually has its canonical point in b with the correct tag; the packet-gap construction gives the same required order and separation there. Together these estimates give the eventual row inequalities on all blocks.

Give the new row the requested color. For each new finite diagram, eventually its old cap points and canonical anchors lie in b and agree with b 's tags outside the corresponding old diagram exceptions. The chosen gap separates the new row from every packet. Finite unions of these exceptions and the midpoint reserve supply new witnesses. This extends the row chain without changing any old record. Apply the top construction again to obtain a new designated top.

Errors and translates. For $a \in \mathcal{J}$, use a record with carrier a , empty support, and exception set its occupied block set. It has no good-block points requiring tags and has load $o(G_m)$. It is compatible with every old finite diagram after adjoining this density-zero exception. Apply the compatible-family version of the top construction.

For a fixed $j \in \mathbb{Z}$ and a listed cap a , take the new carrier

$$a \cup (a + j), \quad a + j = \{k + j : k \in a, k + j \geq 0\},$$

with the same support, retaining the original tags and giving translated points their source tags. On good sufficiently late blocks, core margins and distinct-tag gaps exceed $|j|$ by a super- G amount. Thus the translated points remain in the core and inconsistent distinct-tag overlaps cannot occur. The original copy retains each fixed supported canonical point. Adjoin the old exceptional block set, its adjacent block indices, and a finite initial segment to the new exception set. This covers all boundary crossings. The new load at block m is bounded by a fixed sum of the old loads at $m - 1, m, m + 1$. It is therefore $o(G_m)$, since the ratios of these neighboring G -values are bounded. For a new finite diagram, use the old diagram including a and its other constituents; subtracting $|j|$ from its eventual gaps and margins leaves a super- G reserve. Use these reserves to assign consistent new witnesses without changing any old witness. Finally apply the compatible-family top construction. $\square$

The dense initial-segment presentation $X^{<\omega_1^N}$ of $\mathbb{C}$ projects onto $\mathbb{R}$. Fix the uniform coding of conditions by elements of X and start at the largest condition. At a successor entry, adopt the decoded condition if it is a valid extension of the current one; otherwise keep the current one. At a countable limit take the literal union. This recursion is set-defined in N and requires no global choice of lower bounds. Every extension of the current decoded condition is reached by appending its code, which proves the projection lifting property. By Lemma 4.1(1), the relevant countable X -strings and condition codes are the same in $N[c]$. The old projection therefore works there as well. Write g for the projected generic.

5.1. The finite matching lemma. For an increasing bijection between finite frozen subsets of two integer intervals, including their endpoints, say it is L -balanced if every pair of corresponding consecutive frozen gaps d, d' satisfies

$$\min(d, L) = \min(d', L).$$

Give the endpoints two distinct formal sentinel tags, one below and one above every genuine tag. These tags occur nowhere else.

Lemma 5.4 (Finite tagged interpolation). *Let A_i be finite prescribed subsets of integer intervals $[u_i, v_i]$, including the endpoints, for $i = 0, 1$. Let $h : F_0 \rightarrow F_1$ be an increasing frozen bijection, with $F_i \subseteq A_i$, matching the endpoints. Choose a common finite linear tag order T amalgamating the two tag orders over the frozen tags, and suppose:*

- (1) *the tag maps on A_i are weakly increasing, and frozen matched points have equal tags;*
- (2) *distinct-tag envelope gaps and the sentinel margins in each prescribed configuration are at least the positive integer D ;*
- (3) *h is L -balanced, where $L \geq D$.*

Take T to consist of the tags occurring on at least one side, including the sentinels. Let $n \geq |A_0| + |A_1| + |T|$ and suppose $D \geq 100(n+1)^3$. Then there are finite extensions $A'_i \supseteq A_i$ and an increasing tag-preserving bijection $h' : A'_0 \rightarrow A'_1$ extending h such that, with

$$\varepsilon = \left\lfloor \frac{D}{100(n+1)^3} \right\rfloor,$$

the map h' is ε -balanced and every distinct-tag envelope gap and sentinel margin in the final configurations is at least ε . Every tag occurs on both sides, and the total number of added final points is at most $|A_0| + |A_1|$, in particular $O(n)$. For tags initially present on only one side, the construction supplies the partner packets. Partners are not prescribed independently.

There is also an untagged version. Under the balance hypothesis, the same bound on D and the prescribed point count give an increasing ε -balanced extension retaining all points. The same linear addition bound holds, but there is no tag-separation conclusion.

Proof. We first give a two-pass untyped procedure. Suppose matched endpoint intervals have lengths either equal and less than R , or both at least R , with at most n prescribed interior points in total and $R \geq (n+1)^2$. If the lengths are equal and short, translate the two patterns into one interval, take their union, and translate back. Otherwise put $d = \lfloor R/(n+1) \rfloor$ and insert the prescribed source points, with successive positive gap lengths $g_0, \dots, g_k$, where $k+1 \leq n+1$. Assign target gaps the preliminary lengths $\min(g_j, d)$. Their sum is at most $(n+1)d \leq R$, no greater than the target interval length. At least one source gap is at least d . Otherwise the source interval length, which is their sum, would be less than R . Put all leftover target length in the first such gap. The map copies short gaps exactly. Both members of every other gap pair have length at least d .

Now insert the originally prescribed target points, working backwards in each resulting gap. Translate in equal gaps shorter than d ; in long gaps repeat the allocation with threshold $\lfloor d/(n+1) \rfloor$. The resulting map is balanced at

$$q = \left\lfloor \frac{R}{(n+1)^2} \right\rfloor.$$

Every allocation is positive, so new interior points cannot collide with old matched endpoints. At most one partner is added for each originally prescribed point. This proves the untyped procedure.

Lower the frozen balance threshold from L to D and work separately in consecutive frozen gaps. If the two endpoint tags of such a gap are equal, tag monotonicity forces every prescribed point in the gap to have that tag. Use the untyped procedure with $R = D$ and give all added points that tag. This handles every short frozen gap. Distinct endpoint tags, including a genuine tag next to a sentinel, would force the gap to have length at least D .

It remains to treat a frozen gap with distinct endpoint tags $s < t$. Both corresponding gaps have length at least D . The frozen tags determine the same list of required tags between s and t on both sides. On one side, call the closed envelope of the prescribed points of a represented tag its *packet*. The packets are ordered, with gaps of length at least D between consecutive packets. Put

$$R = \left\lfloor \frac{D}{4(n+1)} \right\rfloor.$$

For consecutive nonempty packets with last and first points $u < v$, let $h_0 \leq n$ be the number of intervening tags missing on this side. Extend the left packet's zone to $u + R$, assign the j th missing tag the zone

$$[u + 2jR, u + (2j+1)R] \quad (1 \leq j \leq h_0),$$

and start the right packet's zone at $v - R$. Every empty guard between successive zones has length at least R , because

$$(2h_0 + 3)R \leq D \leq v - u.$$

At a frozen endpoint, begin or end the corresponding endpoint-tag zone at that endpoint. Thus every zone contains its entire prescribed packet and has length at least R ; an absent tag gets a zone of length R . No packet is compressed, regardless of its diameter. Apply this on both sides with the same ordered tag list and match the two endpoints of corresponding zones and guards. The resulting skeleton extends the frozen endpoint map and is R -balanced. Sentinel zones contain only the original outer endpoint and a temporary artificial inner endpoint.

In each matched same-tag zone apply the untyped procedure to the original prescriptions and label each added point by that tag. Our bound on D gives $R \geq (n+1)^2$. Each active tag occurs on at least one side. If it is missing on the other side, the procedure adds a partner point there. Leave the guards empty. The refined skeleton is balanced at

$$\left\lfloor \frac{R}{(n+1)^2} \right\rfloor = \left\lfloor \frac{D}{4(n+1)^3} \right\rfloor \geq \varepsilon.$$

Every artificial zone endpoint lies in a free interpacket gap and has an artificial partner. Delete these endpoints in matched pairs, retaining the original frozen endpoints and all original prescriptions. A resulting consecutive gap is a sum of corresponding skeleton gaps. If a constituent pair is long at the displayed threshold, both sums are long; otherwise every constituent pair is equal, so the sums are equal. Balance therefore survives deletion. An entire empty guard remains between distinct final tags, including each sentinel and its neighboring genuine tag. Thus the final separation and margins satisfy the same bound.

After removing the artificial skeleton points, every added final point is the partner of an original prescribed point. This gives the asserted linear bound. Adjacent frozen gaps agree at their shared frozen endpoint, so their maps glue. For the untagged version, apply the untyped procedure with $R = D$ separately in each frozen gap and join the resulting maps. No zones are needed. $\square$

Remark 5.5. Translation is used only in short equal same-tag gaps at the *current* threshold. After lowering the threshold from the original L to D , requiring translation in every gap below the original threshold is unnecessary and may be incorrect. The tag-preserving conclusion does not identify two separately prescribed same-tag canonical points unless the frozen map already contains that pair.

To apply the lemma to caps, adjoin the core endpoints as sentinels. Take D_m no larger than the diagram reserve, the frozen map's balance reserve, and w_m . Each selected finite diagram uses entire prescribed cap blocks and only finitely activated canonical anchors. The number of its

active tags is bounded by its point count, so these data have $n_m = o(G_m)$ after sufficiently slow activation. If D_m is super- G , then so is

$$\left\lfloor \frac{D_m}{100(n_m + 1)^3} \right\rfloor : \quad \frac{D_m}{(n_m + 1)^3 G_m^k} \geq \frac{D_m}{G_m^{k+3}} \longrightarrow \infty$$

for each fixed k , on all sufficiently late blocks. Thus the linear load increase remains $o(G_m)$ and the polynomial reserve loss remains super- G . A growing collection of diagram requirements must be activated by an old slow diagonal. The argument does not assert that an arbitrary growing minimum of reserves is super- G . The finite construction is uniform. It uses the displayed leftmost allocations and assigns leftover length to the first long gap, so old finite input arrays produce old output arrays. On exceptional blocks use the untagged version. Any finitely many initial blocks can be handled separately when map extensions are required only eventually, while retaining the prescribed root points.

6. PAIRED FORCING AND LITERAL COMPARISON

Throughout this section, all conditions, rows, cap records, and block maps are old, meaning that they belong to N . A designated top of a condition is a listed cap whose support is the entire row chain and which contains each listed cap and each canonical row carrier modulo finite sets. The last containment holds on all sufficiently late blocks, including exceptional blocks. Cap families are finitely directed under modulo-finite inclusion.

Fix a root condition r_* with a designated top b . To fix finitely many old carriers, first strengthen the root and pad a new top to contain them literally. No existing cap record or assigned witness is changed.

Definition 6.1. A paired condition over r_* consists of conditions $r_0, r_1 \leq r_*$, an order- and color-preserving bijection $\pi : L_0 \longrightarrow L_1$ between their row chains fixing the root rows, designated tops B_0, B_1 containing b literally, and old increasing bijections

$$h_m : B_0 \cap I_m \longrightarrow B_1 \cap I_m.$$

The maps fix b pointwise. Outside the core J_m the two tops agree and the maps are the identity. With core endpoints adjoined as sentinels, the maps are L_m -balanced on every block, for an old super- G sequence $\langle L_m : m < \omega \rangle$.

There is one old density-zero set E containing the exceptional sets of the two tops such that h_m transports all top tags by π whenever $m \notin E$. Moreover, for every fixed $f \in L_0$ there is $M_f < \omega$ such that

$$h_m(f(m)) = \pi(f)(m) \quad (m \geq M_f, m \notin E). \quad (6.1)$$

The designated-top requirement puts these canonical values in their respective tops on all sufficiently late blocks. There is no canonical-match requirement on blocks in E . Each side's row chain still has super- G margins and row inequalities on all sufficiently late blocks.

An extension extends both side conditions and the row bijection, retaining all their immutable data. For some $M < \omega$, its block map extends the earlier map on its entire domain whenever $m \geq M$. The extension may use a new comparison exception set and balance reserve.

The exception in (6.1) is essential. Let $Z \subseteq [m_0, \omega)$ be an infinite old density-zero set, let $p(m)$ be the midpoint of J_m for $m \geq m_0$, and set $p(m) = 2^m$ on the initial blocks. Give $b = \{p(m) - 1, p(m) + 1 : m \in Z\}$ empty support and exception set $Z \cup E_*$. This is a valid root cap. Insert the row p on both sides with different colors. Matching its canonical point on every sufficiently late block, while fixing b , would force its partner to equal $p(m)$ on all sufficiently late $m \in Z$. The differently colored partner must be distinct from the native row p on that side, contradicting their super- G separation. Requiring matches only off E avoids this obstruction.

Lemma 6.2 (Paired-cap lifting). *The paired forcing is countably closed in N . Both coordinate projections have the lifting property for arbitrary one-sided extensions. Any two extensions of r_* can be strengthened to the two coordinates of a paired condition.*

Proof. We first specify a schedule and the treatment of canonical anchors, then construct the extensions and check their records.

Old diagonal scheduling. Consider countably many old density-zero sets, super- G reserves, eventual equalities and containments, and $o(G_m)$ load functions. At stage $n \geq 1$, select finitely many requirements. Increase the selected families so that every fixed requirement is eventually selected. Requirements concerning a particular old top or map selected at stage n may also be included. Let U_n be the union of all exceptions selected up to stage n . Choose strictly increasing M_n such that, for $m \geq M_n$, all selected eventual assertions hold, the selected total point load and number of active anchors are at most G_m/n , and every selected reserve and w_m are at least CG_m^{n+10} . Here C is a fixed sufficiently large constant for Lemma 5.4. Also require

$$|U_n \cap t|/t < 1/n \quad (t \geq M_n).$$

Each choice is possible because it involves only finitely many asymptotic requirements. Set $s(m) = n$ on $[M_n, M_{n+1})$ and

$$E = \{m \geq M_1 : m \in U_{s(m)}\},$$

with finitely many initial blocks added if necessary. If t is in the n th band, all earlier exceptional blocks after M_1 belong to U_n , so $|E \cap t|/t \leq 1/n + M_1/t$. Thus E is old and density zero; each fixed selected exception is contained in E modulo finite.

The bandwise load is $o(G_m)$. The point-and-tag count is bounded by a fixed multiple of the total whole-cap load, the active anchors on both sides, and the sentinel count. The G_m/n schedule therefore gives $\nu_m \leq G_m$ eventually. For this count and input reserve $D_m \geq CG_m^{s(m)+10}$, enlarge the thresholds to accommodate rounding and obtain

$$\left\lfloor \frac{D_m}{100(\nu_m + 1)^3} \right\rfloor \geq G_m^{s(m)+5}.$$

This is super- G and absorbs the polynomial loss in Lemma 5.4. Only finite minima are used at each stage. The exception set is a diagonal pseudounion, not a countable union.

The row order and canonical anchors. For one-sided lifting use the stronger source row order as an abstract order T , identifying old target rows by the existing row bijection. For initial two-sided amalgamation, amalgamate the native orders over the root by comparing their cuts in the root order. Within a single cut, put the foreign left rows before the foreign right rows and preserve each native order. Opposite non-root rows remain distinct abstract labels even when their native old functions coincide. Retain their inherited colors.

Every label is native on at least one side. Labels native on both sides are already identified. Choose increasing finite H_n exhausting T . Include their native canonical-anchor diagrams, and the eventual containments and matches of identified anchors, in the schedule. On a good block in the n th band prescribe the *entire* input top blocks and the native anchors from H_n . Previously identified anchors are frozen matched pairs after their stabilization bounds and off their selected exceptions. Apply the tagged finite lemma. Define a missing canonical value to be the image or inverse image of its native anchor, with the intended tag.

On an exceptional block, retain each side's native anchors from H_n and interpolate the missing labels independently in the order T , using core endpoints as sentinels. Finite native chains have all-block super- G gaps and margins. Divide each gap among its finitely many missing labels; the schedule makes the resulting integer gaps and margins at least G_m^{n+5} . Then include *all these canonical values on both sides* in the prescribed point sets for the untyped finite interpolation.

This puts the canonical points in their tops on exceptional blocks too, although their images need not be their canonical partners there. The additional cost is linear in $|H_n|$.

Before a missing row becomes active, assign it any fixed interior value on blocks $m \geq m_0$, and the required value 2^m on the initial blocks. Each fixed row, and each fixed pair, is permanently active eventually. The two block constructions therefore give old row functions with all-block super- G margins and pairwise inequalities in the order T . A high-index tag in an input block receives its partner *packet* immediately because the whole block is prescribed. Its canonical anchor is activated later.

One-sided lifting. Given a paired condition with tops B_0, B_1 and map h , strengthen its first side arbitrarily and extend it to have a designated top A_0 . On each sufficiently late block prescribe the entire A_0 block, the entire B_0 block if not already included, the entire B_1 block, and the active native anchors. Freeze h and the core sentinels. Outside the core add the union of the prescribed points on both sides and use the identity.

The source diagrams involving A_0, B_0 and active native anchors, and the target diagrams involving B_1 and its native anchors, supply old exceptions and reserves. Include the old paired exception and the selected canonical stabilization bounds. The old map's balance reserve, these diagram reserves, the native-row gaps, and w_m give the finite reserve minimum for the schedule. The prescribed loads are $o(G_m)$, so the constructions for good and exceptional blocks apply. The new maps eventually extend the entire old maps, fix b , and have all-block super- G balance. They define the extended target chain and color-preserving row bijection without changing any native row.

Initial two-sided amalgamation. Extend the two given conditions to have tops A_0, A_1 containing b literally. A previous top contains b modulo finite, so padding a new top by the finite remainder suffices. Amalgamate the abstract row orders as above, freeze the identity on b and the sentinels, and apply the same construction to the two entire top blocks. Include the finite diagrams comparing b and the native input tops. On exceptional blocks interpolate the active anchors separately, insert all their values into the untyped point sets, and retain the identity at b .

In both constructions use the identity on the union of all prescribed points on finitely many initial blocks if necessary. Non-root map extension is only eventual; the root remains fixed throughout. Sentinels are auxiliary. For each matched pair of core endpoints, retain both copies if either was a prescribed carrier point; otherwise remove both copies from the new tops. This preserves the bijection. The margin requirement keeps prescribed good-block points away from the endpoints. Measure balance with the sentinels adjoined. The numerical reserve requirements are asymptotic.

Records and immutable witnesses. Let C_i be the new top carriers, with full new row supports, the good-block tags just constructed, and exception set E . Their loads are $o(G_m)$ and their good-block distinct-tag and margin reserves are super- G . Every fixed canonical row belongs to its top on all sufficiently late blocks, including exceptional ones, and has its own tag on good late blocks. Thus the new records are valid designated tops.

Keep every native old record, labeling, exception set, and assigned finite-diagram witness literally. Each native cap a lies modulo finite in its input top and hence in C_i . A transported record on the other side, if included, has the exact carrier

$$a' = \bigcup_m h_m[a \cap C_i \cap I_m]$$

(or the corresponding inverse image), and transported support. Its exceptions include E , those of a , the old diagram exception comparing a to its input top, and a finite initial containment error. Outside that finite union, its labels agree with the top's labels, and each fixed supported row

eventually has its canonical point. Its load is at most that of a . Do not add canonical corrections on exceptional blocks.

For a new finite diagram, unite the relevant finitely many exceptions and enlarge its finite initial segment. Beyond it, all prescribed points and correctly tagged canonical anchors lie in C_i , and their tag envelopes are sub-envelopes of the new top's packets. The top reserve witnesses the diagram. Assign the new witnesses without changing any old one. Each new top dominates all native and transported records modulo finite, so the cap family is finitely directed. The constructions therefore give side extensions in the stated forcing.

Countable closure. Let $t_0 \geq t_1 \geq \dots$ be an old decreasing sequence of paired conditions. Take the unions of the side conditions, row bijections, and assigned witnesses. Each finite diagram occurred at a stage, so the side unions are conditions. Enumerate their caps, rows, finite diagrams, earlier maps, and exceptions.

At stage n choose a later $t_{k(n)}$ containing the first n enumerated items, with $k(n)$ increasing, and use its *whole* designated tops and map. Include in the schedule their eventual containment of the first n caps and canonical rows on all blocks; their agreement with the first n earlier maps; the first n canonical matches off their comparison exception; the diagrams of the *selected tops* together with the first n old caps and canonical anchors; their top and diagram exceptions; their map and diagram reserves; and their two top loads. Agreement comes from the extension order. Choose M_n by the schedule above.

On the n th band, copy the selected whole top blocks and map. This eventually extends every earlier map and contains every old cap and canonical row. Each fixed canonical match holds eventually off the diagonal E . The root and the identity off the core persist. Loads are $o(G_m)$, and all-block balance reserves are super- G . Copy the selected top labels. For every finite diagram involving a new top and specified old records, the corresponding diagrams with the selected stage top have been scheduled from some stage onwards. Their copied blocks and reserves witness the new-top diagram off E and a finite initial segment. Add only the two new tops and their new witnesses. They restore directedness and give a paired lower bound. Each block copies a single selected top, rather than an infinite pointwise union of cap loads. $\square$

6.1. The objects retained in the inner model. Let $g \subseteq \mathbb{R}$ be generic and let ℓ_g be its union coloring. Define

$$\mathcal{I}_g = \{a \in \mathcal{P}(\omega)^N : a \subseteq^* b \text{ for some cap } b \text{ appearing in } g\}.$$

By finite directedness and Lemma 5.3, this ideal is hereditary under old subsets, is closed under finite unions and integer translates, and contains $\mathcal{J}$. Let $Q(c, g)$ consist of the increasing enumerations of finite modifications, using arbitrary ordinals below κ , of $c[a]$ for infinite old $a \in \mathcal{I}_g$.

Fix $x_0 \in X$. For $q \in Q(c, g)$ set

$$\iota_c(q) = \{n : c(n) \in \text{ran}(q)\}.$$

If a generic row f and an integer j satisfy $\iota_c(q) \triangle (A_f + j) \in \mathcal{J}$, put $F(c, g)(q) = \ell_g(f)$. If there is no such row and translate, put $F(c, g)(q) = x_0$. The index carrier differs finitely from an old carrier, so is old. Distinct rows have disjoint translated $\mathcal{J}$ -clouds: their super- G separation gives distinct canonical points on every sufficiently late block. One point on each such block is not a $\mathcal{J}$ -error, so the definition is unambiguous. Lemma 5.3(2) makes F onto X .

A carrier already supported by a cap cannot acquire the cloud of a later row outside its recorded support. The finite diagram of the whole cap and that row separates the canonical point from all cap points on a density-one set of blocks. The reserve eventually exceeds any fixed translating integer. The difference from the translated row therefore has a point on a density-one set of blocks and is not in $\mathcal{J}$. Hence localized label decisions remain unchanged by later extensions.

Every member of Q is Prikry generic over N . To verify this for the ambient generic, work in the collapse-first choice cover $N[K]$, which contains every old carrier. There c is generic for the lifted measure, so Mathias's criterion applies, together with its infinite-subsequence consequence and finite-change invariance. The old measure base gives genericity for $\mathbb{P}_U$ and hence over N ; see [3, Section 3]. Only the evaluations of the fixed N -names Q, F are retained. The row order, tags, cap records, and master sequence are not separately adjoined.

Lemma 6.3 (Literal common-model comparison). *Let $r_0, r_1 \leq r_*$ and let $p_0, p_1 \in \mathbb{P}_U$ be compatible. Fix finitely many supported Q -parameters whose old carriers lie literally in the root top b , with their finite ordinal decorations. In an auxiliary extension there are genuine $\mathbb{P}_U \times \mathbb{R}$ -generics $(c_0, g_0), (c_1, g_1)$ over N , meeting the respective conditions, such that*

$$Q(c_0, g_0) = Q(c_1, g_1), \quad F(c_0, g_0) = F(c_1, g_1),$$

and the supported parameters are literally equal. In particular the two models $N(Q, F)$ are literally the same inner model there.

Proof. Work in the full choice cover of Section 4. Lemma 6.2 gives a paired condition whose coordinates strengthen r_0, r_1 . Force below it with the *old* paired forcing. Its lifting assertions involve fixed old condition sets, so they remain valid in this cover. Each coordinate is therefore a projection below that condition. Given a dense open subset of the old $\mathbb{R}$ in the cover, strengthen the selected coordinate into it and lift that extension. The resulting marginal filters g_i are genuinely $\mathbb{R}$ -generic over the cover and over N .

The old paired forcing and its data have size below κ in the cover by (4.1). A further forcing of size below κ makes its generic filter countable. Enumerate that filter and use its directedness to take a cofinal decreasing sequence. The sequence need not belong to N . Its maps eventually extend each fixed earlier map. Copy one whole map per block beyond selected agreement thresholds to obtain an increasing block-preserving partial bijection h of ω , fixing b , which eventually agrees with each old top map in the filter.

Old localization of the carrier action. For $a \in \mathcal{I}_{g_0}$ choose an old paired condition t in the filter whose designated top contains a modulo finite. Directedness of the filter and the top property give such a t . Then

$$a' = h^t[a \cap B_0^t] \in N, \quad a' \subseteq B_1^t, \quad h[a \cap \text{dom}(h)] = {}^* a',$$

and $a \setminus \text{dom}(h)$ is finite. Thus a' is an old allowed carrier on the other side. Apply the same argument to the inverse map. Although the real h may be new, its action on each specified carrier is determined modulo finite by one old map.

For an old $d \in \mathcal{J}$ its localized image is again in $\mathcal{J}$. If $n \in I_m$, then

$$|h^t[d \cap B_0^t] \cap n| \leq |d \cap 2^{m+1}| = o(m) = o(\log n).$$

For a fixed generic row f and integer j , choose an old paired condition t containing f whose first top contains both A_f and $A_f + j$ modulo finite. Translate absorption, dense tops, and lifting make this a dense requirement. Write $q = \pi^t(f)$. Once the balance reserve exceeds $|j|$,

$$h_m^t(f(m) + j) = h_m^t(f(m)) + j.$$

Indeed, each consecutive domain-top gap between these points is shorter than the reserve and has its length copied exactly. The core margins keep both endpoints in the same block eventually. The fixed row's canonical match holds eventually off the old E_t . Its image and $A_q + j$ therefore differ, modulo finite, by at most two points in each block in E_t . Up to block m there are at most $2|E_t \cap (m+1)| + O(1) = o(m)$ such points. This is an old $\mathcal{J}$ -error. Combined with the error-image estimate, it shows that the map transports the entire translated cloud and preserves its color. For a particular carrier in that cloud, choose one paired condition whose top covers that

carrier, the canonical translate, and the specified error set. Directedness and dense tops meet these finitely many requirements, so the same old map handles the whole comparison. Apply the argument to the inverse map as well. Only the error for the fixed row is used, not a countable union of errors. Whole exceptional cap blocks are transported literally, not discarded.

Genuine Prikry and product genericity. Choose a common strengthening $p_* = (s, A)$ of p_0, p_1 , including a common reservoir as well as a common stem. Normalize finitely many initial blocks of h to the identity on the whole block, including all integers below $|s|$. The resulting map still eventually agrees with each old map and fixes b .

Amalgamate two ordered copies of ω , identifying n on the first with $h(n)$ on the second. Between identified points, interleave the finite remainders in any fixed order preserving the order of each side; do likewise in an empty block. Since h preserves finite blocks, the resulting order has type ω . Its increasing cofinal embeddings $e_i : \omega \rightarrow \omega$ satisfy

$$e_0(n) = e_1(h(n)) \quad (n \in \text{dom}(h)), \quad e_0(k) = e_1(k) = k \quad (k < |s|).$$

The only cross-side identifications are the specified h -pairs. Thus $e_0(n) = e_1(m)$ if and only if $n \in \text{dom}(h)$ and $h(n) = m$.

All preparation precedes the Prikry forcing and has total forcing size below κ over the original choice ground. The old normal measure generates its normal lift to this prepared choice universe, with the old measure-one sets as a base. Choose a master sequence z Prikry generic over that entire universe through (s, A) , and put $c_i = z \circ e_i$. The maps e_i are in the prepared universe, so Mathias's criterion makes each c_i Prikry generic there. The dense old presentation gives genericity for the old $\mathbb{P}_U$ as well. Each c_i contains s and has all subsequent entries in A , so meets p_i .

Thus c_i is $\mathbb{P}_U$ -generic over $N[g_i]$. Since g_i is $\mathbb{R}$ -generic over N and these are fixed old posets, the product lemma makes (c_i, g_i) $\mathbb{P}_U \times \mathbb{R}$ -generic over N . No Prikry generic was chosen before the comparison preparation.

Literal retained equality. For corresponding carriers, $c_0[a] =^* c_1[a']$. Closure under arbitrary finite ordinal modifications in the fixed definitions gives equality of the sets Q . For any q in this common set, the cross-side identification gives

$$\iota_{c_1}(q) =^* h[\iota_{c_0}(q) \cap \text{dom}(h)].$$

Indeed, all but finitely many values of q lie on its allowed carrier in the first master, and that carrier is almost contained in $\text{dom}(h)$; the only common master values are the identified pairs. Both index carriers are old. Applying the cloud correspondence in both directions gives equality of the two functions F on every such q , including their default-label regions.

Because h fixes b , the two subsequences on every protected carrier $a \subseteq b$ agree literally. The same finite ordinal decorations therefore give the same actual parameters. These are equal evaluations of the fixed N -names Q, F in one ambient universe, so their generated models $N(Q, F)$ are literally equal. The carrier and error ideals and the definition of Q are never recomputed using new auxiliary reals. $\square$

7. RANK PRESERVATION AND DEPENDENT CHOICE

Put

$$M = N(Q, F), \quad Q = Q(c, g), \quad F = F(c, g),$$

using the literal evaluations of the fixed names from the preceding section. Thus $N \subseteq M \subseteq W$, and $M \models \text{ZF}$ by transitive-base constructibility.

Lemma 7.1 (Eliminating the labeling coordinate). *If $Y \in N$, $a \subseteq Y$, and $a \in M$, then $a \in N[c]$.*

Proof. By Lemma 2.2, fix a bounded set-sized evaluator for a . Its inputs are the whole sets Q, F , finitely many actual members $q_0, \dots, q_{d-1}$ of Q , fixed ground parameters, and fixed ordinal codes. Each q_i is the increasing enumeration of a finite ordinal modification of $c[a_i]$, for an old carrier a_i supported by a generic cap. Choose a product condition (p_*, r_*) in the actual generic that verifies these descriptions and the presentation.

By dense tops and finite padding, strengthen this condition *within the actual generic* so that its designated top b contains every a_i literally. First cover the finitely many carriers modulo finite, then add their remaining finite points to the new top. Rename the strengthened condition (p_*, r_*) . Retain the old records and pad only the new top. Keep all finite ordinal decorations as fixed parameters.

Let $\dot{a}$ be the name obtained from this evaluator and the fixed names for its inputs, with a default value outside the chosen localization. For $y \in Y$, define in N

$$E_y^+ = \{p \leq p_* : \exists r \leq r_* (p, r) \Vdash \check{y} \in \dot{a}\},$$

$$E_y^- = \{p \leq p_* : \exists r \leq r_* (p, r) \Vdash \check{y} \notin \dot{a}\}.$$

Their union is dense below p_* . To see this, strengthen (p, r_*) to decide the indicated statement. No member of E_y^+ is compatible in the old Prikry order with a member of E_y^- . Otherwise Lemma 6.3, applied to their labeling witnesses, would give two genuine product-generic realizations with the same literal N, Q, F , the same supported q_i , and the same ground and ordinal parameters. The evaluator in Lemma 2.2 uses satisfaction only for these same set-sized constructibility levels. It therefore has the same value in both realizations, contradicting the opposite membership decisions.

Form the old Prikry name

$$\dot{b} = \{(\check{y}, p) : y \in Y, p \in E_y^+\}.$$

Its actual value is a . For every y , the truth lemma supplies a deciding product condition in the actual generic below the localization. Its Prikry coordinate belongs to the appropriate $E_y^\pm$. The incompatibility just proved excludes the opposite decision from the Prikry filter. Thus $a = \dot{b}^c \in N[c]$. This argument selects no family of maximal antichains. $\square$

Corollary 7.2. $\mathcal{P}(X)^M = \mathcal{P}(X)^N$ and $V_\lambda^M = X$.

Proof. Every M -subset of X belongs to $N[c]$ by Lemma 7.1. Its rank is at most $\lambda + 1 < \kappa$, so it belongs to N by Lemma 4.1(3). This proves the power-set equality. If there were a new member of V_λ^M , choose one of least rank. All its members would belong to $V_\lambda^N = X$. It would then be a new subset of X , contrary to the power-set equality. $\square$

Lemma 7.3 (External countable domination). *For every $\langle q_n : n < \omega \rangle \in W$ with $q_n \in Q$, there is an increasing $q^* \in Q$ such that*

$$\text{ran}(q_n) \subseteq^* \text{ran}(q^*) \quad (n < \omega).$$

Proof. In W , choose old carriers witnessing the q_n and conditions of g containing their supporting caps. Since g is a filter and $W \models \text{ZFC}$, choose a decreasing sequence $\langle r_n : n < \omega \rangle$ in g , with each term below the witnesses chosen so far. The labeling conditions have codes in X , as verified in Definition 5.2. By Lemma 4.1(1), the whole code array belongs to N . Decode it there to obtain the old decreasing sequence of conditions of $\mathbb{R}$.

For any old decreasing sequence $\langle r_n \rangle$, the set

$$D_{\vec{r}} = \{r \in \mathbb{R} : (\exists n \ r \perp r_n) \text{ or } (\forall n \ r \leq r_n)\}$$

is old and dense. Starting with any condition, recursively take common refinements with the successive r_n as long as possible. If a step fails, the current condition is incompatible with that

r_n . Otherwise DC in N gives a descending sequence, and countable closure gives a lower bound below all its terms. Since all r_n belong to g , a member of $g \cap D_{\bar{r}}$ must be below every r_n .

The dense-top lemma gives a further member of g with a top b containing all the carrier witnesses modulo finite. Let q^* be the increasing enumeration of $c[b]$. It belongs to Q , and its range contains each $\text{ran}(q_n)$ modulo the finite carrier errors and finite ordinal decorations. Genericity is used only after the descending sequence has been shown to be old. $\square$

Lemma 7.4. $M \models \text{DC}$.

Proof. We first prove that every W -countable family from Q has an M -wellorderable cover contained in Q . For $\langle q_n : n < \omega \rangle$ choose q^* as in Lemma 7.3, and put

$$u_n = \{k < \omega : q^*(k) \in \text{ran}(q_n)\}.$$

The array $\langle u_n : n < \omega \rangle$ is coded by a single real of W , so it belongs to N by Lemma 4.1(1). Each u_n is infinite. Let v_n be the increasing enumeration of $q^* \restriction u_n$. Equivalently, restrict q^* to u_n and reindex by the increasing enumeration of u_n . The sequence $\langle v_n \rangle$ is definable in M from q^* and the old mask array, and therefore belongs to M .

Take the increasing enumerations of all finite modifications of the ranges of the v_n , allowing arbitrary finite ordinal additions and deletions below κ . Intersect this family with Q and call the resulting set C . It contains every q_n , since only finitely many values of q_n lie outside $\text{ran}(q^*)$. The set C is an image of a wellorderable parameter set consisting of $n < \omega$, finitely many ordinals below κ , and finite operation codes. Least preimages give a wellordering of C in M . This does not imply that the original external sequence $\langle q_n \rangle$ belongs to M .

Let $R \in M$ be a serial relation on a nonempty set $Y \in M$, and prescribe $y_0 \in Y$. Choose a presentation in M

$$\pi : \eta \times Z \times Q^{<\omega} \rightarrow Y, \quad Z \in N,$$

as in Lemma 2.2. Seriality is absolute to W because it quantifies only over the fixed sets Y, R . In $W \models \text{ZFC}$, choose an R -chain starting at y_0 and an argument of π for each entry. By Lemma 4.1(2), an N -wellorderable set $B \subseteq Z$ covers the resulting countably many ground coordinates. Intersect the cover with Z if necessary. Combine the finite Q -tuples into one countable family and apply the preceding paragraph to obtain an M -wellorderable cover $C \subseteq Q$. Then

$$Y_0 = \pi[\eta \times B \times C^{<\omega}]$$

belongs to M , is wellorderable there by least preimages, and contains the chosen external chain.

Inside M , form the tree T of finite R -chains in Y_0 starting at y_0 . Write $t \triangleleft s$ when t properly extends s . A rank function for this relation must assign smaller values to longer chains. The tree is wellorderable. If the extension relation were not well founded, a nonempty subset of T would have no terminal member. Using the fixed wellordering, recursively choose the least proper extension within that subset. This gives an infinite branch in M without any choice principle.

Thus, if there were no internal branch, the extension relation would be well founded and would have an ordinal rank function by well-founded recursion in ZF. The same function would remain a rank function in W , contradicting the chosen external branch. Hence an R -chain starting at y_0 exists in M , proving DC. $\square$

Proposition 7.5 (The sequence seed). *The model M satisfies $\text{ZF} + \text{DC} + \text{CM} + \text{SVC} + \text{KW}_1^{*,\omega}$. There are an ordinal μ and a set $A \subseteq \mu^\omega$ with $M = D(A)$. Moreover, D is a symmetric ground of M .*

Proof. Because $N \models \text{SVC}(X)$ and $F : Q \rightarrow X$, the presentation in Lemma 2.2 gives, for every nonempty $Y \in M$, a surjection

$$\eta \times Q^{<\omega} \rightarrow Y$$

after increasing the ordinal factor. To obtain it, cover the ground parameter set Z by $\xi \times X$, replace the X -coordinate by $F(q)$, and combine the two ordinal factors and the finite Q -coordinates. Thus $Q^{<\omega}$ is an SVC witness in M .

For each such Y , choose an ordinal μ large enough to contain η, κ and the finite coding symbols. To inject an ordinal coordinate and a finite tuple of members of $Q \subseteq \kappa^\omega$ into μ^ω , code the tuple length and ordinal in the initial positions, then interleave the finitely many sequences. Transfer the displayed surjection to the image of this injection. Extend it to all of μ^ω by assigning one fixed element of Y outside that image. This proves $\text{KW}_1^{*,\omega}$ with the power μ^ω computed internally in M .

To obtain one sequence seed for the whole model, define

$$R(p, q) \iff F(p) \in F(q) \quad (p, q \in Q).$$

Transitivity of X and surjectivity of F give

$$F(p) = F(q) \iff \forall t \in Q [R(t, p) \leftrightarrow R(t, q)].$$

Indeed, distinct sets in the transitive set X have a distinguishing member in X , and that member has an F -preimage. Equality of predecessor sets is therefore an equivalence relation. Its quotient relation is well founded and extensional, and the map $[q] \mapsto F(q)$ is an isomorphism to membership on X .

Use different finite tags for domain and relation codes, and interleave pairs of sequences. The resulting set $A \subseteq \mu^\omega$ codes the whole domain Q and binary relation R , for one sufficiently large ordinal μ . Both sets can be decoded in $D(A)$. Their quotient relation remains well founded and extensional. Its unique Mostowski collapse therefore recovers the same literal X and the map F . Hence $D(A)$ contains $N = D(X)$ and then $M = N(Q, F)$. Conversely, $A \in M$ and $D \subseteq M$, so minimality of transitive-base constructibility gives $D(A) \subseteq M$. Therefore $M = D(A)$.

The full choice cover $D[G][c][K]$ is a set-forcing extension of D . Above the size of its forcing, every normal measure in D has its normal generated lift in this cover. For each such ground measure $\mathcal{U}$ on δ , its restriction to M is the set

$$\{a \in \mathcal{P}(\delta)^M : \exists b \in \mathcal{U} (b \subseteq a)\},$$

which belongs to M by Separation. It is a nonprincipal δ -complete ultrafilter. Every M -sequence of fewer than δ members is also a sequence in the full cover, where the generated lift is complete. Its intersection belongs to M and satisfies the same ground-witness definition. Hence $M \models \text{CM}$. Only a proper-class tail is needed, although the selected κ was singularized.

Finally, $D \subseteq M \subseteq D[G][c][K]$ and $M \models \text{SVC}$. By the intermediate symmetric-model theorem, in the form of [7, Corollary 5.10], D is a symmetric ground of M . This uses the intermediate-model theorem; the notation $D(A)$ alone would not give the conclusion. $\square$

Proof of Theorem 1.5. Start with the $\text{ZF} + \text{DC} + \text{SVC} + \text{CM}$ model N in the local statement. By Proposition 3.1, this same N has a symmetric-ground presentation over a choice ground $D \models \text{ZFC} + \text{CM}$. Choose the protected rank $X = V_\lambda^N$ and the measurable κ as in (4.1), with λ above the required rank. The construction gives $N \subseteq M$. Corollary 7.2 proves $\mathcal{P}(X)^M = \mathcal{P}(X)^N$ and $V_\lambda^M = X$, while Lemma 7.4 and Proposition 7.5 give the required theory and the presentation $M = D(A)$ with D a symmetric ground. This proves the local theorem. Sections 8–9 remove SVC and Foundation from the starting assumptions of the structural conservativity theorem. $\square$

8. THE CORE-MODEL PREPARATION REMOVES SVC

8.1. The standard real-kukri and core alternatives. All mouse constructions in this section take place in a choice background. The real is an amenable bottom predicate in the usual relative J -hierarchy, which starts from the empty set. It is not adjoined with its transitive closure as a

separate initial base. Every measure critical point is above ω , and every comparison map fixes the bottom real. We use the standard Jensen–Zeman kukri for the positive alternative and the classical Mitchell true core for the negative alternative. We do not identify their fine indexing or their resulting models.

An r -kukri is the least sound active r -mouse whose top normal measure has critical point κ and whose lower measurable cardinals are unbounded below κ . It projects to ω and is countable in the background. The standard least-sound extraction is the rank-one instance of [9, Definition 1.4 and Proposition 1.6]. The corresponding top iteration is described in [10, Section 2, Definition 2.1 and Facts (i)–(v)]. Only lower measurables will be used as witnesses in the resulting class model.

Lemma 8.1 (Noncreation of the least real-kukri). *Set forcing over a choice background does not create an r -kukri for a ground real r .*

Proof. Use the standard initial-segment comparison for sound mice projecting to their common real base, and fixed-countable-code absoluteness of linear iterability. These are the real-bottom forms of the ordinary comparison and absoluteness arguments; see [8, Lemmas 3.32 and 3.39] and [11, Chapters 4–5]. All structures in each comparison use one chosen standard category.

Suppose a condition p forces that the least such mouse exists, and let $\dot{P}$ name its whole transitive fine structure, including its predicates. In the product forcing below (p, p) the two interpretations P_0, P_1 are sound iterable mice over the same real. Comparison makes them equal or makes one a proper fine initial segment of the other. If $P_0 \triangleleft P_1$, the factor containing the whole P_1 also contains the whole fine initial segment P_0 , including its active record, obtained from P_1 and the ordinal height of P_0 . It is countable in that factor. Choose a real code there. Its iterability in the product returns to the factor by fixed-code Π_2^1 absoluteness. Soundness, projection and the top-limit property are absolute tests on the same set structure. This contradicts the factor's minimality of P_1 . The reverse proper-segment alternative is excluded in the other factor. No assertion of minimality in the product is required.

Thus (p, p) forces the two copies of $\dot{P}$ equal. The product-name lemma gives a ground set P with $p \Vdash \dot{P} = \check{P}$. This applies to the whole fine structure, not just to a real code. Its sound first-projectum- ω presentation gives a canonical Skolem surjection from ω , using its finite standard parameter. Satisfaction for the whole set structure belongs to the ground, so this surjection belongs there too. Hence P is already countable in the ground. Choose a ground real code and use fixed-code absoluteness once more to obtain its iterability there. This contradicts the supposed nonexistence. Conversely an existing countable iterable eligible mouse persists, so the existence test is invariant under set forcing. $\square$

Lemma 8.2 (The fixed-real classical core). *Let $B \models \text{ZFC}$ and let $r \in B$ be a real. If B has no r -kukri, the classical true core $K(r)^B$ is a uniformly r -definable inner model of ZFC containing r . It is unchanged by set forcing over B . In this same core, every ambient singular cardinal that is regular in the core is measurable.*

Proof. We apply the classical core construction, generic-invariance theorem and full-covering theorem over the fixed real. The construction and covering framework are in [8, Sections 1, 3 and 4]; the singular-cardinal consequence is its Theorem 2.5. The classical true-core generic-invariance application is also recorded in [5, Section 4, Lemma 4.2] and [13, proof of Theorem 3.11]. This theorem concerns the true core, not clausewise absoluteness of its preliminary countably complete construction.

The standard obstruction is excluded by the absence of the r -kukri. In the usual ordering of mice, an order-one upper measure supplies an eligible order-zero active mouse with cofinally many lower measurables; taking its least sound core gives the kukri. The stronger μ -measurable obstruction in the full-covering statement supplies the same configuration. Here the source

convention is a P -internal elementary embedding $i : P \rightarrow M$ with M an inner target of P . If its critical point is $\kappa > \omega$ and its whole derived normal measure U belongs to M , then

$$\mathcal{P}(\kappa)^P = U \cup \{\kappa \setminus a : a \in U\} \subseteq M.$$

Thus P and M have the same κ -subsets; normality and completeness restrict to M , which regards κ as measurable. Hence

$$\{\xi < \kappa : P \models \text{“}\xi \text{ is measurable”}\} \in U.$$

The standard relative mouse extraction and least-sound extraction therefore supply an r -kukri. Here i fixes the real in P . This argument excludes the relative obstruction; it makes no claim about unrelativized mice that omit r .

For completeness, the fixed-real application keeps the real as one amenable predicate throughout the rudimentary hierarchy, fine codes and comparisons. All hulls used for covering or for the true-core construction contain ω and r . Their collapses fix the real, and all comparison critical points lie above it. Including this fixed countable base leaves the covering cardinal bounds and thick-hull argument unchanged. The failure configuration is consequently a mouse over that same real. This is the ordinary real-bottom instance of the classical theorems, not a comparison of varying arbitrary oracles. Lemma 8.1 keeps the obstruction absent in the required set-forcing extensions. The true-core theorem gives literal equality of the ground and extension cores. $\square$

Lemma 8.3 (Measurable capture by singularization). *In the negative alternative of Lemma 8.2, every measurable cardinal of the choice background is measurable in its fixed core.*

Proof. Let κ be measurable in the choice background C , and choose a normal measure there. Ordinary Prikry forcing preserves cardinals and makes $\text{cf}(\kappa) = \omega$ [3]. Before forcing, κ was regular in the inner model $K = K(r)^C$. In the Prikry extension $C[h]$, generic invariance gives $K(r)^{C[h]} = K$. Thus κ is an ambient singular cardinal regular in this same core. Apply the full-covering consequence in Lemma 8.2. Its witnessing measure belongs to the unchanged ground class K .

This is an internal forcing proof in ZFC of an assertion about the unchanged core. It assumes no actual generic filter in the original ambient universe. It uses neither a trace of the original measure on K nor an extra weak-amenability assertion. This auxiliary Prikry argument adds no stage to the main Prikry-labeling construction. $\square$

Lemma 8.4 (Measurables in one fixed relative core). *In a ZF universe with unbounded measurables, put $B = \text{HOD}(r)$. If B has no r -kukri, then $K(r)^B \models \text{CM}$.*

Proof. Fix one ambient measurable κ and one ambient κ -complete nonprincipal ultrafilter μ . Set

$$C = \text{HOD}(r, \mu), \quad u = \mu \cap \mathcal{P}(\kappa)^C.$$

The parameterized Vopěnka theorem applies in ZF: take $Y = r$, $X = V_\theta \ni \mu$ and $x = \mu$ in [15, Lemma 3.6]. Thus C is a set-generic extension of B and satisfies ZFC.

The whole u belongs to C , since it is ordinal definable from r, μ, κ and every member is already in C . This does not require $\mu \in C$. Every C -subset of κ is decided by u . Every whole C -sequence of fewer than κ members of u is also an ambient sequence, so its intersection belongs to u by ambient completeness. Nonprincipality also transfers. Consequently C regards κ as measurable.

The negative alternative persists by Lemma 8.1. Apply Lemma 8.3 inside C and use $K(r)^C = K(r)^B$. This makes the original ambient κ measurable in the one fixed core. Repeat the existential argument separately for each ambient measurable. No simultaneous choice of measures, ambient normality or ambient Dependent Choice is needed. $\square$

8.2. The positive choice model and recoding.

Lemma 8.5 (The top-iteration choice model). *If a choice background contains an r -kukri, its canonical top iteration gives a uniformly r -definable inner model S_r containing r and satisfying ZFC + CM.*

Proof. Use the least sound mouse and its standard top-only iteration (P_i, κ_i) , with the real predicate fixed. The construction in [10, Section 2, Facts (ii)–(v)] gives the inner model

$$S_r = \bigcup_{i \in \text{Ord}} H_{\kappa_i}^{P_i}, \quad |P_i| = H_{(\kappa_i^+)^{S_r}}^{S_r}.$$

Its critical points are unbounded, and its displayed lower cuts are coherent elementary cuts. The theorem applies with the fixed amenable real predicate included in every hull and fixed by every map; see the fine-structure convention in [11, Chapter 1].

The whole H -cut identification supplies all final subsets and short graphs below κ_i . Top normality makes κ_i inaccessible in S_r , so the lower cut is $V_{\kappa_i}^{S_r}$. Given any bound below κ_i , choose a lower measurable δ above it in P_i . Its whole normal measure and all S_r -subsets of δ , regressive graphs and short families of such subsets have rank below κ_i . They occur in the same unchanged cut. The measure therefore remains full and normal in S_r . These lower witnesses are unbounded, giving CM. No measurability of the moving top itself is used.

The published construction supplies ZFC and canonical set-length iterates. Its defining recursion takes place inside the choice background, so it defines one class without class choice. The bottom real is retained throughout. $\square$

Lemma 8.6 (Recoding covariance). *Mutually recursive reals r, s in the same choice background give the same pure output in each of the two alternatives above.*

Proof. We first check the pure carriers in the declared predicate convention. The first J -block is HF for either real. The second is the appropriate rudimentary closure of $\text{HF} \cup \{\text{HF}\}$. In $J_2[s]$, both ω and the whole set HF are present, as is the whole trace s . If program e computes r from s , then

$$r = \{n \in \omega : \exists c \in \text{HF } c \text{ is an accepting } s\text{-oracle computation of } e(n)\}.$$

This is bounded Separation with the old set HF as the computation bound. Thus $r \in J_2[s]$. Since intersection with this old r is rudimentary, $J_2[s]$ is closed under the r -predicate operations. Reverse the programs and use minimality of rudimentary closure to obtain the same J_2 . Induction gives the same subsequent J -carriers when the measure predicates are held fixed. The finite auxiliary S -stages need not agree.

The inverse formula translations preserve Σ_1 truth with each fixed ordinal parameter tuple. Above HF, computation witnesses are bounded by the old set HF; its uniform Σ_1 name can be witnessed by the whole pure finite-stage construction tower, whose graph is a bounded subset of HF at J_2 . At an HF reduct, bounded quantifier domains are finite, so one finite evaluation tree collects the needed finite computation witnesses. This supplies the same translations without assuming that ω is a set in that reduct.

Consequently missing-subset projecta and good parameter classes agree. Each canonical Skolem graph is a uniquely defined partial Σ_1 function in the other language. The two Skolem hulls are closed under one another's such functions and hence are identical. Very-good parameter classes agree; the unchanged ordinal ordering gives the same least standard parameters. Repeat this argument on the theory-code reducts. Their formula indices translate recursively while tuple coordinates stay fixed. Whole bounded theory witnesses translate by bounded image or preimage under the old recursive translation graph. At an HF reduct the finite witnesses are hereditarily finite in both presentations. Thus the code recovery and pure sound-core collapses agree. These are the standard fine-code operations of [11, Chapter 1], in each chosen category separately.

The transport is invertible on the eligible mice and commutes with their comparisons, measure quotients and direct limits. It therefore preserves the least kukri and its top-iteration cuts. In the negative alternative, inverse transport preserves the class of relative candidate mice, their comparison relation and the ambient thick-class tests in this same background. Apply the standard true-core definition and uniqueness theorem to these corresponding data. The pure true cores are consequently equal; no literal equality of preliminary auxiliary constructions is required. $\square$

8.3. Canonical refinement inside a choice model. We now obtain the own-HOD presentation directly from a choice model with unbounded measurables. This construction uses ordinary normal measures and their iterated ultrapowers. It applies inside either choice model supplied above. For a set selection F of measures, $L[r, F]$ denotes construction with its domain and filter-incidence predicates, rather than with the whole vector and its transitive closure as an initial seed.

Lemma 8.7 (Bounded uniqueness and exhaustion). *Work in a choice universe B . Let d be a set of measurable cardinals above ω such that $d \cap \kappa$ is bounded below κ for every $\kappa \in d$. Let F and G be whole selections of full normal B -measures on d , and let r be a real. Then*

$$L[r, F] = L[r, G],$$

and the two predicates have the same restricted kernels in this model. The model's only normal measures are these listed restricted measures.

Proof. The restricted kernel of a bounded selection is a whole set in its predicate model: form the model's bounded powerset and tuple carrier, take one later construction-stage trace of the incidence and domain predicates on that carrier, and regroup the trace by Replacement. Each restricted measure is full and normal there, since all its internal test sequences and functions are background sets.

For uniqueness put $T = \text{ran}(F) \cup \text{ran}(G)$. If $U \in T$ is on κ , then

$$\kappa \in j_U(d) \iff d \cap \kappa \in U,$$

which is false by spacing. Thus $j_U(T)$ has no measure at the old κ . The whole earlier family has rank below κ and is fixed. Consequently T is a closed set of measures in the sense of [14, Lemma 19.35], and every member has relative T -rank zero. This permits several different measures at one critical point.

Apply that lemma to two copies of (B, T) , obtaining ordinary iteration maps $i : B \rightarrow M$ and $j : B \rightarrow N$, with, say, $i(T)$ an initial segment of $j(T)$. Write $\ell(e) = \sup\{\delta + 1 : \delta \in e\}$. The initial-segment test compares *all* cross-pairs of measures at a common critical point and relative rank on all common subsets [14, (19.20)]. Both $i(F)$ and $i(G)$ have domain $i(d)$, including any new positions in this image domain. The restriction $j(F) \upharpoonright \ell(i(d))$ has the same domain. Thus simultaneous induction through the three relative hierarchies gives

$$L[r, i(F)] = L[r, i(G)] = L[r, j(F) \upharpoonright \ell(i(d))].$$

At each common level every predicate argument belongs to both terminal models, so the incidence answers agree. The first two predicates also agree on every query in their common construction. These are first-order assertions inside M about its two set parameters $i(F), i(G)$. Pulling them back through the single map i proves both assertions of uniqueness. All critical points fix r .

For clarity, this comparison terminates at a set stage. Choose γ above the original domain and put $\mu = 2^{|\gamma|}$ and $\Theta = \mu^+$ in B . Finite-support representations bound each current image of γ and $\mathcal{P}(\gamma)$, at stage $\nu < \Theta$, in size by $\mu \cdot |\nu|^{<\omega} < \Theta$. All choices can therefore be made in one wellordered set carrier. If comparison continued through Θ , Fodor's lemma at limit stages, followed by stationary thinning, would make the earlier representatives of the disagreement ordinal

constant on both sides. In the two-measure case it also makes the representatives of a common distinguishing subset constant. Slot erasure makes comparison critical points strictly increase. At a one-sided step, the padded side consequently fixes the old seed while the other moves it. At a two-sided step, normality puts the old seed into exactly one image of the distinguishing subset, and later maps preserve that disagreement. Both conclusions contradict the equal later representatives. This is the set version of the comparison argument in [14, Lemma 19.35].

For exhaustion, suppose a predicate model has an extra normal measure. Choose its first critical prefix with this property, allowing the whole final prefix. Replace that selection by its whole internal kernel E and write $M_0 = L[r, E]$. Every proper critical prefix of E has no extra normal measure in its own predicate model. These prefix constructions are literal inside M_0 , so M_0 satisfies this assertion. Choose there the least κ carrying an extra normal measure, then an extra W of least Mitchell rank, breaking ties by the canonical $L[r, E]$ order. This choice is definable from r, E alone; the Mitchell order is well founded [14, Lemma 19.32].

Let $j_W : M_0 \rightarrow N_0 = \text{Ult}(M_0, W)$ and $E_1 = j_W(E)$. Compare (M_0, E) and (N_0, E_1) by the same closed-family lemma, with terminal maps

$$i : M_0 \rightarrow P = L[r, I], \quad k : N_0 \rightarrow Q = L[r, J].$$

If I were a proper critical prefix of J , literal hierarchy induction would give $P = L[r, J \restriction \ell(\text{dom } I)] \subseteq Q$. Its restricted kernel is exactly I . The transferred minimal-prefix assertion in Q forbids an extra normal measure in this very model P , whereas $i(W)$ is such a measure. No fullness over Q is being inferred here. The reverse orientation is excluded in the same way. Thus $P = Q$ and $I = J$ as whole restricted kernels.

Put $l = k \circ j_W$. The maps i, l have the same target and the same images of r, E . The canonical definitions give

$$i(\kappa) = l(\kappa), \quad i(W) = l(W).$$

The normal ultrapower N_0 has the same subsets of κ as M_0 . If E_1 has a measure T at the old κ , then $T \in N_0 \subseteq M_0$. Codes as κ -subsets for short families and regressive graphs show that T is a full normal M_0 -measure. It is a Mitchell predecessor of W , so minimality of W 's rank forces T to be the sole listed measure at κ .

The two initial sequences agree below κ . If they agree at κ as well, all comparison critical points exceed κ ; then $i(\kappa) = \kappa < j_W(\kappa) \leq l(\kappa)$, a contradiction. A target-only slot is an unlisted predecessor and has already been excluded. The remaining case is a source slot $U = E(\kappa)$ and no target slot there. The first source step uses U , the target pads, and every later critical point exceeds κ . Choose the first subset x distinguishing U from W in the canonical $L[r, E]$ order. The images of κ, W, U under i, l agree, so this definition gives $i(x) = l(x)$. But normality and the fixed old seeds give

$$x \in U \iff \kappa \in i(x), \quad x \in W \iff \kappa \in l(x),$$

contradicting the choice of x . This proves exhaustion. All iterations here use images of a whole set family of full normal measures inside a ZFC model, so ordinary set-family iterability applies [12, Lecture 2]. $\square$

Lemma 8.8 (Uniform canonical refinement). *There is one first-order formula $\varphi(x, r)$ such that in every $B \models \text{ZFC} + \text{CM}$, for each real $r \in B$, the class*

$$C_r^B = \{x \in B : B \models \varphi(x, r)\}$$

is an inner model containing r and satisfying $\text{ZFC} + \text{CM} + V = \text{HOD}(r)$. If r, s are recursively interdefinable in this same B , then $C_r^B = C_s^B$. Moreover C_r^B is a set-forcing extension of its own HOD .

Proof. Work in B . Recursively let κ_α be the least measurable strictly above

$$\max(\omega + 1, \sup_{\beta < \alpha} (\kappa_\beta + 1)), \quad d_\alpha = \{\kappa_\beta : \beta < \alpha\}.$$

Each set-length recursion is uniquely definable. Every earlier listed domain is bounded strictly below the next listed cardinal, including at limit steps. For each set α , Replacement and Choice in B supply a whole selection F of full normal measures on d_α . Set $N_\alpha^r = L[r, F]$. By Lemma 8.7 this model is independent of the selection. We use the single formula

$$\varphi(x, r) \iff \exists \alpha \exists F (F \text{ is such a selection on } d_\alpha \wedge x \in L[r, F]).$$

All quantifiers and measure tests in this formula are evaluated in B . No proper-class selection of measures is made.

For $\alpha < \beta$ choose one F on d_β . Inside N_β^r take the whole prefix kernel on d_α , using the entire bounded powerset and tuple carrier. Its domain is the old domain cut at the ordinal κ_α . Literal hierarchy induction identifies construction from this kernel with $L[r, F \upharpoonright d_\alpha]$. Hence $N_\alpha^r \subseteq N_\beta^r$. The kernels agree on every older positive and negative decision.

Write $C = \bigcup_\alpha N_\alpha^r$. This uniformly definable transitive class contains all ordinals and is closed under the Gödel operations. For any B -set of its elements, Replacement bounds their least appearance stages by one ordinal; a rank set in a later N_α^r covers them. Thus C is almost universal and satisfies ZF [14, Theorem 13.9]. Whole wellorders from the prefix models give Choice in C .

For a listed κ , the entire set $X_\kappa = \mathcal{P}(\kappa)^B \cap C$ exists in B . Bound the appearance stages of all its members by a_κ already listing κ . Every N_β^r with $\beta \geq a_\kappa$ then has exactly this final power set. Compatibility identifies their whole restricted filters as one set W_κ . Every short family or regressive graph belonging to C belongs wholly to a later prefix, where that same W_κ is normal and complete. Thus it is a full normal C -measure. The listed cardinals are unbounded, so $C \models \text{CM}$.

Choose a_κ least. Replacement collects these ordinals on each bounded listed domain. At one later prefix all its powersets and measures have stabilized, and its whole kernel collects that entire stabilized prefix. Consequently the class predicate

$$E(\kappa, x) \iff x \in W_\kappa$$

is uniformly definable in B , and each bounded prefix is a C -set. Include its listed domain as part of the predicate.

We claim $C = L[r, E]$. A set-length construction queries only bounded critical indices, where E agrees with one original selection on every query in its prefix model. Literal induction gives $L[r, E] \subseteq C$. The global relative constructor is itself uniformly definable, closed under Gödel operations, and almost universal: bound the appearance stages of a background set of its elements and use a later construction level. Hence it satisfies ZF. Whole later traces reconstruct its stage wellorders, giving Choice.

For the reverse inclusion fix a prefix and a construction length γ . In $L[r, E]$ take a whole rank carrier V_ρ high enough for the levels and tuple queries of a recursion of length γ . Choose ρ from the elementary rank bounds for a relative construction of length γ , before identifying any internally produced level with an external prefix level. A later global level supplies the whole E and domain trace on that carrier. Restrict this set trace at the ordinal cut for the desired prefix and run the entire prefix recursion internally. Every produced level is old and has the chosen rank bound. External induction identifies it with the actual prefix level. Thus every prefix is contained in $L[r, E]$, proving the claim.

Now let W be any whole normal C -measure on κ , and choose N_α^r containing W . Every C -subset of κ is in W or is the complement of a member of W . Transitivity and Separation therefore give

$\mathcal{P}(\kappa)^{N_\alpha^T} = \mathcal{P}(\kappa)^C$. Normality restricts to the prefix. Bounded exhaustion identifies W with its listed kernel, and compatibility identifies it with W_κ . Therefore the internal pure formula “ U is the unique normal measure on κ ” recovers exactly E . Since $C = L[r, E]$, its relative stages and their canonical wellorders are internally definable from r and ordinals. Every set, and every hereditary member, is consequently ordinal definable from r in C . Thus $C = \text{HOD}(r)^C$.

Apply Vopěnka’s theorem inside C , with its own HOD as ground and r as the parameter [15, Lemma 3.6]. It gives set forcing and a generic g such that $C = \text{HOD}^C[g]$. Finally, if r, s are recursively interdefinable in the same B , the measurable recursion is identical, and for every same bounded selection F the pure universes $L[r, F]$ and $L[s, F]$ agree. Their canonical prefix models and their unions therefore agree. $\square$

Corollary 8.9 (Refinement of a uniform choice envelope). *Suppose one formula in a ZF universe defines, for each real r , an all-ordinal inner model E_r containing r and satisfying $\text{ZFC} + \text{CM}$. Suppose $E_r = E_s$ whenever r, s are recursively interdefinable in that same universe. Then there is a uniform assignment $r \mapsto C_r$ with the same properties such that $C_r \models V = \text{HOD}(r)$ and C_r is a set-forcing extension of its own HOD.*

Proof. Relativize the fixed formula φ to the definition of E_r . Apply Lemma 8.8 inside this actual choice model. When $E_r = E_s$, both evaluations take place in the same background containing both reals and their recursive interdefinitions. The lemma’s same-background recoding conclusion gives $C_r = C_s$. $\square$

Lemma 8.10 (Core-model preparation). *In a ZF universe with a proper class of measurables there is a uniform assignment, for every real r , of a choice inner model C_r such that:*

- (1) $r \in C_r$ and $C_r \models \text{ZFC} + \text{CM}$;
- (2) the pure universe of C_r is unchanged by the finite recursive recodings induced by permuting a collapse enumeration;
- (3) C_r is a set-forcing extension of $D_r = \text{HOD}^{C_r}$.

Proof. Work in $B_r = \text{HOD}(r)$. If there is an r -kukri, let $E_r = S_r$ be the top-iteration choice model of Lemma 8.5. If there is no r -kukri in B_r , let $E_r = K(r)^{B_r}$. Lemmas 8.2 and 8.4 give the required $\text{ZFC} + \text{CM}$ choice envelope in the negative branch. Both envelopes contain r .

For mutually recursive r, s , we have $B_r = B_s$. Lemma 8.6 identifies the branch and the pure output envelope: $E_r = E_s$. All recipes are uniformly definable in the background, without selecting a real code for a mouse or an additional wellorder. Apply Corollary 8.9 inside these envelopes. It supplies C_r with all three properties, including its presentation over $D_r = \text{HOD}^{C_r}$. $\square$

8.4. Rank and dependent choice.

Proposition 8.11 (Rank-preserving passage to SVC). *Suppose $V \models T_0$. For every ordinal β there is an inner model $N \subseteq V$ satisfying $T_0 + \text{SVC}$ such that*

$$V_\beta^N = V_\beta^V, \quad \mathcal{P}(V_\beta^V)^N = \mathcal{P}(V_\beta^V)^V.$$

One can arrange a choice ground $D \models \text{ZFC} + \text{CM}$ and a choice set-forcing cover $C = D[g]$ with $D \subseteq N \subseteq C$.

Proof. Choose $\lambda > \beta + \omega$ with $\text{cf}^V(\lambda) > \omega$, and let $B = V_\lambda^V$. DC supplies such a limit ordinal. For example, start sufficiently high and add ω_1^V . Collapse B to be countable, and let r code membership and equality of entries in its generic enumeration. By Lemma 2.1, this fixed collapse preserves a proper-class tail of measurables. Apply Lemma 8.10 in the collapse extension and put $C = C_r$.

Inside C , first quotient the coded enumeration by its equality relation and then take its well-founded extensional collapse. This recovers the same transitive set B . Define

$$N = \text{HOD}_{B \cup \{B\}}^C,$$

allowing individual elements of B as parameters as well as the whole set B . This relative hereditary-definability construction satisfies ZF and contains B and all its elements.

Finite permutations of the ω -domain witness weak homogeneity of the collapse by moving the finite domain of one condition away from that of another. They replace r by exactly the recordings covered by Lemma 8.10(2) and fix the actual ground set B and its members. Thus the group preserves every fixed ordinal-and- B definition over C . The usual hereditary rank induction gives $N \subseteq V$. Consequently

$$V_\lambda^N = B = V_\lambda^V.$$

In particular, both the rank V_β and all its subsets have been retained.

For every nonempty $Y \in N$, the relative-HOD definition gives a surjection in N from $\eta \times B^{<\omega}$ onto Y , for some ordinal η . The surjection enumerates all bounded definition codes and finite parameter lists, without choosing a definition for each element of Y . Finite lists from the limit rank B can be coded into B , so $N \models \text{SVC}(B)$.

For DC, let $R \in N$ be serial on Y , fix $y_0 \in Y$, and take $f : \eta \times B \rightarrow Y$ in N . At a current y put

$$\xi(y) = \min\{\xi < \eta : \exists b \in B (y R f(\xi, b))\}.$$

Use DC in V to choose successive b_n at these least ordinal coordinates. The array $s = \langle b_n : n < \omega \rangle$ belongs to B because $\text{cf}^V(\lambda) > \omega$ and λ is a limit. Hence it belongs to N . Within N , recursion from s , y_0 , and the least-coordinate rule reconstructs the entire chain. Thus $N \models \text{DC}$.

Finally, $D = \text{HOD}^C \subseteq N \subseteq C = D[g]$. Since the forcing is fixed, downward Lévy–Solovay gives a measurable tail in D from $C \models \text{CM}$. For every sufficiently high ground measure U , its generated lift in C restricts to the set

$$\{a \in \mathcal{P}(\delta)^N : \exists b \in U (b \subseteq a)\}$$

in N . This is the complete nonprincipal ultrafilter required for CM^N . We fix the ground D and cover C before taking this tail. If desired, the intermediate symmetric-model theorem and SVC^N make D a symmetric ground of N . $\square$

Proof of Theorem 1.4. Apply Proposition 8.11 above $\alpha + 1$ to obtain $N \subseteq V$. Then apply the local Theorem 1.5 to N , protecting the same rank and its power set. The resulting M has the displayed rank and power-set equalities with V , together with the stated sequence-seed presentation. Choose all required generics over the original collapse extension. The construction then takes place in a set-generic extension of V . The inclusion asserted is $N \subseteq M$; the argument does not assert $V \subseteq M$. $\square$

9. REMOVING FOUNDATION BY ATOMIZATION AND FIBERS

We now reduce the Foundation-free starting theory to the preceding case. The atomization and fiber purification follow the earlier argument of [2, 23 September, 06:07 UTC]. We first prove the version of the local forcing theorem needed here. In this section, ZFA has Foundation above a fixed set $\mathcal{A}$ of atoms, and Ker denotes the pure kernel.

9.1. The local construction with atoms.

Lemma 9.1 (The choice-envelope version). *Let $N \subseteq E$ be ZFA models with the same ordinals and the same set $\mathcal{A}$ of atoms. Suppose:*

- (1) $N \models \text{DC} + \text{SVC}(X)$, where $X = V_\lambda^N(\mathcal{A})$ is a sufficiently high limit rank, $\text{cf}^N(\lambda) > \omega$, and $\mathcal{A} \subseteq X$;
- (2) E satisfies Choice, $\mathcal{A}$ is countable in E , and $\text{Ker}(E) = D[g]$ for a pure $D \subseteq N$ satisfying $\text{ZFC} + \text{CM}$ and fixed set forcing;
- (3) a measurable $\kappa > \lambda + \omega$ of D is chosen above the size of this fixed forcing and the sizes in E of X , the old collapse, and the cap and comparison posets.

Then the local Prikry-labeling construction produces $N \subseteq U$ with

$$U \models \text{ZFA} + \text{DC} + \text{CM} + \text{SVC} + \text{KW}_1^{*,\omega}, \quad \mathcal{P}(X)^U = \mathcal{P}(X)^N, \quad V_\lambda^U(\mathcal{A}) = X.$$

Proof. This version requires only the pure D and the choice envelope E . Use the old pure Prikry poset from D with its old wellordering. The old normal measure base generates the normal lift in E and its small choice extensions. Compute the countably closed collapse in N and choose the product generic over E , as in Section 4. Internal countable closure and condition coding use $\text{cf}^N(\lambda) > \omega$. No requirement is imposed on $\text{cf}^E(\lambda)$.

The small-forcing assertions with atoms reduce to their pure counterparts. Fix an enumeration of the atoms in E . Recursively code each atom by its tagged index and each set by the tagged set of its members' codes. Every resulting code is pure and belongs to $\text{Ker}(E)$. Choice in E codes any set forcing by an ordinal condition set and a pure order. Translate names using the same atom indices. Decoding a name for a pure value uses no atoms. The pure kernel after forcing is therefore the corresponding set-forcing extension of the old pure kernel. We use this coding only in the envelope and do not adjoin its atom enumeration to N or U .

The collapse-first proof of Lemma 4.1(1)–(2) therefore applies to atomful X and arbitrary ground values. SVC makes $N[K]$ a choice model with atoms. Since Prikry forcing adds no bounded subsets, we obtain the same preservation of countable sequences from X and wellorderable covers of ground families.

For part (3), we need preservation of atom-ranks. Since $|\mathcal{A}|^E < \kappa$ and κ remains inaccessible in the choice envelope, every old atom-rank below κ has size below κ . Prikry forcing preserves these ranks because it adds no bounded subsets. A new set of least rank would be a subset of a small old rank; its pullback by an old wellordering would be a new bounded subset of κ . The atoms are unchanged. The generic-intersection proof proceeds by rank over the common atom set, starting with the atoms, which already belong to N . Thus

$$V_\kappa^{N[c]}(\mathcal{A}) = V_\kappa^N(\mathcal{A}).$$

Use transitive-base constructibility with atoms retained at every successor stage, for example

$$J_{\xi+1} = \mathcal{A} \cup \text{Def}(J_\xi; \in, \text{Atom}),$$

so that taking definable subsets does not discard the atoms. Lemma 2.2 still gives finite expressions. Atoms are ground leaves, and the atom predicate has the same interpretation in the compared universes. The presentation lemma does not use an equality $N = D(X)$.

All cap and paired constructions concern integer carriers and their literal colors in X . A color may be an atom. Since the constructions never recover a color from its members, the proofs in Sections 5–6 are unchanged. The common atom predicate also gives bounded-evaluator absoluteness. Labeling elimination and the atom-rank equality just proved yield $\mathcal{P}(X)$ preservation.

The old condition-array argument, the real-coded masks, and the wellorderable-tree argument prove DC as before. Bounded presentations and $F : Q \twoheadrightarrow X$ give SVC and KW for every nonempty set of U , including sets containing atoms. Finally the full choice envelope after these set forcings has pure kernel a set-forcing extension of D , by the coding paragraph above. A sufficiently high tail of ground measures has generated lifts there, and the old-base definition used in Proposition 7.5 restricts those lifts to measures belonging to U . This proves CM^U .

The atom lemma does not include the pure sequence-seed and symmetric-ground conclusions of Proposition 7.5. The predecessor-set quotient does not distinguish distinct atoms, and a pure $D(A')$ cannot contain them. Neither conclusion is needed below. $\square$

9.2. Preparation of the protected atom structure. Start with $V \models T_-$ and a full second-order counterexample $\mathcal{M}$ to a fixed sentence. Take its relational multisorted presentation, with a first-order sort S_0 and, for each arity k in the sentence, a sort representing all V -subsets of S_0^k with the incidence relation. The two-sorted presentation is the unary or tuple-coded case. Replace the combined domain by a set $\mathcal{A}$ of formal atoms, with distinct tags for the sorts.

Form the full interpreted cumulative hierarchy $\mathcal{H}(\mathcal{A})$ using the original universe's power sets. Treat its atoms as opaque labels whose original members do not become members of the atoms. Non-well-founded membership in V therefore introduces no cycles in the interpreted hierarchy. The hierarchy satisfies ZFA. It also satisfies DC, since an ambient chain through a set of its coded objects has bounded atom-rank and belongs to the hierarchy. Its pure kernel has the original ordinals and their full power sets. It therefore retains the original measures and the ordinal-indexed families testing their completeness. Hence

$$\mathcal{H}(\mathcal{A}) \models \text{ZFA} + \text{DC} + \text{CM}.$$

The atomized structure remains full because inverse images under the atom-label bijection identify its subsets and finite-arity relations with all the original ones.

Choose a limit λ of uncountable cofinality in this hierarchy, high enough to contain the structure and the full power sorts, and set

$$B = V_\lambda^{\mathcal{H}(\mathcal{A})}(\mathcal{A}).$$

Collapse B to be countable. Its real code r records membership, equality of repeated enumeration entries, and which entries are atoms. The small-forcing proof of Lemma 2.1 works with fixed atoms, so a measurable tail remains. Apply the earlier core-model Lemma 8.10 in the pure kernel of this collapse extension, obtaining $C_r = D[g]$.

Construct the choice companion E as follows. Let I be the least enumeration indices of the distinct atoms, as determined by r . Inside C_r , form the hierarchy over the formal atom set indexed by I . Use the collapse enumeration to relabel those formal atoms by the actual atoms. The resulting E satisfies Choice, has pure kernel C_r , and contains the literal B . To prove the last assertion, quotient the code by its equality predicate and take its well-founded collapse with the specified atom labels. These labels distinguish the atoms from one another and from the empty set. Uniqueness of the collapse gives the same B .

The finite permutation group used in Proposition 8.11 preserves this companion. The pure C_r is unchanged, and the induced bijection between the two atom-index sets belongs to the same pure model. Transporting the hierarchy along this bijection gives the identical actual companion. Therefore

$$N = \text{HOD}_{B \cup \{B\}}^E,$$

with the atom predicate retained, descends to $\mathcal{H}(\mathcal{A})$ by the same hereditary homogeneity argument. It contains B and has $V_\lambda^N(\mathcal{A}) = B$. Relative-HOD presentations give $\text{SVC}(B)$, and the least-ordinal-coordinate DC proof in Proposition 8.11 applies using the countable closure of B in $\mathcal{H}(\mathcal{A})$.

The pure $D = \text{HOD}^{C_r}$ is contained in N , because definitions over the definable pure kernel of E are available over E . Ground measure lifts restrict to N , giving CM^N . We have a pure ground inside N and a choice envelope above it. This does not assert a choice ground with atoms inside N , and it does not apply the pure SVC symmetric-ground theorem to an atom model.

Lemma 9.1 now applies with $X = B$. In particular, $\text{cf}^N(\lambda) > \omega$ because $N \subseteq \mathcal{H}(\mathcal{A})$. The envelope may see B and λ as countable, which the lemma permits. The old condition sets have

codes in B and are small in E . Once these set-sized bounds are fixed, choose a sufficiently high D -measurable κ . The lemma yields

$$U \models \text{ZFA} + \text{DC} + \text{CM} + \text{SVC} + \text{KW}_1^{*,\omega}$$

with the protected power sets unchanged. The original multisorted counterexample is consequently still full in U .

9.3. Fiber purification and the final theorem.

Lemma 9.2 (Fiber purification). *If $U \models \text{ZFA} + \text{DC} + \text{CM} + \text{SVC} + \text{KW}_1^{*,\omega}$, then its pure kernel P satisfies $T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$. Every full second-order structure in U has a full isomorphic copy in P .*

Proof. For the nonempty combined domain S of a relational presentation, take a surjection $f : \kappa^\omega \twoheadrightarrow S$ in U and put

$$j(s) = f^{-1}\{s\}.$$

The nonempty disjoint fibers make j injective. Each fiber is a set of ordinal sequences and is therefore pure. Thus the whole range $j[S]$, its sort predicates, and all transported finite-arity relations are pure sets in P . The graph of j may contain atoms and need not belong to P . Empty domains can be treated directly.

To check fullness at each relevant arity, let $Y \in P$ with $Y \subseteq j[S_0]^k$. Its pullback under j^k is a U -subset of S_0^k , represented by an element of the full power sort in U . The image of that element represents exactly Y . Conversely, each represented subset in the transported structure is pure and belongs to P . In the application, preservation of the protected power sets identifies these U -relations with the relations of the original counterexample. The copied structure therefore has the same full second-order truth.

The pure kernel satisfies ZF, including Foundation. A DC chain through a pure set is itself pure. Measures on ordinals and all families testing their completeness are pure as well, so CM descends. The models have the same κ^ω ; a surjection from it onto a pure set has pure graph. Hence KW descends.

Finally, fix one SVC witness C in U and fiber-code it as above to obtain a pure copy $C^\flat$. For any nonempty pure Y , compose an SVC surjection $\eta \times C \twoheadrightarrow Y$ with the inverse of this fiber bijection. The resulting map $\eta \times C^\flat \twoheadrightarrow Y$ has pure graph and belongs to P . The same $C^\flat$ works for every Y , proving SVC in P . $\square$

Proof of Theorem 1.8. Given a full second-order counterexample in a model of T_- , atomize it, carry out the core-model preparation, and apply the local atom theorem as above. Lemma 9.2 gives a pure model of $T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$ containing a full counterexample to the same sentence. The countermodel contrapositive proves structural conservativity. The reverse implication of provability follows because the target theory extends T_- . All constructions are interpretations and set-forcing constructions over the starting model. No external transitive-model hypothesis is used. $\square$

10. EXTERNAL HOD AND HOMOGENEOUS DESCENT

Lemma 10.1. *If $U \models \text{ZFC}$, then $\mathcal{H}^U \models \text{ZF} + \text{DC}$. It contains every ordinal ω -sequence of U and is closed under U -countable sequences of its elements.*

Proof. The usual hereditary-definability proof of ZF applies to the definable class of allowed parameters. For example, the internal power set of $x \in \mathcal{H}^U$ is

$$\{y \subseteq x : y \in \mathcal{H}^U\},$$

which is definable from a definition of x and has only hereditary members of the class. The same argument gives Separation and Replacement.

Let $\langle x_n : n < \omega \rangle \in U$, with each $x_n \in \mathcal{H}^U$. In U , choose a definition of x_n from an ordinal sequence and ordinal parameters. Evaluate each definition at a rank level where its formula has the required unique value. Interleave the sequence parameters, the rank levels, the ordinal parameters, and the formula codes. This gives one $q \in (\text{Ord}^\omega)^U$ from which the entire sequence $\langle x_n \rangle$ is definable in U . Its hereditary members already belong to $\mathcal{H}^U$, so the sequence belongs to $\mathcal{H}^U$. Every $q \in (\text{Ord}^\omega)^U$ is itself hereditarily definable from q . To prove DC, choose a dependent-choice chain in U through a serial relation belonging to $\mathcal{H}^U$ and apply the countable closure just proved. $\square$

Lemma 10.2 (Homogeneous descent). *Let $M \models \text{ZF}$, and suppose $U = M[G]$ is a weakly homogeneous set-forcing extension with*

$$(\text{Ord}^\omega)^U = (\text{Ord}^\omega)^M.$$

Then $\mathcal{H}^U \subseteq M$. No ordinal-definability assumption on the forcing is needed.

Proof. Induct on hereditary rank. Suppose $x \in \mathcal{H}^U$ and every member of x is in M . Choose β with $x \subseteq V_\beta^M$. A definition of x in U uses an allowed parameter q belonging to M and ordinal parameters. By weak homogeneity, the largest condition decides the corresponding uniqueness assertion and, for each $y \in V_\beta^M$, whether y belongs to its unique value. The forcing relation therefore defines x as a subset of V_β^M inside M . $\square$

Proposition 10.3. *For every σ , $\text{Con}(E_1(\sigma))$ implies $\text{Con}(E_3(\sigma))$.*

Proof. Let $U \models E_1(\sigma)$, put $H = \mathcal{H}^U$, and choose κ with $V_\kappa^H \models \sigma$. Choose $\lambda > \kappa + \omega$ of uncountable cofinality in U , and set $X = V_\lambda^H$. This X is ordinal-definable in U , belongs to H , and is closed under U -countable sequences of its elements by Lemma 10.1.

Collapse X to be countable over U , coding both equality and membership and quotienting repetitions before collapse. Apply Lemma 8.10, and again set $W = \text{HOD}_{X \cup \{X\}}^{C_r}$. The collapse-homogeneity argument gives $W \subseteq U$ and controls the parameters in each descended definition. The collapse forcing is definable from the OD set X , so these definitions use X , finitely many elements of X , and ordinals. The set X is OD in U , and each of those finitely many elements is hereditarily definable from an ordinal ω -sequence. Combining their parameters gives $W \subseteq H$.

Since $X \in W$, we have $V_\lambda^W = X$. The presentation $\text{SVC}(X)$ and the ambient countable closure of X give DC in W . The same generated-measure argument gives CM. Therefore $W \models E_3(\sigma)$. $\square$

11. SPARSE EVALUATED COHEN CAPS

We work over a choice model B and code a *family of coherent rows*. The code records which bit positions belong to each row, rather than only their union.

We use the convention $\text{Add}(\gamma, 1) = 2^{<\gamma}$, ordered by extension; the dense partial-function presentation gives the same forcing. Throughout this section, the superscript specifies the model in which each poset is computed.

Theorem 11.1 (Fixed-base spectrum). *Let $B \models \text{ZFC}$ have a proper class of inaccessible cardinals, and let $\emptyset \neq F \subseteq 2^\tau$ belong to B , where τ is infinite. There are a schedule of inaccessible coding cardinals, uniformly defined forcings Q_e for $e \in 2^\tau$, and a cap cardinal ρ , all definable from B and ordinal parameters independently of F , such that the weakly homogeneous forcing P_F defined below has the following spectrum in $B[G]$:*

$$e \in F \iff \text{there is a } B\text{-generic } h_e \subseteq Q_e, \text{ and an } \text{Add}(\rho, 1)^{B[h_e]}\text{-generic over } B[h_e]. \quad (11.1)$$

The first coding cardinal can be placed above any prescribed bound, and the resulting forcing is correspondingly distributive.

11.1. The schedule and the actual forcing. Order the physical sites $s = (\xi, i)$, for $\xi < \tau$ and $i < 2$, lexicographically with ξ first. Distinct rows have independent copies at a shared physical site. Choose inaccessible cardinals γ_s recursively, with the first above $2^{|\tau|}$ and the prescribed protection bound, and with

$$|\text{RO}(S_{<s}^{\text{univ}})|^B < \gamma_s. \quad (11.2)$$

The universal prefix includes copies for every row in 2^τ , including those outside F . The schedule is therefore independent of the family being coded. At each step, choose the least inaccessible above the displayed bound.

For $e \in 2^\tau$, let

$$Q_e = \prod_{\xi < \tau}^{\text{full}} \text{Add}(\gamma_{(\xi, e(\xi))}, 1)^B.$$

Let ρ be an inaccessible above the Boolean completion of the entire universal lower product. Define

$$P_F = \prod_{d \in F}^{\text{full}} (Q_d * \text{Add}(\rho, 1)^{B[h_d]}). \quad (11.3)$$

Compute the terminal forcing in its *own row extension*. Form the actual extension from the evaluated terminal generics, without adjoining the larger raw termspace generics.

The full lower product

$$S_F = \prod_{d \in F}^{\text{full}} Q_d$$

can be grouped by physical sites. Denote the aggregate generic at s by G_s . Since $|F| < \gamma_0 \leq \gamma_s$, each site forcing is a product of fewer than γ_s copies of $\text{Add}(\gamma_s, 1)^B$. It is $< \gamma_s$ -closed in B and adds no shorter ordinal sequences over B .

For each d , let T_d be a full set of total names for cap conditions: the largest condition of Q_d forces each member to be a cap condition, and every total name has an equivalent member. Use all bounded Boolean-name codes for graphs on $\rho \times 2$ with this property, allowing equivalent codes. The resulting set is defined canonically from Q_d and ρ ; the forcing has no auxiliary wellorder as a parameter. Smallness of $\text{RO}(Q_d)$ and regularity of ρ bound the possible lengths of each name below ρ . Normalize forced unions and Boolean mixes to these Boolean-name codes. Lower automorphisms act on their Boolean values and preserve this full name system. The product

$$S_F \times \prod_{d \in F} T_d$$

with the ground termspace orders projects onto P_F and preserves the lower coordinate. To verify the lifting property, mix each desired stronger term on its strengthened lower cone with the old term elsewhere. Thus this projection commutes with the projection to S_F . Each ground termspace is $< \rho$ -closed. After the lower forcing, which has size below ρ , the full product is $< \rho$ -distributive by Easton's lemma. The actual cap quotient therefore adds no ordinal sequences of length below ρ over the lower extension. Here a closed *dominating* forcing proves preservation; the actual extension need not equal the larger termspace extension. See [17, Section 3] for the term-forcing facts.

This also proves preservation. If the first site is above a regular θ , the dominating forcing is $< \theta$ -closed in B , so the actual forcing is $< \theta$ -distributive. Taking $\theta > |V_\eta^B|$ preserves V_η^B by rank induction. The canonical lower-row automorphisms lift through the name for the recomputed

Cohen forcing, which is forced homogeneous. Coordinatewise automorphisms then witness weak homogeneity of the full product.

11.2. Localization to physical sites.

Lemma 11.2. *Suppose a B -generic $h_e \subseteq Q_e$ belongs to the P_F -extension. Then*

$$B[h_e] \subseteq B[G_s : s \text{ is selected by } e].$$

Proof. The whole lower generic h_e can be coded by a subset of a ground ordinal below ρ . The cap quotient adds no such subsets, so h_e is in the lower extension. Let $k_s \subseteq \gamma_s$ be its component at an e -selected site.

If there is a site after s , its first cardinal exceeds the Boolean size of the prefix through s . The ground full tail is closed up to that cardinal. Easton's lemma over the smaller prefix therefore makes it add no subsets of γ_s ; at a final site there is no tail. Thus the required tail distributivity follows from (11.2), and

$$k_s \in B[G_s][G_{<s}].$$

The forcing producing $G_{<s}$ has size below γ_s . Consequently the pair

$$(B[G_s], B[G_s][G_{<s}])$$

has the $< \gamma_s$ -approximation property. To check that k_s is approximated over $B[G_s]$, take a $B[G_s]$ -set of ordinal positions of size below γ_s . It belongs to B , since G_s adds no short ordinal sequences. The restriction of the B -Cohen generic k_s to that set also belongs to B . Therefore

$$k_s \in B[G_s]. \tag{11.4}$$

For each site, fix in B a wellordering of a bounded set of names for subsets of γ_s . Choose the least name whose value is the actual k_s . The indices form an ordinal sequence of length $|\tau|$. The lower forcing adds no such sequence, so the whole vector of indices belongs to B . Hence the vector of all k_s belongs to the extension by the selected sites alone.

This vector determines the full product generic h_e . A ground product condition belongs to a generic filter exactly when it is compatible with the union of the generic bits at every coordinate. Otherwise, the filter's meeting the dense set deciding compatibility with that condition gives a contradiction. Hence h_e belongs to the selected-site extension. $\square$

11.3. The quotient and its term order. The spectrum depends on the following quotient calculation. Replacing this order by raw term forcing would destroy the spectrum.

Let $E = B[h_e]$. Use the old vector of names in Lemma 11.2 to name h_e over the selected sites. Restrict to an actual positive ground cone which forces that this name is Q_e -generic over B . This is a forcing statement: the ground dense subsets of Q_e form a set. The membership Boolean values of the named filter generate a complete subalgebra of that selected-site completion, on the cone, whose induced extension is exactly E . Its size is below ρ ; injectivity of the map from the whole Q_e completion is not required. The omitted sites remain independent product directions over this algebra. Quotient positivity depends only on the selected lower projection, since every terminal name has been totalized. Restricting to this cone leaves the actual extension unchanged. Let P be a dense old-condition presentation of the quotient. Let

$$T = \left(\prod_{d \in F} T_d \right)^B$$

be the old set of vectors of row terms. The quotient of the *actual* forcing (11.3) is represented by $P \times T$ with order

$$(s', t') \leq (s, t) \iff s' \leq s \text{ and } \forall d \in F \ s'_d \Vdash_{Q_d}^B t'_d \leq t_d. \tag{11.5}$$

The right side uses the original ground forcing relation. It does not quantify over all new E -names for terminal conditions.

We apply Jakob's criterion [16, Definition 3.5 and Theorem 3.8]: an iteration-like order with a square- ρ -cc base and a strongly $< \rho$ -distributive term order has the $< \rho$ -approximation property. Strong $< \rho$ -distributivity means that every ρ -long sequence of dense open sets has a descending sequence meeting them. A final lower bound is not required. We check these hypotheses for (11.5).

Lemma 11.3. *If $e \notin F$, the quotient order (11.5) has the $< \rho$ -approximation property over E .*

Proof. The quotient base and its square have size below ρ . For a fixed old lower condition s , the section order on T is $< \rho$ -closed in B . In each row, take a name for the union of the descending cap conditions below s_d and use a default value off that cone. The extension E/B is by forcing of size below ρ . By the preservation observation preceding Jakob's criterion, this chain-condition forcing preserves strong $< \rho$ -distributivity of the fixed ground section orders. Their disjoint union is the term order, so it is strongly $< \rho$ -distributive in E .

The order is based, and the forcing relation gives its refinement property. For the projection property, suppose $(s', t') \leq (s, t)$. In row d , mix the name t'_d below s'_d with t_d off that Boolean cone. The resulting old vector t'' is a section extension below s , and below s' it is equivalent to t' . All inputs to the mixing belong to the ground, and the mixing takes place in B .

To verify mixing, take two section extensions t^0, t^1 below t at a fixed s . For every $d \in F$, choose the least ξ_d with $d(\xi_d) \neq e(\xi_d)$. The site

$$s_d = (\xi_d, d(\xi_d))$$

is omitted by e and active in row d . Keep the selected sites unchanged, and refine s in two ways by assigning opposite values to a fresh bit in this d -copy, simultaneously for every $d \in F$. This gives legal old conditions because $|F|$ is below the first coding cardinal and the product has full support. Both refinements remain positive in the quotient because only omitted sites changed. Their projections are incompatible in every actual row.

In row d , Boolean-mix t_d^0 and t_d^1 below the incompatible row projections, retaining t_d elsewhere. The resulting old vector t^* is a section extension of t at s . Below the respective quotient refinements it specializes to t^0 and t^1 , as required for mixing. Jakob's theorem gives the approximation property. $\square$

Proof of Theorem 11.1. If $e \in F$, its planted row and cap witness the right side of (11.1). Conversely, suppose that side holds and $e \notin F$. Lemma 11.2 and Lemma 11.3 give the $< \rho$ -approximation property for $(B[h_e], B[G])$. A recomputed $\text{Add}(\rho, 1)^{B[h_e]}$ -generic would be a new subset of ρ whose every $B[h_e]$ -small restriction belongs to $B[h_e]$, contradicting approximation. Preservation and homogeneity were proved after (11.3). $\square$

Corollary 11.4 (Coding an orbit of presentations). *Let $x \in B$. There is an arbitrarily highly distributive, weakly homogeneous forcing, uniformly definable in B from x and ordinal parameters, which makes x OD relative to the class B . If B is ordinal-definable in the final extension, then x is OD there.*

Proof. Choose an infinite cardinal τ large enough to code $(\text{TC}(\{x\}), \in, x)$. Let $F_x \subseteq 2^\tau$ be the family of all pointed well-founded extensional presentation codes of this structure. The codes include domain and root data and allow unused positions. This nonempty family is defined from x, τ without choosing a presentation.

Apply Theorem 11.1. Relative to the class B , the first-order capped-history predicate recovers exactly F_x . Each member has pointed Mostowski collapse x , so x is OD relative to B . The

schedules use the universal family 2^τ for their bounds. Thus the decoder can define the candidate forcings without knowing F_x . $\square$

Lemma 11.5 (GCH preservation). *If $B \models \text{GCH}$ in Theorem 11.1, then the actual evaluated forcing P_F preserves all cardinals and GCH. Consequently the coding forcing in Corollary 11.4 can be chosen to have these properties over a GCH base.*

Proof. Write $S = S_F$ for the lower product and let ν be the cardinality of the physical-site set. The schedule gives $\nu, |F| < \gamma_0$, where γ_0 is its first inaccessible. At an active site γ , the aggregate forcing has the dense partial-function presentation $\text{Add}(\gamma, m_\gamma)$ for some $m_\gamma < \gamma_0$. It has size γ and is $< \gamma$ -closed in B . Thus the full product S is $< \gamma_0$ -closed and adds no ordinal sequences of length less than γ_0 .

All arithmetic and cofinalities in the lower-product argument are computed in B . For an infinite cardinal λ , let $S_{\leq \lambda}$ contain the sites at most λ . If $\lambda \geq \gamma_0$ and $\text{cf}(\lambda) \geq \gamma_0$, then GCH gives

$$|S_{\leq \lambda}| \leq \lambda^\nu = \lambda.$$

This includes every regular $\lambda \geq \gamma_0$. If a tail remains, its first site η is greater than λ , and the old tail is $< \eta$ -closed. Easton's lemma makes it $< \eta$ -distributive over the head extension. Hence every ordinal sequence of length λ in the full extension already belongs to the head extension. The same conclusion is immediate when the head is all of S .

We first prove cardinal preservation. Successors below γ_0 are preserved by closure. For $\lambda \geq \gamma_0$ with $\text{cf}(\lambda) \geq \gamma_0$, the head has size at most λ and preserves λ^+ ; the tail adds no λ -sequences and cannot collapse it. Now suppose that λ is singular and $\epsilon = \text{cf}(\lambda) < \gamma_0$. Fix a cofinal sequence $\langle \lambda_j : j < \epsilon \rangle$ of infinite regular cardinals between γ_0 and λ . If $f : \lambda \rightarrow \lambda^+$ belongs to the full extension, then $f \restriction \lambda_j$ belongs to the $S_{\leq \lambda_j}$ -extension. A name for this restriction has at most λ_j possible values at each argument, so its range is bounded below the ground λ^+ . In the full extension, let $\beta_j < \lambda^+$ be the least strict bound on that range. The whole sequence $\langle \beta_j : j < \epsilon \rangle$ belongs to B by $< \gamma_0$ -closure. Ground regularity gives $\sup_{j < \epsilon} \beta_j < \lambda^+$, so f is bounded and cannot collapse λ^+ . Every ground successor cardinal is therefore preserved. A ground limit cardinal cannot collapse while all its smaller successor cardinals remain cardinals, so S preserves all cardinals.

When $|S_{\leq \lambda}| \leq \lambda$, Boolean-name counting gives at most $(2^\lambda)^\lambda = \lambda^+$ names for subsets of λ , and tail distributivity returns every such subset to the head extension. This proves GCH for the cases already covered by that size bound. Below γ_0 no subsets are added. For singular λ with $\epsilon < \gamma_0$, use the same regular sequence $\langle \lambda_j : j < \epsilon \rangle$. In B , fix for each j a list of at most λ_j^+ head names covering all subsets of λ_j in its head extension. For each actual $x \subseteq \lambda$, choose the least index representing $x \cap \lambda_j$ under the actual head generic. The resulting index vector has length $\epsilon < \gamma_0$, so the whole vector belongs to B . Its entries are below λ , and GCH and König's theorem give $\lambda^\epsilon = \lambda^+$ in B . A vector determines at most one x , by union of its evaluated restrictions. Cardinal preservation now gives $2^\lambda = \lambda^+$ in the full lower extension.

Finally, $|S| < \rho$ by the choice of the inaccessible cap cardinal, so ρ remains inaccessible in the lower extension. For each row d , the Boolean algebra $\text{RO}(Q_d)^B$ has size below ρ . The possible lengths of a total cap-condition name are therefore bounded below ρ . For each fixed length bound there are fewer than ρ bounded Boolean graph codes, by strong limitness. Taking all bounds gives $|T_d|^B \leq \rho$, and hence

$$\left| \left(\prod_{d \in F} T_d \right)^B \right| \leq \rho^{|F|} = \rho.$$

Together with $|S| < \rho$, this gives a dense old-condition presentation of the actual cap quotient of size at most ρ . The term-space projection established after (11.3) makes this quotient $< \rho$ -distributive over the lower extension. It preserves all smaller power sets and all cardinals at most ρ ; its ρ^+ -chain condition preserves higher cardinals. For $\lambda \geq \rho$, Boolean-name counting in the lower extension gives at most

$$(2^\rho)^\lambda = 2^\lambda = \lambda^+$$

names for subsets of λ . Thus the actual evaluated quotient also preserves GCH, completing the proof. $\square$

Remark 11.6. Theorem 11.1 uses no continuum-function comparison, large cap width, or singularization. Sparse site localization supplies the omitted quotient directions. A universal Cohen source is not retained. The ground term-spaces prove preservation, and the quotient argument applies to the evaluated forcing (11.5).

12. A FRESH ENVELOPE AND A SECURE GROUND

Lemma 12.1 (A fresh choice envelope). *Suppose $M \models \text{ZF} + \text{DC} + \text{SVC}$. There is a weakly homogeneous set-forcing extension $B = M[K] \models \text{ZFC}$ with*

$$(\text{Ord}^\omega)^B = (\text{Ord}^\omega)^M.$$

Proof. Choose an SVC witness S , and force with countable partial functions from ω_1^M to S . Countable closure and ground DC imply that no countable sequences of ground elements are added. Permutations of ω_1 moving two countable domains apart witness weak homogeneity. The generic union maps ω_1 onto S , so least preimages wellorder S . The SVC surjections then wellorder every old set. Each new set has an old name whose collection of possible named elements is now wellorderable. Evaluating that collection gives a wellordering of the new set. Thus the extension satisfies Choice. $\square$

Definition 12.2 (Secure grounds). A choice model D is *secure* if, in every set-forcing extension U of D , we have $D \subseteq \text{HOD}^U$ and D is the mantle of U . The mantle is the intersection of the set-forcing grounds and is uniformly definable.

We obtain a secure ground by coding with $\diamond^*$. For an uncountable regular λ , this principle asserts the existence of a sequence $\langle \mathcal{D}_\xi : \xi < \lambda \rangle$ such that $\mathcal{D}_\xi \subseteq \mathcal{P}(\xi)$ has size at most $|\xi|$ and, for each $x \subseteq \lambda$, there is a club $C \subseteq \lambda$ on which $x \cap \xi \in \mathcal{D}_\xi$. Write *eventual GCH* for GCH at every cardinal above some bound.

Lemma 12.3 (Secure preparation with eventual GCH). *Suppose $D_0 \models \text{ZFC} + \text{CM}$. Given a regular cardinal χ and an ordinal α with $|V_\alpha^{D_0}| < \chi$, there is a progressively closed class-forcing extension D such that*

$$D \models \text{ZFC} + \text{CM} + \text{eventual GCH}, \quad V_\alpha^D = V_\alpha^{D_0},$$

and D is secure. Every set-sized initial segment of the preparation can be taken $< \chi$ -closed in D_0 .

Proof. Use the $\diamond^*$ option of [18, Appendix, Lemma 20], starting above χ . The cited iteration combines GCH forcing with coding on blocks of triple successors of both fixed points. It preserves measurables; using $\diamond^*$ for the codes permits GCH. We record the effect of omitting its bounded initial part.

Make every stage through χ trivial and retain the stated iteration above it, including its Cohen stages at inaccessible cardinals. All remaining factors are $< \chi$ -closed. A union of fewer than χ Easton supports still satisfies the support bounds, since all active locations exceed χ . Thus every initial segment is $< \chi$ -closed. The progressively closed set-name construction gives ZFC, with

every set in an initial extension. It adds no ordinal sequences of length less than χ and hence preserves the indicated rank.

The GCH calculation applies above the omitted segment. For measurables $\kappa > \chi$, the cutoff is fixed by the normal ultrapower embedding. The iteration and its image agree below κ and have the same stage- κ Cohen preparation and closed upper tail used in the cited lift. Removing the bounded initial stages introduces no further lifting requirement. Consequently a class of measurables survives. These conclusions give eventual GCH and CM; they impose no GCH requirement on the protected rank.

The coding density argument applies above each prescribed bound. Every ordinal set appearing in an initial extension can be written on a fresh block by choosing the appropriate coding factors. Subsequent tails preserve that block's truth pattern. It follows that every ordinal set of D has arbitrarily high $\diamond^*$ codes. We next verify that these codes make D secure.

First, if $W[g]/W$ is set forcing and λ is regular above its size, then $\diamond_\lambda^*$ has the same truth value in the two models, by [18, Lemma 19]. The locations of sufficiently high codes also agree. Indeed, if the forcing has size ν , cardinal preservation and the bound of $(2^\kappa)^W$ on names for subsets of κ show that the cardinal-valued continuum functions agree for cardinals $\kappa > \nu$. Let b_ξ and b'_ξ be the two beth sequences. Above a fixed initial bound, monotonicity gives fixed ordinal shifts θ, θ' with

$$b_\xi \leq b'_{\theta+\xi}, \quad b'_\xi \leq b_{\theta'+\xi} \quad (\xi \in \text{Ord}).$$

For example, choose the shifted initial value beyond the disturbed continuum segment and induct using their common continuum function there. If δ is a beth fixed point in either model above these shifts, then δ is an infinite cardinal and $\theta + \xi, \theta' + \xi < \delta$ for every $\xi < \delta$. The inequalities put all values of the other beth sequence below δ . Continuity, together with $\beth_\delta \geq \delta$, makes it a fixed point there too. Thus the high beth fixed points and their triple successors agree. Describe a coding block by its actual first beth fixed point and its ordinal length. This avoids any change in the enumeration below the agreement bound.

Now let U be a set-forcing extension of D . For any ordinal set $x \in D$, choose one of its codes above the forcing size and the bound for agreement of locations. The unchanged truth pattern defines x in U from the starting point and length. Ordinal codes for transitive closures then give $D \subseteq \text{HOD}^U$.

The same reasoning gives the Ground Axiom in D . If $D = W[g]$ were a nontrivial set extension, a sufficiently high code for each ordinal set of D could be read in W . Separation in W would reconstruct every such set. Choice then gives $D = W$, a contradiction. Finally, let W be any ground of U . Downward directedness of grounds gives a common ground below D and W ; see [22, 7]. The intermediate-model theorem makes it a ground of D . The Ground Axiom makes this common ground equal to D . Since D itself is a ground of U , its mantle is exactly D . $\square$

The Continuum Coding Axiom also gives secure grounds by the same high-code and Ground Axiom argument [21]. Its continuum codes entail unbounded failures of GCH. The $\diamond^*$ preparation is what allows the later choice envelope to satisfy eventual GCH.

Lemma 12.4 (A fresh GCH envelope). *Suppose $M \models \text{ZF}$ and $B = M[K] \models \text{ZFC} + \text{CM}$ is a weakly homogeneous set-forcing extension with no new ordinal ω -sequences. If B satisfies eventual GCH, there is a further set-forcing extension $B' = B[H]$ such that*

$$B' \models \text{ZFC} + \text{GCH} + \text{CM}, \quad (\text{Ord}^\omega)^{B'} = (\text{Ord}^\omega)^M.$$

Moreover B'/M is a weakly homogeneous set-forcing extension.

Proof. In B , choose an inaccessible δ above all GCH exceptions. Use the following presentation of $L_\delta = \text{Col}(\omega_1, < \delta)$. A condition is a countable partial function p on $\delta \times \omega_1$ with $p(\beta, \xi) < \beta$,

ordered by reverse inclusion. All ordinals below δ are coordinates; the definition does not use a model's list of cardinals.

In B this forcing is countably closed, has size δ , and is δ -cc. For the last assertion, the generalized Δ -system lemma gives a common root for δ many countable domains. Inaccessibility bounds the number of value assignments on that root below δ . Two conditions therefore agree on the root and are compatible. Each nonzero coordinate adds a surjection from ω_1 onto that ordinal. Hence ω_1 is preserved, δ becomes ω_2 , and all cardinals at least δ are preserved.

There are no new reals, and the old continuum has size less than δ , so CH holds. A nice name for a subset of ω_1 uses ω_1 antichains, each of size less than δ . The entire name has size less than δ , so $\delta^{<\delta} = \delta$ bounds their number by δ . Thus $2^{\omega_1} = \delta = \omega_2$. For each old cardinal $\lambda \geq \delta$, a name for a subset of λ is coded by a subset of $\lambda \times L_\delta$. There are at most $(2^\lambda)^B$ such codes, and that cardinal is preserved. The old subsets give the opposite bound, so the continuum cardinal at λ is unchanged. This proves global GCH. Small forcing preserves all B -measurables above δ .

The models M and B have the same countable ordinal sequences, so they agree on ω_1 and on this literal presentation of L_δ . An enumeration of a condition is coded by one ordinal ω -sequence and therefore belongs to M . The same observation applies to the countable graphs of the permutations used below. No inaccessibility assumption on δ in M is needed.

Permute the second coordinate ω_1 to move the countable domain of one condition away from that of another. A permutation with countable support suffices, and belongs to M . It witnesses weak homogeneity of L_δ there. If $\mathbb{C} \in M$ is the old forcing for K , the product lemma identifies $B' = M[K, H]$ as an extension by the fixed product $\mathbb{C} \times L_\delta$. This product is weakly homogeneous. Countable closure is used in the choice model B , so it adds no ordinal ω -sequences. This proves all assertions. $\square$

Lemma 12.5 (Recognizing an envelope). *Suppose D is secure and $B = D(A, K)$ is a set-forcing extension of D satisfying ZFC. Choose η with $A, K \in V_\eta^B$. If U is a further set-forcing extension preserving V_η^B , then the actual class B is definable in U from the ordinal η by*

$$B = D(V_\eta^U).$$

Proof. The right-hand model is constructed inside B when $V_\eta^U = V_\eta^B$, and contains D, A, K . Thus it equals B . Security identifies D inside U as its mantle. $\square$

Remark 12.6. Definability of the class B does not imply $B \subseteq \text{HOD}^U$. Similarly, V_η^U is OD but need not belong to HOD. The decoder can therefore recognize the choice envelope without placing its new wellorderings in the target inner model.

13. EXACT EXTERNAL REALIZATION

Proposition 13.1 (Realization over a secure ground). *Suppose*

$$D \subseteq M = D(A) \subseteq B = M[K],$$

where $D \models \text{ZFC} + \text{CM}$ is secure, $A \subseteq \mu^\omega$, B is a set-forcing extension of D satisfying ZFC, and B/M is weakly homogeneous with no new ordinal ω -sequences. Then there is a further set-forcing extension U of B such that

$$U \models \text{ZFC} + \text{CM}, \quad \mathcal{H}^U = M, \quad M \models \text{SVC} + \text{CM}.$$

If B satisfies eventual GCH, one may also require $U \models \text{GCH}$.

Proof. If B satisfies eventual GCH, first apply Lemma 12.4 to obtain $B' = B[H]$ with GCH. Otherwise put $H = \emptyset$ and $B' = B$. In either case B'/M is weakly homogeneous and adds no ordinal ω -sequences. Since B is a set extension of D , so is B' , and we have $B' = D(A, K, H)$.

Choose η in this actual envelope B' so that $A, \langle K, H \rangle$ belong to $V_\eta^{B'}$. Apply Corollary 11.4 over B' , starting above $|V_\eta^{B'}|$, to code A , and call the extension U . Lemma 12.5, with the pair $\langle K, H \rangle$ as its second generator, gives

$$B' = D(V_\eta^U).$$

Thus the capped-history spectrum defines A in U . In the GCH case, Lemma 11.5 gives $U \models \text{GCH}$.

The combined forcing from M to U is weakly homogeneous. The name for the coding forcing is uniformly defined from the old A and ordinal parameters. Automorphisms of the first forcing carry it to a forced equal name and therefore lift to the iteration. The second forcing is forced weakly homogeneous. For a local name for a second-stage automorphism, use the identity off a condition forcing it to be an automorphism. This requires only localized Boolean mixing, without a choiceless maximum principle.

By hypothesis, the first extension adds no ordinal ω -sequences; the coding extension is highly distributive over the choice model B' . Lemma 10.2 gives $\mathcal{H}^U \subseteq M$. Conversely, $D \subseteq \text{HOD}^U \subseteq \mathcal{H}^U$ and $A \in \mathcal{H}^U$. The latter holds because it is OD in U and every member of A is an allowed ordinal-sequence parameter. Minimality of $D(A)$ gives $M \subseteq \mathcal{H}^U$.

The entire extension U/D is by set forcing. All sufficiently high normal D -measures lift to U , and their restrictions (2.1) belong to M . Thus both models have a proper class of measurables.

Finally, $B = M[K] \models \text{ZFC}$ is a set-forcing extension of M . The truth lemma gives a condition forcing Choice. Restricting the forcing to that condition shows that Choice is forceable over M . Blass's equivalence of this property with SVC, as stated in [7, Theorem 3.14], gives $M \models \text{SVC}$. $\square$

The next preparation allows us to apply this proposition without assuming that class forcing preserves DC in arbitrary choiceless intermediate models.

Proposition 13.2 (Installing the secure ground). *Suppose $M = D(A) \models \text{ZF} + \text{DC} + \text{CM} + \text{SVC}$, where $D \models \text{ZFC} + \text{CM}$ is a symmetric ground and $A \subseteq \mu^\omega$. Given an ordinal α , there are*

$$D^* \subseteq M^* = D^*(A) \subseteq B^* = M^*[K]$$

satisfying the hypotheses of Proposition 13.1, with B^ satisfying eventual GCH and*

$$V_\alpha^{M^*} = V_\alpha^M.$$

Proof. Take a fresh envelope $B = M[K]$ from Lemma 12.1. The symmetric-ground relation composes with set-forcing extensions. Since D and B satisfy Choice, D is a set-forcing ground of B by [7, Lemma 5.8]. Write $B = D[b]$ using a forcing presentation in D .

Let $\mathbb{P} \in D$ be the presentation with generic b . Choose in D a regular $\delta > \max(|\mathbb{P}|, \omega)$, above a uniform $\mathbb{P}$ -name bound for the cardinality of the protected B -rank. Such a bound is available before evaluating b , using a set of names for that rank. Choose the preparation $\mathbb{Q}$ of Lemma 12.3 in D , independently of b , with every set-sized initial segment $\mathbb{Q}_\beta < \delta$ -closed in D . Take a coherent class generic J over B for this fixed D -iteration in the set-name framework just specified. For every β , the ordinary product lemma gives

$$D[b][J_\beta] = D[J_\beta][b].$$

Since $\mathbb{P}$ is δ -cc, Easton's lemma makes the fixed $\mathbb{Q}_\beta < \delta$ -distributive in B . This does not assert that its ground closure persists in B . This commutation of small forcing with class preparation is used in [21, Appendix, Lemma 121].

Put $D^* = D[J]$. Every dense subset of the fixed $\mathbb{P}$ in D^* appears in some $D[J_\beta]$, where b is generic by the product lemma. Thus b is generic over D^* , and

$$B^* = D^*[b] = \bigcup_{\beta} B[J_\beta] \models \text{ZFC}, \quad M^* = D^*(A).$$

Every short ordinal sequence in B^* appears at a set stage. Distributivity at each such stage therefore shows that B^*/B adds none of length less than δ . It also preserves the protected rank: a first new set in that rank would be a new subset of an old set of cardinality less than δ , coded by its old wellorder as a short ordinal sequence. Thus, by freshness of B/M and the inclusions $M \subseteq M^* \subseteq B^*$,

$$(\text{Ord}^\omega)^{M^*} = (\text{Ord}^\omega)^{B^*} = (\text{Ord}^\omega)^M. \quad (13.1)$$

We claim

$$M^* \cap B = M. \quad (13.2)$$

Every set of $D^*(A)$ belongs to $M[J_\beta]$ for some set-sized initial segment of the class iteration. Indeed, take a bounded transitive-base construction witnessing membership, then a stage at which the required rank of D^* has stabilized. The old set forcings for K and J_β are mutually generic over M , so

$$M[J_\beta] \cap M[K] = M.$$

The usual product-intersection proof works over ZF : compare two names for a first new set of a given rank and use product conditions deciding membership. This proves (13.2). Together with preservation of the protected B -rank, it gives $V_\alpha^{M^*} = V_\alpha^M$.

The same filter K is generic over M^* for the *fixed* old collapse forcing. Each dense set in M^* appears in some $M[J_\beta]$, where mutual genericity makes K meet it. Also $B^* = M^*[K]$. One inclusion holds inside B^* . For the reverse inclusion, $D^* \subseteq M^*$ and $b \in B = M[K] \subseteq M^*[K]$ give $D^*[b] \subseteq M^*[K]$. The old collapse automorphisms still witness weak homogeneity, and (13.1) shows that there are no new allowed parameters.

By Lemma 12.3, D^* is secure and satisfies $\text{ZFC} + \text{CM}$ and eventual GCH. Its set-forcing extension B^* also satisfies eventual GCH: above the forcing size, the cardinals and continuum cardinal values are unchanged. Thus the GCH clause of Proposition 13.1 applies. The set-forcing identity $B^* = M^*[K]$ also yields $M^* \models \text{SVC}$ by Blass's equivalence. We have not assumed that the class preparation preserves DC in M^* . Once M^* is identified with $\mathcal{H}^U$, Lemma 10.1 gives DC. $\square$

Proof of Theorem 1.3. From SVC, take a choice symmetric ground D and a transitive generator T with $M = D(T)$; see [7, Proposition 5.4]. The choice ground may be taken to have a proper class of measurables: make a set-generic choice envelope, use Lemma 2.1, and apply downward Lévy–Solovay to the resulting choice-ground presentation. Alternatively, in the application from Theorem 1.4, the ground D already has this property.

By $\text{KW}_1^{*,\omega}$, choose a surjection $f : (\mu^\omega)^M \rightarrow T$. Code its domain and the relation $f(p) \in f(q)$ by a single set A of ordinal ω -sequences. Equality of predecessor sets and the well-founded extensional collapse, as in Proposition 7.5, recover T and f . Hence $M = D(A)$. Apply Proposition 13.2 and then Proposition 13.1. $\square$

14. THE THREE CONSISTENCY IMPLICATIONS

Proof of Theorem 1.1. The implication $\text{Con}(E_2(\sigma)) \Rightarrow \text{Con}(E_1(\sigma))$ follows directly from the definitions. If $U \models E_2(\sigma)$, Lemma 10.1 gives $\mathcal{H}^U$ as a model of $E_3(\sigma)$. Hence $\text{Con}(E_2(\sigma)) \Rightarrow \text{Con}(E_3(\sigma))$.

Proposition 10.3 gives $\text{Con}(E_1(\sigma)) \Rightarrow \text{Con}(E_3(\sigma))$ without assuming that the original external HOD already contains a proper class of measurables.

For the remaining direction, let $V \models E_3(\sigma)$ and choose κ with $V_\kappa^V \models \sigma$. Apply Theorem 1.4 above κ to obtain a sequence-generated model M with $V_\kappa^M = V_\kappa^V$. Apply Theorem 1.3 while protecting that rank. The resulting U satisfies $\text{ZFC} + \text{GCH} + \text{CM}$, and

$$\mathcal{H}^U \models \text{SVC} + \text{CM}, \quad V_\kappa^{\mathcal{H}^U} = V_\kappa^M = V_\kappa^V.$$

Hence $U \models E_2(\sigma)$, completing the cycle of implications. $\square$

Why this is an external theorem. The original Prikry ambient universe contains the master sequence c . Every external $\mathcal{H}$ contains all ambient ordinal ω -sequences, including c . We therefore cannot simply take the final external HOD in that universe. The master can expose real-coded information omitted from the intermediate model. Lemma 12.1 provides a *fresh* choice envelope with no new ordinal ω -sequences.

The high witnesses in Section 11 need not belong to the final $\mathcal{H}$. Their ambient existence makes A definable, and weak homogeneity controls hereditary descent. The proof requires neither tree-Prikry completions of the Cohen caps nor a unique high witness for each code.

Proof dependencies and scope. The construction uses two nonstandard forcing results, Lemma 6.2 and Theorem 11.1. The first uses tagged finite interpolation and a countable diagonal argument. The second applies an existing approximation criterion to the displayed quotient order. Both are needed, for different parts of the proof. Lemma 8.10 supplies the separate fine-structural reduction. The following qualifications describe their scope.

- (1) The global rank-preservation proof uses literal equality of Q, F and of supported parameters, not merely isomorphism of their indexing structures.
- (2) The DC proof uses external countable domination and wellorderable search spaces; it does not assert that every ambient countable sequence belongs to the inner model.
- (3) The cap spectrum is computed over the actual recognizable envelope. It requires no classification of arbitrary incomparable grounds.
- (4) The secure preparation uses repeated $\diamond^*$ codes and retains a proper class of measurables. A subsequent countably closed set collapse gives global GCH in the choice envelope; the final evaluated cap forcing preserves it. The protected rank belongs to the target inner model and need not satisfy GCH.
- (5) The conclusion concerns first-order sentences true in a rank segment. It neither identifies the complete theories of the three ambient universes nor proves the internal assertion $V = \text{HOD}_{\text{Ord}^\omega}$ in the original model.

15. A CANONICAL RELATIVE INNER MODEL WITH UNBOUNDED WOODIN CARDINALS

The reverse consistency argument needs a choice model containing a prescribed real, with unbounded Woodin cardinals, whose definition is unchanged by a finite recursive recoding of that real. It also needs this model to be a set-forcing extension of its own HOD. We establish these requirements together. The fine structure used below is the classical Mitchell–Steel short-extender fine structure of pure-extender premice. The model we construct has no strategy predicate.

For a real r , put

$$a(r) = \text{HYP}(r) = L_{\omega_1^{\text{CK}, r}}[r].$$

Here $\text{HYP}(r)$ denotes the entire transitive admissible set, including its ordinals, rather than its collection of reals. It is countable, transitive, and self-wellordered in the usual relative-premouse sense. The normalization by $a(r)$ will remove the choice of a real code from the construction.

Convention 15.1. We use classical premice over the named transitive base a . Every relative Skolem hull contains $a \cup \{a\}$, and every extender has critical point above $\text{rank}(a)$. Cores, projecta, standard parameters, and ultrapowers are taken in this one convention throughout. An enumeration or wellorder of a may be used to prove a property of a premouse; it is not an additional predicate of that premouse.

For a class premouse C , *countable iterability* means that every countable elementary pullback of a set initial segment, with the base fixed pointwise, has the ordinary Mitchell–Steel stack

iterability in the applicable degrees. The game allows taking an initial segment at the start of a round. This is the countable-hull convention of [40, Section 1.1]; the relative fine-structural Łoś and copying conventions are those of [41, Section 2.5].

Theorem 15.2 (Canonical Woodin envelope). *There is a uniform first-order class definition $F(a)$ such that $\text{ZFC} + \text{CW}$ proves the following, with assertions of ZFC understood schematically. For every real r , if $a = \text{HYP}(r)$ and $C = F(a)$, then:*

- (i) C is a transitive inner model containing a as a set, and $C \models \text{ZFC} + \text{CW}$;
- (ii) the definition uses only the pure set a , with no chosen enumeration, background witness, or iteration strategy;
- (iii) C is a set-forcing extension of HOD^C .

Consequently, in the same ambient universe,

$$r \equiv_T s \implies F(\text{HYP}(r)) = F(\text{HYP}(s)).$$

The equality in the last assertion is literal equality of the resulting classes. It is a same-universe assertion; no generic absoluteness of F is asserted or needed. We prove the theorem in four steps.

15.1. The standard background construction and its stopping rule. We use the fully backgrounded, minimum-support Mitchell–Steel construction, with its usual core and limit clauses [34]. The explicit version in [39, Section 2.3] fixes the transition rule we need. Suppose that the current sound premouse M_ξ is passive and has height α . An active extension with fine extender E and support $\nu < \alpha$ is eligible when (M_ξ, E) is a premouse and there is a total background extender E^* over the ambient universe such that

$$V_{\nu+\omega} \subseteq \text{Ult}(V, E^*), \quad E \restriction \nu = E^* \cap ([\nu]^{<\omega} \times M_\xi). \quad (15.1)$$

Among eligible extensions, the construction takes one of least support. It first forms the raw active level $N_{\xi+1}$, verifies reliability, and then takes its ω -core. If there is no active extension, or the current level is active, it instead forms the usual raw passive Jensen successor before the same reliability and core operations.

At a limit λ , the raw level has height

$$\eta_\lambda = \liminf_{\xi \rightarrow \lambda} (\rho_\omega(M_\xi)^+)^{M_\xi};$$

its bounded initial segments are the stabilized bounded initial segments of the earlier levels. Reliability is again checked before taking the core. We retain the raw levels and the decoded cores as distinct objects. This distinction matters when a core is represented by a type III code.

We add diagnostic stopping tests, without changing a transition which passes those tests:

- (a) Test each raw level for nontameness before testing reliability or taking a core. Test the decoded core as well.
- (b) At a passive core, stop if any eligible active head is nontame, before selecting a minimum-support head.
- (c) Stop if a core already has a sound tame active type I proper initial segment whose top critical point is a limit of Woodin cardinals above the base.

At a stop, we will choose the least index of a segment as in (c) on the already determined sequence. The extraction argument below proves that such a segment exists in every genuine stopping case.

We record precisely the classical fine-structure inputs to this construction. The tame extension of Mitchell–Steel fine structure gives reliability, solidity and universality of the relevant cores, and countable-hull stack iterability for tame backgrounded levels and their associated comparison objects [35]. In particular, it applies to the associated bicephali used in next-extender uniqueness;

this scope of the tame theory is also explained in [36, p. 9]. The assertion is about tame levels of the construction and their associated objects. It does not require that every premouse in the universe be tame.

Before we continue at a passive core, test (b) has excluded every nontame eligible head. Two eligible heads of minimum support therefore give an associated tame bicephalus to which the joint iterability and next-extender uniqueness theorems apply. The classical category check is useful here: type II has successor natural support, whereas types I and III have limit natural support. Equal minimum support places the two heads in the same category covered by the classical uniqueness theorem. Thus the fine extender is unique. We use joint iterability of the associated bicephalus, not an inference from separate iterability of its two heads.

The witness E^* in (15.1) is only existential data. Once the fine head is unique, different witnesses do not change the next premouse. Jensen successor, core-taking, the $\liminf$ clause, and the diagnostic tests are uniform from the current set structures. The resulting recursion is therefore definable from a alone. No wellorder of the possible background witnesses is used.

Since a is countable in the ambient universe, the critical point of any total background extender, being measurable there, exceeds $\text{rank}(a) + \omega$. This bound may be used in the relative fine-structure calculations without adding an eligibility restriction to (15.1).

Lemma 15.3. *If the construction never stops, its stabilized class limit is a countably iterable inner model of $\text{ZFC} + \text{CW}$ containing a .*

Proof. Every raw level and decoded core is tame. The tame fine-structure theorem therefore supplies reliability at the point where the ordinary construction requires it, so there is no breakdown. The standard core-stabilization theorem gives the class premouse limit.

We now apply the Woodin-inheritance theorem for this very construction: the fully backgrounded Mitchell–Steel construction, when it does not break down, inherits the ambient Woodin cardinals. This is the theorem recalled at the opening of [38]. In the relative version, the Woodins above the base are inherited, which suffices for CW. All extender critical points fix a , so the usual reflection and background-restriction proof uses exactly the same bounded data above the base. In particular, we have not imposed an extra inaccessible-length or certificate requirement on (15.1). Such a restriction would require a separate inheritance argument.

The classical fine structure and stabilization give ZFC for the proper-class limit. To check the required countable iterability, fix a set initial segment and a countable pullback. A sufficiently late tame stage contains its stabilized segment. Include the segment and the pullback map in the ambient hull used for that stage’s iterability theorem, and use the initial-segment move and copying. This supplies the required stack strategy on the pullback. $\square$

15.2. Extracting a tame top measure at the first overlap. The stopping alternative does not require an iterable nontame mouse. It uses a measure fragment already present on a determined sequence. We give the endpoint details because the Woodin overlapped by the new extender can equal its natural support.

Lemma 15.4 (First-overlap extraction). *Suppose a first nontame active level $Q = (P, E)$ is encountered in the construction above, either as an eligible head or inside a decoded core. Then the already determined sequence below that head has a sound tame active type I segment S whose top critical point is a limit of Woodin cardinals. The countable elementary pullbacks of S have the inherited stack iterability. The least index of such a segment is determined by the preceding core, without choosing a bad head or a countable hull.*

Proof. Choose the first offending extender index in the indicated premouse. Write

$$\kappa = \text{crit}(E), \quad \nu = \nu(E), \quad \alpha = \text{lh}(E), \quad P = J_\alpha[\vec{E}, a].$$

There is a $\delta \geq \kappa$ which P regards as Woodin and which is overlapped by E . All earlier active initial segments are tame. Let i be the degree-zero ultrapower map by E , and put $R = i(J_\kappa^P)$. The coherence and indexing axioms give

$$J_\alpha^R = P, \quad \alpha = (\nu^+)^{\text{Ult}_0(P, E)}, \quad \alpha < i(\kappa). \quad (15.2)$$

Only the wellfounded part required by these axioms is used at this point; no iterability of the newly adjoined nontame extender is assumed.

Acceptability gives the following useful endpoint calculation. If $\tau < \alpha$, every subset of τ in R is already in J_α^R . Otherwise the first later level at which such a subset appeared would give a surjection from τ onto α , contradicting that α is a cardinal of R . Also P has no cardinals above ν in this indexed active situation [37, Remark 2.5]. Hence $\delta \leq \nu < \alpha$. The two structures agree on subsets of δ and on the bounded extender witnesses used to test its Woodinness. Thus R regards δ as Woodin, including when $\delta = \nu$.

The restriction of i to the coded set J_κ^P is fully elementary as a map of set structures. Indeed, for each fixed formula, relativizing all its quantifiers to J_κ^P gives a bounded formula; the degree-zero map preserves that formula and sends the displayed set to R . For every $\beta < \kappa$, R has a Woodin cardinal between β and $i(\kappa)$. Reflecting this assertion to J_κ^P shows that κ is a limit of Woodin cardinals. Acceptability transfers these bounded witnesses to P and to the relevant proper initial segments.

We must have $\delta > \kappa$. If $\delta = \kappa$, the proper segment through $(\kappa^+)^P$ would have the same subsets of κ and would regard κ as a Woodin limit of Woodin cardinals. That segment is tame and has iterable countable pullbacks. This contradicts the tame-smallness fact recorded immediately after [37, Definition 6.15]: a tame mouse cannot have a Woodin limit of Woodin cardinals.

Set $\eta = (\kappa^+)^P$. Then

$$\kappa < \eta < \delta \leq \nu.$$

Apply the initial-segment condition to $E \restriction \eta$. Its natural support is η , so this is not the type Z exceptional fragment. The condition puts its trivial completion either earlier on the sequence of P , or on the one-step ultrapower using a nonempty extender indexed at η [37, Definition 2.4(3)]. The second alternative is impossible: η is a cardinal of P , whereas a nonempty extender at that index makes it noncardinal at the next Jensen stage, which still lies below α . Thus the completion occurs at an earlier index $\gamma < \alpha$ on the sequence of P .

The active segment at γ is the desired S . Its extender is the normal-measure fragment with critical point κ , hence is type I. It agrees on the bounded sets below κ , so its top critical point remains a limit of Woodin cardinals. It is tame by the choice of the first offending index, and it is sound as a proper initial segment.

Here is the iterability accounting. When P is a passive decoded core of a tame background level N , a countable hull of P maps weakly into $C_\omega(N)$. The tame level theorem supplies its stack iterability, and the initial-segment choice before the first round supplies that of the corresponding pullback of S .

If instead the decoded core itself is nontame, take an ambient countable hull containing the *unsquashed* core, its code, the chosen overlap witness, and the relevant set satisfaction relations. Collapse this data together. The overlap witness survives even when it uses a type III top extender and the endpoint $\delta = \nu$ lies outside the ordinal domain of the squashed code. The restriction on the codes is the weak core embedding to which the tame predecessor's iterability theorem applies. We may therefore take the first bad active initial segment and then the proper segment S just found. This uses iterability inherited from the preceding tame level, not an assumed strategy on a newly adjoined bad head.

Finally, the fragment belongs to the fixed preceding sequence. The existence of a bad eligible head is used only to prove that a suitable index exists there. Taking the least suitable index

makes S independent of the choice of that head. Countable hulls are used to verify iterability, never to select the output. $\square$

For completeness, the raw-level tests create no further stopping case. A first nontame passive successor would already have its local overlap witness on the previously tested indexed extender sequence. At a first nontame limit, that witness is bounded and occurs in a stabilized earlier segment. Thus neither can be the first new failure. The tests are nevertheless made before reliability, so the proof never takes a core whose existence has not been justified.

15.3. Iterating the top measure out of the model.

Lemma 15.5 (Top-measure envelope). *Let S be a sound tame active type I premouse over a , with inherited countable-hull stack iterability, whose top critical point κ is a limit of Woodin cardinals above the base. There is a class premouse C_S , uniformly definable from S , containing a , whose pure universe satisfies $\text{ZFC} + \text{CW}$. Its set initial segments have the countable iterability of Convention 15.1.*

Proof. Form the linear degree-zero iteration of the top measure. Every set-length history is wellfounded. For if a first bad history existed, take a countable elementary hull of a sufficiently large ambient H_Θ containing the history, its maps and direct-limit data, and an actual descending ω -sequence in the illfounded last structure. The collapse is a bad countable linear history of a countable pullback of S , contradicting its inherited stack strategy. This argument also covers an uncountable S : the starting model in the collapsed history is its iterable pullback, rather than S itself. The unique compatible set recursions therefore define the linear iteration through all ordinals.

Write S_ξ for the iterates, $i_{\xi\zeta}$ for their maps, and $\kappa_\xi = i_{0\xi}(\kappa)$. We will show that

$$B_\xi = J_{\kappa_\xi}^{S_\xi} = V_{\kappa_\xi}^{S_\xi}, \quad B_\xi \prec B_\zeta \quad (\xi < \zeta), \quad (15.3)$$

where the elementary map in the second assertion is literal inclusion. Some care is needed because the entire active premouse S has not been assumed to satisfy ZFC .

First, κ is internally regular in the sense required below. If S contained a cofinal $f : \lambda \rightarrow \kappa$ with $\lambda < \kappa$, the next degree-zero map j would send it to a cofinal map from the same λ to $j(\kappa)$. This statement is bounded in its displayed parameters. But $j(f)(\xi) = j(f(\xi)) = f(\xi) < \kappa$ for every $\xi < \lambda$, a contradiction. The same argument rules out a bijection from a smaller ordinal onto κ .

The cofinal Woodin cardinals and acceptability give strong-limit closure below κ . Given $\tau < \kappa$, choose a Woodin between τ and κ . Its bounded inaccessible-rank calculation locates the power set of τ and a bound on its size below that Woodin; acceptability excludes new subsets at later levels. Consequently the usual acceptable-hierarchy calculation above a self-wellordered base gives $J_\kappa^S = V_\kappa^S$. Each of its elements has an internal wellordering of order type below κ . All its power sets agree with those computed in S and belong to J_κ^S .

Put $B = J_\kappa^S$. The type I top index lies beyond κ^+ , so B itself and the finite Jensen successor closures used next are sets in S . Fix a formula φ and parameters from B , and suppose that, in B , it defines a function on $x \in B$. Relativize all quantifiers of this one formula to the set B . The resulting graph is a bounded-definable subset of $x \times B$, and belongs to a finite Jensen successor closure above B , still inside S . There is no appeal to Replacement in S in forming this graph.

Assign to each value its least construction level below κ . The hierarchy through κ and this least-level graph also lie in the finite successor room above B . If these levels were unbounded in κ , composition with an internal enumeration of x of length $\lambda < \kappa$ would give, in S , a cofinal map $\lambda \rightarrow \kappa$. Internal regularity excludes this. Thus the range is bounded in some J_η^S , $\eta < \kappa$. Coding that level by a set of size below κ and using the power-set calculation puts the range in

B . This proves each Replacement instance. The same bounded-graph argument followed by the subset calculation proves Separation. The remaining elementary axioms and Choice follow from the acceptable relative hierarchy and the wellordering of the base. Hence $B \models \text{ZFC}$.

The Woodins cofinal in κ remain Woodin in B : their subsets, functions, and bounded extender witnesses agree. Therefore $B \models \text{CW}$.

Fine ultrapower coherence gives, for $\xi < \zeta$,

$$i_{\xi\zeta}(J_{\kappa_\xi}^{S_\xi}) = J_{\kappa_\zeta}^{S_\zeta}.$$

These hierarchy sets are actual elements of the preimage. Bounded relativization to them proves full elementarity of the maps between the set structures B_ξ and B_ζ , and transfers $\text{ZFC} + \text{CW}$ from B_0 to every B_ξ .

The maps fix B_ξ pointwise. To see this, argue by rank and take an internal enumeration $e : \lambda \rightarrow x$ with $\lambda < \kappa_\xi$ for $x \in B_\xi$. By induction its values are fixed, while bounded elementarity says that $i_{\xi\zeta}(e)$ maps the same domain onto $i_{\xi\zeta}(x)$. Thus $i_{\xi\zeta}(x) = x$. The base is fixed by convention. This proves (15.3) with inclusion as the elementary map.

The critical sequence is strictly increasing and unbounded in the ordinals. Define

$$C_S = \bigcup_{\xi \in \text{Ord}} B_\xi.$$

It is transitive and contains all ordinals. The elementary-chain argument applies to each formula: an existential witness in the union lies in a later stage and can be reflected to any earlier stage containing its parameters. Thus $B_\xi \prec C_S$, and $C_S \models \text{ZFC} + \text{CW}$. The stabilized extender sequence presents it as the class premouse obtained by iterating out the top measure.

For countable iterability of a set initial segment of C_S , include that segment, its pullback map, a sufficiently high top iterate, and the preceding linear history in an ambient countable hull. The collapsed starting premouse has its given stack strategy. Following the collapsed top history, and then taking its tail and the indicated initial segment, supplies the required strategy. This proves the countable-hull assertion. It does not assert a single strategy on the entire class premouse, and none is needed. $\square$

15.4. The fixed-base own-HOD argument. The own-HOD step requires more than choosing a canonical extender sequence externally. We must define that sequence over the pure universe of the resulting premouse from bounded set parameters. We use the direct condensation-stack argument of [43, Theorem 4.1 and Remark 4.3], with the trivial forcing. The following proof gives the fixed-base and local-iterability details of that application.

Proposition 15.6. *Let C be a proper-class classical pure-extender premouse over a transitive self-wellordered base a . Suppose that its pure universe satisfies ZFC , that a is countable both in C and in the ambient universe, and that C has the countable iterability of Convention 15.1. Then, for some set of ordinals $t \in C$,*

$$C = \text{HOD}_{\{t\}}^C.$$

In particular $C = \text{HOD}^C[h]$ for set forcing in HOD^C .

Proof. Choose in C an enumeration e of a . Fix cardinals $\gamma < \theta$ of C , with θ regular, so far above the base that a , e , and a hierarchy segment coding them are bounded below γ . Put $m = C \restriction \theta$. These are parameters for this internal proof; they do not change the external definition of C .

Local condensation. The ordinary finite-degree solidity, universality, and sound-source condensation theorems hold with the base fixed [34, 40]. Each required instance on an uncountable set segment follows by reflection to a countable ambient hull containing the structures, the map, their set satisfaction relations, and a pointwise. Countable iterability supplies the target strategy there. This derives local condensation without selecting a class strategy for C .

For the sound-hull construction we also need the stronger condensation theorem for a ρ -sound source, which need not yet be fully sound. We use the passive degree-zero case of [42, Theorem 5.2]. Retain its lifting, projectum, cardinal, and ρ -soundness hypotheses, place a in each hull and fine theory, and require the maps to fix a pointwise. The proof gives the same alternatives in this relative setting. We describe why its stronger conclusion, rather than just ordinary sound-source condensation, is available.

The solid-core parameters and fine theories are formed using tuples from a as well as the ordinal parameters. Relative $\aleph_0$ and the classical hull calculations preserve them. In the equal-successor-cardinal case of that proof, the required agreement of power sets and generalized solidity witnesses is unchanged, since the base is fixed and has its enumeration below our cardinal floor. The condensation used for already sound proper segments is the ordinary theorem just established; the stronger conclusion is not assumed at this point.

In the other case, the bicephali and cephalanxes have the same named base. The iterability proof starts with the source embedding and the identity into one iterable target, as in Claim 5 of the proof of the cited theorem. The Shift maps send $[u, f]$ to $[\pi(u), \pi(f)]$ and fix every element of a . Model and degree drops take initial segments over the same base; direct limits retain the fixed base. If a type III structure occurs in a comparison, the usual squashed copying and unsquashing are performed with this same fixed-base $\aleph_0$ calculation.

The subsequent comparison analysis [42, Theorems 4.3, 4.5, 4.17 and the proof of Theorem 5.2] uses least disagreement, extender lengths and generators, cardinal cutpoints, and the solid-core calculations. All these computations take place above the base. Replacing an ordinal hull by its hull together with a does not repair an omitted-generator contradiction: the maps already have every base element in their range. The same terminal alternatives follow. In the passive degree-zero application these are the initial-segment and active-ultrapower possibilities, including the type I normal-measure exception. The active type II final alternative is absent. This supplies the stronger local condensation needed below. The argument has used classical relative premise throughout; no conversion to an operator-premise convention is required.

Sound hulls above the base. Repeat the finite sound-hull construction of [43, Lemma 2.3]. In a passive 1-sound target H with $\rho_1^H \geq \gamma$, put a into each generating set and include the desired finite data among its prescribed parameters. The recursion adjusts a finite decreasing ordinal parameter and a cardinal bound γ_i , maintaining $\gamma \leq \gamma_i$. At a nonterminal step it chooses a noncardinal new generator $\eta > \gamma$ while putting the earlier solidity witnesses in the appropriate preceding hull. The resulting collapse W is ρ -sound, with

$$\rho = \gamma_i^{+W}, \quad \rho_1^W = \gamma_i;$$

it has not at this stage been asserted 1-sound.

Apply the stronger condensation just proved in the specialization used by that lemma. Either W is an initial segment of H , or the remaining exceptional case presents W as a type I degree-zero ultrapower with critical point γ_i . The exception contradicts the choice of the new generator: that ultrapower is generated by the old range and its normal-measure seed γ_i , with the standard-parameter tail carried along. Since the old range already contains a , the proposed new generator would be in the hull generated by a , $\gamma_i + 1$, and that tail.

Thus W is an initial segment. Its description as the least relevant projecting segment makes the next solidity witness available in the next hull. Each nonterminal step strictly decreases the cardinal bound. The bound remains at least γ , so the recursion terminates with a sound collapse containing the prescribed objects in its range. In particular it never crosses the base. This proves soundness for these adjusted hulls, rather than for arbitrary hulls.

The simultaneous version is obtained by alternating these parameter adjustments for two passive successor structures with the same pure universe and the same initial segment m . Include

a, e , the two preceding structures, and a prescribed element x in the common parameters. At each adjustment retain the previously arranged ordinal agreement and mutual containment. The construction in [43, Theorem 4.1, Subclaim 1.1], with trivial forcing, then gives sound collapses of the same ordinal height whose uncollapse maps agree on ordinals and on a . The enumeration e ensures that the common pure-universe collapse also fixes the whole base.

The internal regular stack. Let $\eta \geq \theta$ be regular in C . Compute inside C the Jensen stack of sound classical a -premise extending $C \restriction \eta$, projecting to η , and satisfying the condensation properties above. This stack is exactly $C \restriction \eta^{+C}$.

Here both the internal qualification and the equality are important. For comparability, put the base pointwise into the common hull in the proof of [43, Fact 3.1]. Regularity makes the hull's intersection with η an ordinal. The two collapse maps have the corresponding common critical point, and their projecta collapse to it. Sound-source condensation places the collapses in the same initial segment of the root, or in the same exceptional ultrapower of that root, giving comparability. For a candidate coded in C , compare with a sufficiently high true sound projecting segment containing it. The argument of [43, Fact 3.2] puts the candidate on the true sequence. Conversely, the acceptable-hierarchy calculation gives cofinally many sound projecting levels below η^{+C} : the bounded power sets are already fixed below η , and the levels adding the relevant subsets of η project to η . Their union is the true successor-cardinal segment. Its pure universe is $H_{\eta^+}^C$. We have made no assertion about a stack containing every externally available mouse.

Uniqueness over the pure universe. Use the excellent-candidate definition in the direct proof of [43, Theorem 4.1], restricted to the fixed base a and trivial forcing. A candidate extends m , has regular height η , has the local condensation just used, and has a successor-cardinal stack whose pure universe is $H_{\eta^+}^C$. The true $C \restriction \eta$ is a candidate by the preceding paragraph.

If R is another candidate, alternately take the least appropriate projecting extensions on the two sides containing the preceding structure on the other side. The resulting ω -unions, and then their passive successors, have the same pure universe. For each x in this universe, take the simultaneous sound hulls just constructed. Condensation places both collapsed mice in the common initial segment m . They have the same ordinal height, hence are the same premouse. Their uncollapse maps agree on ordinals and on a ; relative Skolem generation makes the maps agree everywhere. Varying x identifies the two successor structures and hence identifies R with $C \restriction \eta$.

Excellence is expressed using set-coded premeice, their satisfaction relations, and their bounded stacks. Thus this uniqueness defines $C \restriction \eta$ over the pure universe C from m, a, e , uniformly for all sufficiently large regular η . Taking their union defines the extender sequence of C from those set parameters. No iteration strategy for a possibly spurious candidate is assumed: its required condensation is part of the candidate definition, and the adjusted-hull argument uses those properties directly.

The relative Jensen hierarchy, together with the fixed enumeration of the base, now gives a global wellorder definable over C from m, a, e . Every element is therefore hereditarily ordinal-definable from this tuple. Choice in C permits coding the tuple and its transitive closure by one set of ordinals $t \in C$, giving $C = \text{HOD}_{\{t\}}^C$. Vopěnka forcing over HOD^C , for the bounded ordinal code t , gives a generic filter $h \in C$ with

$$\text{HOD}^C[h] = \text{HOD}_{\{t\}}^C = C.$$

This is the coarse own-HOD observation used in [44, Introduction]; no additional hypothesis for its subsequent refined conclusions is used here.

The proof assumed ZFC only in the pure universe. All candidates, local satisfaction relations, and bounded stacks were sets. Codes for true initial segments belong to C by its premouse hierarchy and amenability. After extender definability has been proved, formulas in the expanded

premouse language can be translated into pure formulas with the fixed parameters. Expanded-language Separation and Replacement thus follow at the end; they were not assumed to obtain the definition. $\square$

15.5. Completion and recoding invariance.

Proof of Theorem 15.2. Run the construction of Section 15.1 over the pure set $a = \text{HYP}(r)$. If it never stops, use its stabilized limit from Lemma 15.3. If it stops, choose the least qualifying type I segment on the determined sequence, using Lemma 15.4, and use its top-measure envelope from Lemma 15.5. The transition rule, stopping tests, least-index choice, and unique linear iteration are all first-order uniform from a . This defines $F(a)$ without any additional choice parameter.

In both cases $C = F(a)$ is a countably iterable classical premouse containing a , and its pure universe satisfies $\text{ZFC} + \text{CW}$. Since $r \in a \in C$, we have $L[r] \subseteq C$. The ordinal $\omega_1^{\text{CK},r}$ and the set $L_{\omega_1^{\text{CK},r}}[r]$ are countable in $L[r]$; their enumerations therefore belong to C . Thus a is countable in C as well as in the ambient universe. Proposition 15.6 applies and supplies the own-HOD set-forcing presentation.

Finally, if $r \equiv_T s$, every admissible set containing r contains s , and conversely: the recursive computation is available inside the admissible set. The least transitive admissible sets containing the two reals are consequently identical. Thus $\text{HYP}(r) = \text{HYP}(s)$, and the uniform pure-set definition gives the stated equality of envelopes. $\square$

Remark 15.7. This construction retains fine structure. Its role is confined to the tame background construction, the first-overlap fragment, and the condensation needed for the own-HOD presentation. It requires neither the iterability of an arbitrary nontame premouse nor a strategy on the ambient universe. The enumeration used in the own-HOD proof is internal evidence about an already determined model; it is not part of the definition whose recoding invariance is used in the consistency argument.

16. FROM EXTERNAL HOD TO A WOODIN SYMMETRIC GROUND

We apply the relative inner-model theorem to a real coding a rank of external HOD. The recoding invariance is needed inside a single collapse extension. No absoluteness of the relative construction between different universes is used.

Proposition 16.1. *Suppose $U \models \text{ZFC} + \text{CW}$, let $H = \mathcal{H}^U$, and fix an ordinal α . In a set-generic extension of U there are transitive classes $D \subseteq N$ such that*

$$D \models \text{ZFC} + \text{CW}, \quad N \models \text{ZF} + \text{DC} + \text{SVC}, \quad V_\alpha^N = V_\alpha^H.$$

Moreover, N has a set-symmetric presentation over D .

Proof. Choose $\lambda > \alpha + \omega$ with $\text{cf}^U(\lambda) > \omega$, and put $X = V_\lambda^H$. By Lemma 10.1, H is closed under U -countable sequences of its members. The cofinality condition then gives

$${}^\omega X \cap U \subseteq X. \tag{16.1}$$

Indeed, the ranks of the countably many entries have a bound below λ , and adding the finitely many levels needed to code the sequence still leaves a rank below λ . The set X is ordinal-definable in U from λ .

Force over U with the finite partial functions from ω to X . Let $e : \omega \rightarrow X$ be the generic enumeration. Let r code both relations

$$i E_r j \iff e(i) = e(j), \quad i R_r j \iff e(i) \in e(j).$$

Set forcing preserves the sufficiently high Woodins, so $U[e] \models \text{ZFC} + \text{CW}$ by [45]. Apply Theorem 15.2 there to obtain

$$C = F(\text{HYP}(r)), \quad C \models \text{ZFC} + \text{CW}, \quad C = \text{HOD}^C[h].$$

Since $r \in C$, the extensional quotient of (ω, R_r) by E_r and its Mostowski collapse belong to C . Their collapse is the actual transitive set X . Thus $X \in C$. Define

$$N = \text{HOD}_{X \cup \{X\}}^C.$$

Here definitions may use X , finitely many of its elements, and ordinals. Relative hereditary definability gives $N \models \text{ZF}$, $X \in N$, and $\text{HOD}^C \subseteq N$.

Descent to the original external HOD. Finite permutations of ω witness weak homogeneity of the collapse. If π is such a permutation, the real coding $e \circ \pi$ is Turing equivalent to r . Consequently

$$\text{HYP}(r_{e \circ \pi}) = \text{HYP}(r_e), \quad F(\text{HYP}(r_{e \circ \pi})) = F(\text{HYP}(r_e)).$$

Both equalities are evaluated in $U[e]$, which is also $U[e \circ \pi]$. The class C and the parameters X and $x \in X$ are therefore fixed by these recodings.

Induct on hereditary rank to show $N \subseteq H$. If $y \in N$ and all members of y have already descended to U , then y is a subset of a ground rank V_β^U . Choose a relative-HOD definition of y over C , evaluated at a sufficiently large set-sized rank. The formula uses X , finitely many elements of X , and ordinals. The preceding invariance and weak homogeneity make its membership decisions independent of the collapse condition. The forcing relation defines y in U from these same parameters. Since X is OD in U and its finitely many specified elements belong to H , their ordinal-sequence definitions can be combined into a single ordinal-sequence parameter. Hence y is definable in U from an allowed parameter. The induction hypothesis supplies the hereditary requirement, proving $y \in H$.

Transitivity and $X \in N \subseteq H$ now give

$$V_\lambda^N = X = V_\lambda^H, \quad V_\alpha^N = V_\alpha^H.$$

Small violations of choice and dependent choice. The bounded evaluation map for relative-OD definitions gives $\text{SVC}(X)$ in N . To spell out its parameter bound, fix a nonempty $Y \in N$ and a member $y_* \in Y$. Collection in C bounds ranks witnessing relative-OD definitions of all members of Y . Formula codes, stage ordinals, and ordinal parameters are absorbed into an ordinal η . Finite tuples of elements of X can be coded by elements of X . Thus an evaluation map in N gives a surjection

$$f : \eta \times X \twoheadrightarrow Y.$$

Send invalid codes to y_* . The graph is relatively OD in C from Y , y_* , X , and the ordinal bounds, and all its hereditary constituents belong to N . Thus $f \in N$. This does not choose a wellordering of X .

Let $R \in N$ be a serial relation on a nonempty set $Y \in N$, and fix $y_0 \in Y$. Use such a surjection f . In U , having chosen y_n , take the least $\xi_n < \eta$ such that

$$\exists x \in X \quad y_n R f(\xi_n, x),$$

choose one such x_n using choice in U , and put $y_{n+1} = f(\xi_n, x_n)$. All these operations take place in U , since $R, Y, f \in N \subseteq U$. By (16.1), the whole sequence $\langle x_n : n < \omega \rangle$ belongs to X , and hence to N . Inside N , reconstruct the y_n by recursion using this sequence, y_0 , f , R , and the least-ordinal rule. Its defining quantifiers range over the same sets in N and U , so the reconstruction gives exactly the chosen chain. This proves DC in N . No bound on η by λ is needed.

The Woodin ground. Set $D = \text{HOD}^C$. Theorem 15.2 gives a set forcing $P \in D$ and a D -generic h such that

$$D \subseteq N \subseteq C = D[h].$$

Let $\tau = |P|^D$. Every sufficiently high Woodin of C , in particular each above the small-forcing bound, is already Woodin in D by [45]. Thus $D \models \text{ZFC} + \text{CW}$. Since $N \models \text{ZF} + \text{SVC}$, the symmetric-intermediate theorem [7, Corollary 5.10] implies that N is a symmetric extension of D . Its set-forcing presentation is supplied by [7, Proposition 4.4]. The presenting forcing need not be the particular P used above. This is sufficient for the claimed existence of a set-symmetric presentation. $\square$

17. WOODIN REALIZATION BY SET FORCING

Write CW for the assertion that the Woodin cardinals form a proper class. For a rank-restricted conclusion, recognizing an entire choice envelope is unnecessary. We shall recognize enough of that envelope to evaluate the capped-history predicate of Theorem 11.1. This permits a construction using only set forcing over the original Woodin ground.

17.1. Keeping the original symmetric ground.

Lemma 17.1 (The old-ground version of local transfer). *Suppose $D \models \text{ZFC} + \text{CW}$, and N is a set-symmetric extension of D satisfying $\text{ZF} + \text{DC}$. Given an ordinal α , there are models and sets*

$$D \subseteq N \subseteq M = D(A) \subseteq B = M[K] = D[b]$$

such that

- (1) $V_\alpha^M = V_\alpha^N$ and $M \models \text{ZF} + \text{DC} + \text{SVC}$;
- (2) $A \subseteq \mu^\omega$ for some ordinal μ ;
- (3) $B \models \text{ZFC} + \text{CW}$, and B/M is weakly homogeneous and adds no ordinal ω -sequences;
- (4) $B = D[b]$ is an ordinary set-forcing extension of this same D .

There is no large-cardinal hypothesis on N in this statement.

Proof. Fix a symmetric presentation $D \subseteq N \subseteq E_0 = D[G]$, with forcing $T \in D$. By [7, Lemma 4.1 and Proposition 5.4], there is $z \in N$ with $N = D(z)$, and N satisfies SVC . Fix an SVC witness $S \in N$. The ground is already given; the measurable capture argument in Proposition 3.1 is not needed.

Choose an uncountable regular D -cardinal $\vartheta > |T|^D$ above the presentation bounds. Choose a sufficiently large ground beth fixed point $\lambda > \alpha$ of D -cofinality ϑ , with $z, S \in V_\lambda^N$, and put $X = V_\lambda^N$. The small forcing T preserves this cofinality. The old cofinal sequence belongs to N , and the full cover has no shorter one, so $\text{cf}^N(\lambda) = \vartheta > \omega$. The transitive-base convention gives $N = D(X)$, and extending each SVC surjection by a default value gives $N \models \text{SVC}(X)$.

A Woodin cardinal is a limit of measurable cardinals, so D has arbitrarily large measurable cardinals. After bounding the ground names for X and the old collapse below, choose a D -measurable $\kappa > \lambda + \omega$ above these bounds. Fix a normal D -measure $\mathcal{U}$ on κ . Let $\mathbb{P}_\mathcal{U}^D$ be the old Prikry poset and let

$$\mathbb{C}_0 = \{k : k \text{ is an } N\text{-countable partial function } \omega_1^N \longrightarrow X\}^N.$$

Both the presentation and $\mathbb{C}_0$ have size below κ in the full cover. Take (c, K_0) generic for the fixed product $\mathbb{P}_\mathcal{U}^D \times \mathbb{C}_0$ over E_0 , and set

$$W = N[K_0][c] = N[c][K_0], \quad E = E_0[K_0][c].$$

Here the use of the old measure is essential. In each small choice cover Y of D , the normal generated lift is

$$\mathcal{U}_Y = \{a \in \mathcal{P}(\kappa)^Y : \exists u \in \mathcal{U} (u \subseteq a)\}.$$

Every tail in $\mathcal{U}_Y$ contains an old tail, so $\mathbb{P}_\mathcal{U}^D$ is dense in the Prikry forcing recomputed from $\mathcal{U}_Y$ in Y . We do not require $\mathcal{U}$ itself to measure all the new subsets of κ in N .

In N , DC makes $\mathbb{C}_0$ countably closed. Its generic union enumerates X ; SVC then wellorders every old set and every set of names. Hence $N[K_0] \models \text{ZFC}$. The generated measure in $E_0[K_0]$ restricts to the normal measure defined by the same ground-witness formula in $N[K_0]$. Product genericity makes c Prikry generic over this choice model, so $W \models \text{ZFC}$. These are exactly the hypotheses of Section 4. In particular Lemma 4.1 gives

$${}^\omega X \cap W = {}^\omega X \cap N, \quad V_\kappa^{N[c]} = V_\kappa^N,$$

and gives an N -wellorderable cover for every W -countable family of N -elements. The countable closure is used before adjoining c ; closure of the old collapse after adjoining c is not assumed.

The old initial-segment collapse projects onto the labeling forcing of Section 5. Let g be its projected generic. The finite interpolation, paired lifting and literal comparison lemmas give $Q = Q(c, g)$ and $F = F(c, g) : Q \rightarrow X$. Put $M = N(Q, F)$. Lemma 7.1 and Corollary 7.2 give $V_\lambda^M = X$; Lemmas 7.3 and 7.4 give DC in M . These results use the old countable arrays and the facts just verified, rather than a measure belonging to N .

For clarity, the SVC and seed conclusions can also be read directly from the proof of Proposition 7.5. Bounded presentations and the surjection onto X give surjections $\xi \times Q^{<\omega} \rightarrow Y$ for every nonempty $Y \in M$, so $Q^{<\omega}$ witnesses SVC. Code Q and the relation

$$R(p, q) \iff F(p) \in F(q)$$

by differently tagged ordinal ω -sequences. Equality of predecessor sets is equality of F -values. The quotient relation's unique Mostowski collapse recovers the literal X and F . Thus the resulting $A \subseteq \mu^\omega$ satisfies $M = D(A)$. No measurable-tail conclusion about M is needed here.

Apply Lemma 12.1 to M , taking its fresh generic K over the full choice cover E as well. Then $B = M[K]$ is a weakly homogeneous choice extension with no new ordinal ω -sequences. Moreover

$$D \subseteq B \subseteq E[K].$$

The outer model $E[K]$ is a set-forcing extension of D : the posets defined in intermediate models have ground names. Since D and B satisfy ZFC, the ordinary intermediate-model theorem gives a set forcing $P \in D$ and a D -generic b with $B = D[b]$; see [4]. Small forcing preserves Woodinness [45]. Every D -Woodin above $|P|^D$ therefore survives in B , proving $B \models \text{CW}$. $\square$

17.2. One bounded homogeneous ordinal code.

Lemma 17.2 (Bounded homogeneous coding). *Let $W \models \text{ZFC}$ have a proper class of inaccessible cardinals, let $d \subseteq \delta$ be a set of ordinals, and let β be an ordinal. There is a weakly homogeneous set forcing, uniformly defined from d, δ and an ordinal bound b , which preserves V_β^W , adds no ordinal ω -sequences, and makes d ordinal definable. No GCH hypothesis is required.*

Proof. If $\delta = 0$, use trivial forcing. Assume $\delta > 0$. Choose $b > \beta, \delta, \omega$ with $|V_\beta^W| < b$. Let $\langle \kappa_i : i < \delta \rangle$ enumerate the first δ inaccessible cardinals strictly above b , in their ordinal order. In particular $\delta < \kappa_0$. All calculations until an extension is specified take place in W .

At an inaccessible $\kappa = \kappa_i$, put

$$\theta = 2^\kappa, \quad \ell = \theta^+, \quad C = \text{Col}(\kappa^+, \theta), \quad A = \text{Add}(\kappa, \ell).$$

Here C consists of partial functions from κ^+ to θ of size $< \kappa^+$, and A consists of partial binary functions on $\kappa \times \ell$ of size $< \kappa$. Both are ordered by reverse inclusion. The factor at i is

$$R_i = \begin{cases} C, & i \notin d, \\ C \times A, & i \in d. \end{cases}$$

The Cohen factor is the ground poset in this displayed product.

Since $\theta^\kappa = (2^\kappa)^\kappa = \theta$, we have $|C| = \theta$. The collapse is $< \kappa^+$ -closed, preserves κ^+ and θ^+ , and makes $|\theta| = \kappa^+$. It adds no subsets of κ . Consequently, in $W[H_C]$,

$$2^\kappa = \kappa^+, \quad \ell = \kappa^{++}.$$

Moreover, C adds no ordinal sequences of length $< \kappa$, so the old A is exactly the recomputed $\text{Add}(\kappa, \ell)^{W[H_C]}$. In this intermediate model $\kappa^{<\kappa} = \kappa$. Regularity of $\ell = \kappa^{++}$ and $(\kappa^+)^\kappa = (2^\kappa)^\kappa = \kappa^+$ give $\ell^\kappa = \ell$, by bounding the range of each function $\kappa \rightarrow \ell$. The Cohen chain condition and the count of nice names therefore give $2^\kappa = \ell$ after A , while preserving cardinals. The two choices of R_i thus force, respectively, $2^\kappa = \kappa^+$ and $2^\kappa = \kappa^{++}$.

For later size calculations, already in W we have $\ell^{<\kappa} = \ell$. Indeed each short function into the regular θ^+ has bounded range and $\theta^\kappa = \theta$. Hence $|R_i| \leq \theta^+$. Each R_i is $< \kappa_i$ -closed and weakly homogeneous. Permutations of the collapse domain, and of the Cohen coordinates, can move either of two condition domains off the other.

Use the full-support set product

$$R(d, \delta, b) = \prod_{i < \delta} R_i.$$

It is $< \kappa_0$ -closed and weakly homogeneous, using componentwise unions and automorphisms. Since $|V_\beta^W| < \kappa_0$, induction on rank shows that it preserves V_β^W . We must also verify that the inaccessible coding sites remain recognizable and that different coordinates do not erase their bits.

Let $\nu \geq \kappa_0$ be any inaccessible of W . There are fewer than ν coordinates below ν , and their number is bounded by $|\delta| < \kappa_0$. Their sites and factor sizes are bounded below ν by regularity and the strong-limit property. For a common size bound $\zeta < \nu$, their product has size at most $\zeta^{|\delta|} < \nu$. The remaining old product is $< \nu$ -closed. Easton's lemma [46, Fact 5.10] makes that fixed tail $< \nu$ -distributive over the small lower product. Small forcing preserves the inaccessibility of ν , and the tail preserves its regularity and all smaller power sets. Thus ν remains inaccessible. We use distributivity of the old tail here; closure of that tail in the intermediate extension is unnecessary.

Forcing cannot create a strongly inaccessible ordinal. An old noncardinality or singularity witness persists. If an old regular cardinal ν was not strong limit, an old injection $\nu \rightarrow \mathcal{P}(\zeta)$ for some $\zeta < \nu$ persists as well. Therefore the first δ inaccessibles above b are the same ordinals in the extension.

Fix i . Force first with R_i . The product of lower coordinates has size $< \kappa_i$, and its old enumeration remains an enumeration after R_i . It preserves the relevant successor cardinals and the calculated value of 2^{κ_i} : names for subsets of κ_i are coded by subsets of the product of its condition set with κ_i , so there are at most the current 2^{κ_i} such names. The old subsets give the opposite inequality. These factors commute. If a tail remains, put $\nu = \kappa_{i+1}$. The initial product through i has size $< \nu$ in W , whereas the old remaining product is $< \nu$ -closed. Easton's lemma makes the tail $< \nu$ -distributive over that initial product. It changes neither $\mathcal{P}(\kappa_i)$ nor the successor cardinals used in the bit test, all of which are below ν .

Consequently the final extension satisfies

$$i \in d \iff 2^{\kappa_i} = \kappa_i^{++} \quad (i < \delta), \quad (17.1)$$

with $2^{\kappa_i} = \kappa_i^+$ at the other sites. Enumerating the inaccessibles above b and applying this test defines d from the ordinals b, δ .

Finally, the entire forcing recipe uses only d, δ, b . It chooses no auxiliary wellorder or enumeration of names. If W is an earlier generic extension and d is an old parameter, the name given by this recipe is invariant, up to forced equality, under every earlier automorphism fixing that parameter and the ordinals. This is the form of uniformity needed for homogeneous composition below. $\square$

17.3. Bounding all capped histories. For a fixed schedule as in Theorem 11.1, write $\text{Hist}_B(e, h, k)$ for the assertion that h is Q_e -generic over B and k is generic over $B[h]$ for the recomputed cap $\text{Add}(\rho, 1)^{B[h]}$. All generic filters in this predicate are actual sets in the universe in which it is evaluated. The next lemma bounds the ground data needed to test the predicate, including the dense sets for the recomputed cap.

Lemma 17.3 (Bounded history transfer). *Fix $B \models \text{ZFC}$, the universal schedule, the candidate forcings Q_e for $e \in (2^\tau)^B$, and the cap ρ from Theorem 11.1. There is an ordinal χ such that every transitive inner model $C \subseteq B$ of ZFC with $V_\chi^C = V_\chi^B$ computes the same schedule and candidate forcings and satisfies*

$$\text{Hist}_C(e, h, k) \iff \text{Hist}_B(e, h, k)$$

for all candidates and all possible outer filters h, k . The bound can be increased arbitrarily.

Proof. For each $e \in (2^\tau)^B$, put $\mathbb{B}_e = \text{RO}^B(Q_e)$. Let T_e be the full bounded system of total Boolean names for cap conditions used after (11.3). For every B -generic $h \subseteq Q_e$,

$$\{t^h : t \in T_e\} = \text{Add}(\rho, 1)^{B[h]}. \quad (17.2)$$

Indeed, localize any name for a cap condition, mix with the empty cap condition off that Boolean cone, and normalize to the bounded Boolean-name codes. This normalization takes place in the choice model B and yields a set T_e independent of h .

Let $\mathcal{N}_e$ consist of all Boolean names for subsets of the ground set T_e . A name has one Boolean value in $\mathbb{B}_e$ for each possible member of T_e , so the codes are contained in $\mathcal{P}^B(T_e \times \mathbb{B}_e)$, together with the fixed check-name coding. Choose χ so large that its rank cut contains, for every candidate e , the whole sets

$$Q_e, \quad \mathcal{P}^B(Q_e), \quad \mathbb{B}_e, \quad T_e, \quad \mathcal{P}^B(T_e \times \mathbb{B}_e), \quad \mathcal{N}_e,$$

their elements, and all the lower universal products and their power sets used to form the schedule and Boolean completions. Include the wellordering and cardinal-bound witnesses from the schedule recursion, and take $\chi > \rho + \omega$. There is one such bound because the candidate family and schedule have set size. Agreement above $\rho + \omega$ includes all bounded witnesses to cardinality, cofinality and the strong-limit tests below ρ ; thus no earlier inaccessible is introduced in the schedule computed in C . The complete Boolean algebras and the indicated name systems are computed identically.

Every B -dense subset of Q_e belongs to C , and every C -dense subset is a B -dense subset of the same poset. Hence C -genericity and B -genericity of h are equivalent. For such an h we have $C[h] \subseteq B[h]$. Absolute evaluation of the names in T_e , together with (17.2), gives equality of the two recomputed cap posets. In the convention $\text{Add}(\rho, 1) = 2^{<\rho}$, membership is absolute for a sequence already present in the smaller model.

In fact the full power sets of these cap posets agree. Given $Z \subseteq \text{Add}(\rho, 1)^{B[h]}$ in $B[h]$, form

$$Y = \{t \in T_e : t^h \in Z\}.$$

Some member of $\mathcal{N}_e$ evaluates to Y . That name belongs to C , so $Y \in C[h]$, and evaluating its members reconstructs Z inside $C[h]$. The opposite inclusion follows from $C[h] \subseteq B[h]$. Therefore the two models have exactly the same dense subsets of the common cap poset. Genericity of any outer filter k is equivalent over them, proving the lemma. $\square$

Lemma 17.4 (A sufficient ground fragment). *Suppose $D \subseteq M = D(A) \subseteq B = D[b]$, where D, B satisfy ZFC and b is generic for a set forcing $P \in D$. Fix ordinals α, χ, η , with $P, b \in V_\eta^B$. There is θ such that*

- (1) every element of V_χ^B is the value of a P -name in V_θ^D ;

$$(2) V_\alpha^M \subseteq L(V_\theta^D, A).$$

Here and below L with set arguments denotes transitive-base constructibility, including all members of its arguments.

Proof. The bounded-rank name lemma supplies the first bound. One can obtain it by recursion in D , using at successor stages all Boolean names whose possible members are in the preceding name set, and taking unions at limits. Their evaluations cover the corresponding ranks in every generic extension. Equivalently, in B choose a ground name for each element of the set V_χ^B and bound their ranks. The ground D is definable in its set extension from a set parameter; the chosen collection need not itself belong to D for its ordinal rank bound to do so.

For the second bound, use the convention and bounded-witness argument of Lemma 2.2. Each $y \in V_\alpha^M$ belongs to a set-sized construction level over $\text{TC}(\{V_\xi^D, A\})$ for some ξ . Collection in B bounds the ground ranks ξ needed for the set V_α^M . Increase θ past this bound and the name bound. The inner model $L(V_\theta^D, A)$ contains every one of the indicated levels: recursion through the definable-subset operation on set structures is absolute. This proves the second inclusion without claiming that an arbitrary directed union of inner models satisfies ZF. $\square$

17.4. Realizing the protected rank.

Theorem 17.5 (Woodin rank realization). *Let $D \models \text{ZFC} + \text{CW}$, let N be a set-symmetric extension of D satisfying $\text{ZF} + \text{DC}$, and let α be an ordinal. In a common outer universe there is an ordinary set-forcing extension U of D such that*

$$U \models \text{ZFC} + \text{CW}, \quad V_\alpha^{\mathcal{H}^U} = V_\alpha^N.$$

In particular, a rank sentence true in such a symmetric extension is true in the corresponding rank of external $\text{HOD}_{\text{Ord}^\omega}$ in a model of $\text{ZFC} + \text{CW}$.

Proof. Apply Lemma 17.1. We obtain

$$D \subseteq N \subseteq M = D(A) \subseteq B = M[K] = D[b], \quad V_\alpha^M = V_\alpha^N,$$

with $A \subseteq \mu^\omega$ and the stated homogeneous sequence-preserving envelope. Fix its forcing presentation $P \in D$ and choose η with $A, K, P, b \in V_\eta^B$.

Choose an infinite τ large enough to code $(\text{TC}(\{A\}), \in, A)$, and let $F_A \subseteq 2^\tau$ be the family of all its pointed well-founded extensional presentations, with domain, root and unused-position data as in Corollary 11.4. This family is defined from A, τ ; no presentation is selected. Apply Theorem 11.1 over B , placing the first site above a regular bound greater than $|V_\eta^B|$. With its universal schedule and cap ρ , use the actual forcing

$$S = P_{F_A} = \prod_{e \in F_A}^{\text{full}} (Q_e * \text{Add}(\rho, 1)^{B[h_e]}), \quad U_0 = B[G_S].$$

The terminal forcing is recomputed in its own row extension. The extension retains the evaluated row and cap generics; the larger raw termspace generics are not adjoined. Theorem 11.1 gives the spectrum F_A , weak homogeneity and preservation of V_η^B . Since U_0 is a set extension of D , it still has a proper class of Woodins, and hence of inaccessibles.

Next choose $\chi > \eta + \omega$ satisfying Lemma 17.3, and choose θ by Lemma 17.4. In D , code the transitive set V_θ^D by a set of ordinals $d \subseteq \delta$. Include domain and root information in its well-founded extensional presentation. The code d belongs to D , hence to M , regardless of how these ordinal bounds were selected.

In U_0 choose a rank bound ζ above η and all possible capped-history witnesses for this schedule. Such a bound depends only on the set family of lower posets and on ρ : a lower filter is a subset of some Q_e , and a cap filter is a subset of $2^{<\rho}$. It bounds the ranks of all potential witnesses, not

just of the generics originally planted by S . Enlarge it to include the candidate codes. Apply Lemma 17.2 over U_0 to d , preserving $V_\zeta^{U_0}$, and call the resulting extension U .

We now define the decoder inside U . The set d is OD, and $V_\eta^U = V_\eta^B$. Thus the class

$$C = L(d, V_\eta^U) \quad (17.3)$$

is definable from ordinal parameters in U . Its bases are sets of B , so the absolute transitive-base construction gives $C \subseteq B$. Decoding d puts V_θ^D in C ; the second base contains P, b . Evaluation of the names in Lemma 17.4 therefore gives

$$V_\chi^C = V_\chi^B.$$

A B -wellordering of V_η^B has rank below χ after the stipulated increase of χ , and hence belongs to C . Together with the ordinal-set code d , it wellorders the transitive base of (17.3). Consequently $C \models \text{ZFC}$.

Lemma 17.3 identifies the history predicate relative to C with the predicate relative to B . The last forcing preserved every possible history witness and changed neither C nor its ground-name and dense-set families. Thus in U the spectrum of histories relative to the definable class C is exactly F_A . Taking the pointed Mostowski collapse of any member recovers A and makes A OD in U . We have transferred a bounded predicate; we have not reapplied Theorem 11.1 to C , which need not have a proper class of inaccessible cardinals.

Put $H = \mathcal{H}^U$. The forcing from M to U is weakly homogeneous. The spectrum name is uniformly defined from the old A and ordinals, and the final code name is uniformly defined from the old d and ordinals. Earlier automorphisms therefore send each name to a forced equal name and lift to the iteration. The local Boolean-mixing argument in the proof of Proposition 13.1 combines these automorphisms: a locally forced automorphism is given the identity value off its Boolean cone. This uses no global maximum principle over the choiceless M and introduces no wellorder from the choice envelope as a parameter.

The first stage adds no ordinal ω -sequences, and both later stages are highly distributive. Lemma 10.2 gives $H \subseteq M$. Conversely, the ordinal code $d \in \text{HOD}^U$ and its unique Mostowski collapse put V_θ^D , with all its hereditary elements, in H . Also $A \in H$: it is OD and each of its members is an allowed ordinal ω -sequence. Since $H \models \text{ZF}$ by Lemma 10.1, it contains $L(V_\theta^D, A)$. Hence

$$V_\alpha^M \subseteq L(V_\theta^D, A) \subseteq H \subseteq M,$$

which proves the required rank equality.

The definability of C places none of the extra choice wellorderings of B in H . In particular V_η^U is used as an OD set defining a class, not as a hereditarily OD parameter. This is the distinction of Remark 12.6. The object coded hereditarily is the ground fragment V_θ^D , not the envelope rank V_χ^B .

Finally, U/D is a composition of set forcings. Every sufficiently high ground Woodin survives by small-forcing preservation [45]. Thus $U \models \text{ZFC} + \text{CW}$. The bounds were chosen in the order

$$\eta, \quad S \text{ and } \rho, \quad \chi, \quad \theta \text{ and } d, \quad \zeta, \quad \text{the last coding block.}$$

There is no demand that S preserve V_θ^B : the ground fragment is recovered from d . All preparation and coding in this proof are set forcing, and no conclusion that N or M has Woodin cardinals is used. $\square$

18. GCH REALIZATION AS A SYMMETRIC GROUND

We now strengthen the consistency conclusion by using the special ground produced in Proposition 16.1. The additional hypotheses on that ground allow a bounded GCH-preserving

code to identify all its sets inside the final HOD. We then recover the sequence-generated model exactly and exhibit the ambient universe as its set-forcing extension.

18.1. The normalized choice ground. The reverse construction supplies more structure than is needed for Theorem 17.5. We use that structure to obtain global GCH and to identify the entire external HOD.

Lemma 18.1 (Properties of the normalized ground). *Let $D \subseteq N$ be the models constructed in Proposition 16.1, and let C be its canonical choice envelope, so that*

$$D = \text{HOD}^C, \quad D \subseteq N \subseteq C = D[h].$$

Then D satisfies eventual GCH, and there is a set of ordinals $t \in D$ such that

$$D = \text{HOD}_{\{t\}}^D.$$

Every set-forcing extension of D satisfies eventual GCH.

Proof. The model C supplied by Theorem 15.2 is an acceptable relative pure-extender premouse over a bounded self-wellordered base. In either branch of that construction, the usual acceptability calculation applies above the size and height of the base. Subsets of an infinite cardinal κ occur below J_{κ^+} , and the standard surjections from the base times κ bound the earlier levels by κ . Thus C satisfies GCH above a fixed bound; compare [37, Section 2.2].

Fix a forcing presentation $C = D[h]$ by $T \in D$. Above $|T|^D$, the relevant cardinals are preserved. If GCH failed in D at an arbitrarily large cardinal κ , its old subsets of κ would still contradict GCH in C . Hence D satisfies eventual GCH. Conversely, for any fixed set forcing $P \in D$ and every sufficiently large cardinal κ , names for subsets of κ can be coded by subsets of $\kappa \times P$. When GCH holds in D there, this bounds their number by $(\kappa^+)^D$, which the forcing preserves. The old subsets give the opposite inequality. This proves eventual GCH in every set extension of D .

Replace T by an ordinal-coded presentation in D , and let t be one ordinal-set code for its domain and order. If $x \in D$, then x is ordinal definable in C . There are a formula φ , a finite tuple of ordinals $\vec{\xi}$, and $p \in h$ such that

$$p \Vdash_T \text{“}\check{x} \text{ is the unique } y \text{ satisfying } \varphi(y, \check{\xi})\text{”}.$$

Inside D , the set x is therefore the unique y for which $p \Vdash_T \varphi(\check{y}, \check{\xi})$. The forcing relation is definable in D from T , and p is represented by an ordinal. Thus x is definable over D from t and ordinals. The same argument applies to every member of its transitive closure, proving $D = \text{HOD}_{\{t\}}^D$. $\square$

18.2. Bounded coding while preserving GCH. The continuum code of Lemma 17.2 can be replaced, over a GCH model, by a code in the $\diamond^*$ pattern. We need a bounded set product whose definition uses only the set being coded and ordinal parameters. We also need its actual closure, rather than only distributivity, for the ground-recognition argument.

Lemma 18.2 (Bounded GCH-preserving coding). *Let $W \models \text{ZFC} + \text{GCH}$ have a proper class of inaccessible cardinals. Suppose $d \subseteq \delta$, and let β, χ be ordinals. For every sufficiently large ordinal b there is a set forcing $R(d, \delta, b)$, uniformly defined from these three parameters, which is locally homogeneous and $< \chi$ -closed. It preserves all cardinals, GCH, the class of inaccessible cardinals, and V_β^W , adds no ordinal ω -sequences, and makes d ordinal definable from b, δ .*

Proof. If $\delta = 0$, use trivial forcing. Otherwise take

$$b > \delta, \beta, \chi, \omega, |V_\beta^W|,$$

and let $\langle \kappa_i : i < \delta \rangle$ enumerate the first δ inaccessible cardinals above b . Put

$$\lambda_i = \kappa_i^{+++}.$$

Until an extension is specified, all posets and cardinal calculations in this proof are computed in W . In particular $\delta < \kappa_0$.

We first specify the two possible factors at an infinite successor cardinal $\lambda = \theta^+$. For the positive factor P_λ , use pairs (X, ξ) with

$$\emptyset \neq X \subseteq {}^\lambda 2, \quad |X| \leq \theta, \quad \xi < \lambda,$$

and a separate largest condition. Declare $(Y, \zeta) \leq (X, \xi)$ if $Y \supseteq X$, $\zeta \geq \xi$, and

$$\forall f \in Y \exists g \in X \quad f \restriction \xi = g \restriction \xi.$$

The nonempty-branch convention is part of this presentation. An empty branch set would give a degenerate component under the displayed order and is excluded.

This is the positive coding forcing of [19, Section 2]. It forces $\diamond_\lambda^+$, and hence $\diamond_\lambda^*$. The relevant feature of its proof is that the generic tree captures every old binary λ -sequence up to a bounded modification. Indeed, below any nonempty condition, alter the initial part of a given sequence to match a represented node and then add that branch. Nice names for new binary λ -sequences use only λ many conditions and branches, so they have old binary λ -sequence codes. The tree recovers the generic filter. Together with a code for the old $\mathcal{P}(\theta)$, these facts give the hypothesis of the positive coding theorem in [19, Section 2]. That argument uses only the nonempty component specified here.

The negative factor is $Q_\lambda = \text{Add}(\lambda, \lambda^+)$, which forces $\neg \diamond_\lambda^*$; see [18, Appendix, Lemma 18]. Both factors are $< \lambda$ -closed, have size λ^+ , and satisfy the λ^+ -chain condition under GCH. For P_λ , unions of increasing branch sets and suprema of heights give the closure. At each fixed height $\xi < \lambda$ there are at most λ possible sets of at most θ binary ξ -prefixes. Conditions with the same prefix set are compatible by taking the union of their branch sets. This proves its chain condition. The size bound follows from $(2^\lambda)^\theta = 2^\lambda = \lambda^+$. The usual Cohen calculation gives the same bounds for Q_λ . For either factor, closure, the chain condition, and nice-name counting preserve all cardinals and GCH.

Define

$$R_i = \begin{cases} P_{\lambda_i}, & i \in d, \\ Q_{\lambda_i}, & i \notin d, \end{cases} \quad R(d, \delta, b) = \prod_{i < \delta} R_i,$$

using full support and the old factors. The product is $< \lambda_0$ -closed by coordinatewise lower bounds in W . Since $\lambda_0 > b > \chi$, $|V_\beta^W|$, it is $< \chi$ -closed, adds no ordinal ω -sequences, and preserves V_β^W . For the rank assertion, a first new set below that rank would be a new subset of an old set of size $< \lambda_0$, contrary to closure and Choice in W .

Local homogeneity. Given two nontrivial conditions in P_λ , extend both distinguished branch sets to have size θ . Choose a common height $\gamma < \lambda$ above their heights which separates every pair of distinct branches in either set. Match their θ many nodes at height γ bijectively, together with the unique distinguished branch through each node. Within each matched pair of binary cones, replace the prefix and translate the tail by coordinatewise addition modulo two so that the distinguished branches correspond. The resulting map preserves equality of initial segments at every height at least γ . Applied to branch sets, it is an isomorphism between the two forcing cones below the strengthened conditions. Thus P_λ is locally homogeneous. The Cohen factor Q_λ is weakly homogeneous. Choice in W combines the component cone isomorphisms into a cone isomorphism for the full product. This proves the required local homogeneity.

Factorization estimates. For each $i < \delta$, the product below i has size $< \kappa_i$. Its number of coordinates is less than κ_i , and regularity bounds their sizes below κ_i . The strong-limit property

then bounds the full product below κ_i . Consequently the product through i has size at most λ_i^+ and is λ_i^+ -cc. To verify the latter claim, an antichain of size λ_i^+ could be thinned to have the same lower-product coordinate, contradicting the chain condition of R_i .

If a tail remains after i , it is $< \lambda_{i+1}$ -closed in W . The prefix has size $< \lambda_{i+1}$. Easton's lemma [46, Fact 5.10] makes this fixed old tail $< \lambda_{i+1}$ -distributive over the prefix extension. Thus it adds no subsets to the cardinals at or immediately above λ_i . This use of distributivity does not require the old tail to remain closed in the intermediate model.

Cardinals and GCH. At an active site λ_i , the preceding size and chain condition bounds give at most

$$(\lambda_i^+)^{\lambda_i} = \lambda_i^+$$

nice prefix names for subsets of λ_i . At its successor and at larger cardinals above that prefix's size, a subset name is coded by a subset of the product of the domain cardinal with the forcing. Ground GCH gives the corresponding successor bound. Upper-tail distributivity transfers these calculations to the full extension. At cardinals below the active site, one may instead force with R_i first and then with the small lower product. These factors commute, the first preserves GCH, and the second preserves the continuum function above its size by the same name count. This also handles the gaps between a completed prefix and the next active site.

We give the limit calculation separately. Suppose

$$\mu = \sup_{i < j} \lambda_i, \quad 0 < j \leq \delta, \quad j \text{ is a limit ordinal.}$$

The underlying inaccessibles are cofinal in μ , so μ is strong limit and

$$\varepsilon = \text{cf}^W(\mu) \leq |j| \leq |\delta| < \kappa_0.$$

Ground GCH gives

$$\left| \prod_{i < j} R_i \right| \leq \mu^{|j|} = \mu^+. \quad (18.1)$$

If $j < \delta$, then $\kappa_j > \mu^+$, and the remaining tail is distributive beyond μ^+ after this prefix.

The earlier factorization calculations preserve all cardinals below μ , so μ remains a cardinal. Its old short cofinal sequence shows that it remains singular. If the old μ^+ ceased to be regular, its new cofinality would be a regular cardinal $\nu < \mu$. Choose $i < j$ sufficiently large that $\nu < \lambda_{i+1}$ and the prefix through i still has size $< \mu$. This prefix preserves the regularity of the old μ^+ , while its remaining old tail is $< \lambda_{i+1}$ -distributive by Easton's lemma. The tail cannot add the purported cofinal ν -sequence. Hence μ^+ is preserved. The prefix size bound (18.1) preserves the higher cardinals and gives GCH at μ^+ and above, up to the next active site.

To count subsets of μ , fix in W a cofinal sequence $\langle \lambda_{i_\xi} : \xi < \varepsilon \rangle$ of active cardinals below μ . For each ξ , the isolated-site nice-name calculation supplies a fixed ground set T_ξ of names of size at most $\lambda_{i_\xi}^+$. Every subset of λ_{i_ξ} in the full extension is the value of a name in T_ξ , because the upper tail adds no such subsets. Wellorder these sets of names in W . For each $x \subseteq \mu$ in the extension, select the least index of a name evaluating to $x \cap \lambda_{i_\xi}$, for each $\xi < \varepsilon$. This is an ε -sequence of ordinals below μ .

The whole product is $< \lambda_0$ -closed and $\varepsilon < \kappa_0 < \lambda_0$, so this entire sequence of indices already belongs to W . There are at most

$$(\mu^\varepsilon)^W = \mu^+$$

such ground sequences. Evaluating the named restrictions and taking their union gives all subsets of μ in the extension. Since μ^+ has been preserved, this proves $2^\mu = \mu^+$. In particular no μ^+ -chain-condition claim about the limit product is needed.

An induction through the active sites and their limits now proves cardinal preservation and GCH at every cardinal. Away from a limit of active sites, the sites below a given cardinal are bounded and the prefix-size calculation applies. Above the size of the entire set product it applies directly.

Preservation and recognition of the code. Let $\nu \geq \kappa_0$ be any inaccessible of W . Fewer than ν coordinates lie below ν , their sizes are bounded below ν , and their product has size $< \nu$. The remaining old tail is $< \nu$ -closed. Small forcing preserves the inaccessibility of ν , and Easton's lemma makes the tail $< \nu$ -distributive over that small extension. It preserves regularity and all smaller power sets. Inaccessibles below κ_0 are preserved by closure. Conversely, old cofinality witnesses, noncardinality witnesses, and injections witnessing a failure of strong limit persist. Thus the class of inaccessible cardinals is unchanged.

Fix $i < \delta$ and first force with R_i . The old lower product has size $< \kappa_i < \lambda_i$ and therefore preserves the truth value of $\diamond_{\lambda_i}^*$ by [18, Appendix, Lemma 19]. The upper tail adds no subsets of λ_i . Under GCH below λ_i , each candidate $\diamond^*$ sequence and its club witnesses can be coded by subsets of λ_i . Hence the upper tail preserves that truth value as well. We have proved

$$i \in d \iff \diamond_{\lambda_i}^* \quad (i < \delta) \quad (18.2)$$

in the final extension. Its first δ inaccessibles above b , and their triple successors, are the same ordinals as in W . Equation (18.2) defines d from b, δ .

Finally, the formula for the forcing chooses no enumeration or wellorder of a ground power set. The choices of matches and wellorders in the proof establish homogeneity and counting; they are not parameters in the forcing definition. If W is an earlier generic extension and d is old, this uniform definition gives an invariant name under any earlier cone isomorphism fixing the old parameters and ordinals. $\square$

18.3. Local homogeneity and hereditary descent. For composition, we understand local homogeneity on separative quotients, equivalently on regular-open cones. Thus two arbitrary conditions have stronger conditions whose forcing cones are isomorphic. Transporting names gives isomorphisms in this sense; it need not biject redundant syntactic iteration names.

Lemma 18.3 (Local homogeneous descent). *Let $M \models \text{ZF}$ and suppose $U = M[G] \models \text{ZFC}$ is a locally homogeneous set-forcing extension with*

$$(\text{Ord}^\omega)^U = (\text{Ord}^\omega)^M.$$

Then $\mathcal{H}^U \subseteq M$.

Moreover, local homogeneity is preserved by finite iterations whose next forcing is forced locally homogeneous and is uniformly defined from fixed old parameters and ordinals. For this assertion the defining names are taken invariant, up to forced equality, under the preceding cone isomorphisms. The composition argument requires no maximum principle in M .

Proof. An isomorphism of forcing cones transports check names to check names. If two conditions forced opposite truth values for a sentence with only ground parameters, local homogeneity would give stronger isomorphic cones with the same opposite decisions, a contradiction. Thus the largest condition decides every such sentence. This is the local-homogeneity argument of [18, Appendix, Lemma 16].

The rank induction of Lemma 10.2 now applies. Suppose $x \in \mathcal{H}^U$ and every member of x belongs to M . Choose ξ with $x \subseteq V_\xi^M$. A definition of x uses an ordinal ω -sequence already in M and ordinal parameters. The top condition forces the uniqueness assertion and uniformly decides membership of each ground $y \in V_\xi^M$ in its unique value. Separation using the forcing relation in M recovers x . This proves $\mathcal{H}^U \subseteq M$.

For composition, consider two conditions in a two-step iteration. First strengthen their first coordinates to isomorphic cones using local homogeneity of the first forcing. Transport one second-coordinate name to the other cone. Uniformity of the definition identifies the two second-forcing names there. After strengthening the common first-coordinate condition, choose names for stronger second-coordinate conditions and an isomorphism between their cones. Such witnesses exist locally by the forcing existential clause, since the second forcing is forced locally homogeneous. No name for witnesses below all first-coordinate conditions is required.

The resulting named isomorphism induces an isomorphism of the separative iteration cones over that common first-coordinate cone. Compose it with the transported first-coordinate isomorphism. This proves local homogeneity of the two-step iteration, and finite induction proves the stated extension. All these cone maps fix check names for old parameters and ordinals, so the same reasoning applies at the next uniformly defined stage. $\square$

In particular, the weakly homogeneous fresh envelope of Lemma 12.4, the locally homogeneous forcing of Lemma 18.2, and the uniformly defined spectrum forcing can be composed over the original choiceless model. Their definitions use the old sequence generator, the old ordinal code, and ordinal parameters. Closure at the two choice-model stages preserves the allowed ordinal ω -sequences, so Lemma 18.3 applies to the combined extension.

18.4. Closure of the evaluated spectrum. The ground-recognition step below needs closure of the actual forcing, in addition to the distributivity already established for the spectrum.

Lemma 18.4 (Closure of the evaluated spectrum). *In Theorem 11.1, let γ_0 be the first coding cardinal. The actual evaluated forcing P_F of (11.3) is $< \gamma_0$ -closed in B .*

Proof. Each lower row Q_e is $< \gamma_0$ -closed. Consider a descending chain

$$\langle (q_i, \dot{u}_i) : i < \zeta \rangle, \quad \zeta < \gamma_0,$$

in the actual two-step row $Q_e * \text{Add}(\rho, 1)^{B[h_e]}$. Take a lower bound q for the q_i . Below q , the cap names form a descending chain of conditions of the recomputed Cohen forcing. Since $\gamma_0 < \rho$ and Q_e preserves regularity of ρ , their forced union is a cap condition and a lower bound for the chain. Mix its name below q with the empty cap off q , and normalize the resulting total name to the bounded Boolean-name system T_e used after (11.3). If $\dot{u}$ is this name, then $(q, \dot{u})$ is a lower bound in the actual row.

Given a descending chain in the full product P_F , use Choice in B to select these lower bounds across its set of rows. Their product is a lower bound in P_F . The construction uses names for evaluated cap conditions; it does not adjoin generics for the larger ground termspace. $\square$

18.5. Exact realization and ground recognition.

Theorem 18.5 (Exact GCH realization over the normalized ground). *Let $D \subseteq N$ be supplied by Proposition 16.1, and fix an ordinal α . In a common outer universe there are M and U such that*

$$U \models \text{ZFC} + \text{GCH} + \text{CW}, \quad M = \mathcal{H}^U \models \text{ZF} + \text{DC} + \text{SVC}, \\ V_\alpha^M = V_\alpha^N.$$

Moreover, U is an ordinary set-forcing extension both of D and of M . In particular, $\mathcal{H}^U$ is a symmetric ground of U .

Proof. By Lemma 17.1, there are

$$D \subseteq N \subseteq M = D(A) \subseteq B = M[K] = D[b]$$

with $A \subseteq \mu^\omega$, $M \models \text{ZF} + \text{DC} + \text{SVC}$, $V_\alpha^M = V_\alpha^N$, and B/M weakly homogeneous with no new ordinal ω -sequences. The model B satisfies $\text{ZFC} + \text{CW}$. By Lemma 18.1, it also satisfies eventual GCH.

Apply Lemma 12.4. Its measurable-cardinal hypothesis follows from CW. We obtain a choice envelope B' satisfying global GCH, still weakly homogeneous over M and with the same ordinal ω -sequences. It is a set extension of D , so a proper class of ground Woodins survives.

We may arrange that B'/D is a genuinely nontrivial forcing extension. If necessary, append $\text{Add}(\omega_1, 1)^{B'}$ before fixing the presentation of B'/D . Over the GCH model B' this forcing is countably closed, homogeneous, and GCH preserving. Its literal countable conditions and the countably supported permutations witnessing homogeneity belong to M , because M and B' have the same ordinal ω -sequences. The product presentation therefore preserves weak homogeneity over M , and the additional forcing adds no such sequences. Include this extension in the notation B' .

Fix a presentation $B' = D[b']$ by $P \in D$, and put

$$\kappa = |P|^D, \quad \delta = (\kappa^+)^D, \quad s = (<^\delta 2)^D.$$

Take $t \in D$ from Lemma 18.1. In D , choose one ordinal-set code z for (s, t) , including the well-founded extensional relation and the distinguished points needed for its unique Mostowski collapse. This fixes δ and z before any further coding.

Apply Lemma 18.2 over B' to z , placing its sites high enough that the coding forcing is $< \delta^+$ -closed and preserves δ^+ . Let the result be $B'' = B'[J]$. It satisfies GCH and has the same ordinal ω -sequences as M . The set z is ordinal definable in B'' . Its uniquely decoded components s, t are therefore ordinal definable there as well. The forcing is uniformly defined from the old z and ordinal parameters and is locally homogeneous.

Choose η large enough to contain all generators and forcing data for B''/D , and all bounded data of the preceding code. In particular, it bounds the coding sites, their relevant power sets, and the candidate diamond-star sequences and club witnesses. Thus preservation of $V_\eta^{B''}$ preserves the whole code, including its decoding locations.

Choose an infinite ordinal τ large enough to present $(\text{TC}(\{A\}), \in, A)$, and let $F_A \subseteq 2^\tau$ be the family of all its pointed well-founded extensional presentations as in Corollary 11.4. Apply Theorem 11.1 over the actual choice model B'' , placing its first cardinal γ_0 above δ^+ and a regular bound greater than $|V_\eta^{B''}|$. Form the extension

$$U = B''[G], \quad G \subseteq P_{F_A}.$$

The forcing preserves $V_\eta^{B''}$, so the code for z and its components s, t remains ordinal definable in U . Lemma 11.5 gives $U \models \text{GCH}$. By Lemma 18.4, this actual final forcing is $< \gamma_0$ -closed.

Consequently the complete forcing from D to U has the fixed nontrivial stem P , of D -size κ , followed by a $< \delta^+$ -closed tail. The tail preserves $(\delta^+)^D$; the small stem preserves it as well. The closure-point lemma gives the δ -approximation and δ -cover properties for $D \subseteq U$, where $\delta = (\kappa^+)^D$ is regular. The ground-definition theorem now defines D inside U from $s = (<^\delta 2)^D$ and ordinal parameters; see [20, Theorem 6 and Lemma 8]. This uses the fixed approximation bound δ , not a bound exceeding the size of the complete coding forcing.

Since s is ordinal definable in U , the class D is ordinal definable there. Since t is also ordinal definable and $D = \text{HOD}_{\{t\}}^D$, relativizing the internal definitions to this class shows that every member of D , and each of its hereditary members, is ordinal definable in U . Hence

$$D \subseteq \text{HOD}^U.$$

This conclusion uses the parameter t ; definability of the class D by itself would not suffice.

The full choice envelope is now recognizable inside U as

$$B'' = D(V_\eta^U).$$

Indeed $V_\eta^U = V_\eta^{B''}$ contains the generators for B''/D , whereas the generated model on the right is constructed inside B'' . Thus B'' is an ordinal-definable class in U . The fixed-base spectrum of Theorem 11.1, over this very class B'' , recovers F_A . The common pointed collapse of its members is A , so A is ordinal definable in U . Every member of A is an ordinal ω -sequence. It follows that $A \in \mathcal{H}^U$.

In the other direction, the entire forcing U/M is locally homogeneous. Start with its weakly homogeneous choice-envelope factor and apply the composition assertion of Lemma 18.3 to the coding and spectrum factors. Their defining parameters z, A are old elements of M , and their other parameters are ordinals. The local cone maps fix these parameters; no selected wellordering of the presentation codes is part of a forcing definition. None of the factors adds ordinal ω -sequences, so the descent conclusion of that lemma gives

$$\mathcal{H}^U \subseteq M.$$

Conversely, $D \subseteq \text{HOD}^U \subseteq \mathcal{H}^U$ and $A \in \mathcal{H}^U$. Minimality of the generated model yields

$$M = D(A) \subseteq \mathcal{H}^U.$$

Therefore $\mathcal{H}^U = M$, with the required protected rank.

The construction exhibits U as an ordinary set-forcing extension of M . Since U satisfies Choice, M is also a symmetric ground of U by [7, Corollary 5.9]. Every forcing used above is set forcing over an intermediate set extension of D , so their composition is a set forcing over D . All sufficiently high D -Woodins remain Woodin in U by small-forcing preservation [45]. This proves $U \models \text{ZFC} + \text{GCH} + \text{CW}$ and all the conclusions. $\square$

19. THE WOODIN CONSISTENCY IMPLICATIONS

Proof of Theorem 1.2. First suppose $U \models W_1(\sigma)$, and choose α with $V_\alpha^{\mathcal{H}^U} \models \sigma$. Proposition 16.1 supplies a model $D \models \text{ZFC} + \text{CW}$ and a set-symmetric extension $N \models \text{ZF} + \text{DC}$ of D with

$$V_\alpha^N = V_\alpha^{\mathcal{H}^U}.$$

Thus $N \models R_\sigma$, giving the required interpretation of $W_2(\sigma)$.

Conversely, start with a $\text{ZFC} + \text{CW}$ ground D and a set-symmetric extension $N \models \text{ZF} + \text{DC} + R_\sigma$. Choose α witnessing R_σ in N . Theorem 17.5 gives a set-forcing extension U of D such that

$$U \models \text{ZFC} + \text{CW}, \quad V_\alpha^{\mathcal{H}^U} = V_\alpha^N.$$

It follows that $U \models W_1(\sigma)$.

To pass from W_1 to W_3 , use the particular D, N supplied by Proposition 16.1 from the original W_1 model. Lemma 18.1 gives eventual GCH in D and a set of ordinals $t \in D$ with $D = \text{HOD}_{\{t\}}^D$. Theorem 18.5 yields a set-forcing extension U^* of D and a model M such that

$$U^* \models \text{ZFC} + \text{GCH} + \text{CW}, \quad \mathcal{H}^{U^*} = M, \quad V_\alpha^M = V_\alpha^N,$$

and U^* is an ordinary set-forcing extension of M . Thus $U^* \models W_3(\sigma)$. The implication from W_3 to W_1 follows by omitting the two additional assertions. Together with the first two implications, this proves the three consistency equivalences. The passage through the normalized ground is part of the construction; it makes no GCH assumption about the original ground of an arbitrary W_2 model.

These are ordinary syntactic consistency implications. Fix σ before taking either interpretation. The relative construction $F(a)$ is a uniform first-order class definition. Its assertions of ZFC in the resulting class are read as the corresponding axiom scheme. Every iteration history used for a

set-length argument, every relative-HOD definition, and every occurrence of satisfaction is coded in a set-sized structure. The argument requires no truth predicate for the whole universe or an external class iteration strategy.

For W_2 , use a set symmetric system and a condition forcing $\text{DC} + R_\sigma$ in the hereditary-symmetric interpretation. The symmetric forcing theorem gives the interpretation of ZF. Conversely, if a particular generic interpretation satisfies $\text{DC} + R_\sigma$, its truth lemma supplies a condition forcing this fixed conjunction. There is no requirement that the largest condition force it. For W_3 , the construction supplies an actual set-forcing presentation over the definable class $\mathcal{H}$. The ground and symmetric-ground assertions have the first-order formulations of [7]; the choice ground D is recovered by [20, Theorem 6]. All additional coding stages are set forcings. The usual set-forcing and definable-inner-model interpretations therefore give all three consistency implications. Equivalently, the constructions can be carried out internally in countable models supplied by completeness. External transitivity of those models is not an additional hypothesis. $\square$

Theorem 1.2 places the Woodins in the choice ground of W_2 and in the ambient universes of W_1 and W_3 . The symmetric model satisfies dependent choice and has a rank satisfying σ . The general realization theorem preserves that rank. The realization over the normalized ground identifies the whole model $M = D(A)$ with external HOD and makes it a ground of a $\text{ZFC} + \text{GCH} + \text{CW}$ universe. Neither conclusion requires Woodins inside M . The original collapse real and its canonical envelope need not belong to the final universe.

APPENDIX A. TWO BOOKKEEPING DETAILS

A.1. Names, parameters, and nonstandard models. All definability arguments use bounded-stage codes. To say that a set belongs to $D(A)$, specify a set-sized stage over a rank of D and the transitive closure of A . A relative-HOD definition can likewise be evaluated at a sufficiently large rank level, with that ordinal included in the code. No satisfaction predicate for the entire universe is required.

The forcing comparisons concern fixed names and formulas. Auxiliary collapses produce common generic realizations only to refute incompatible decisions. The Prikry-only names in Lemma 7.1 are constructed in the original model N . These facts allow the relative-consistency argument to be formalized over arbitrary countable models, without assuming transitivity.

For class preparation, use the conservative class-theoretic expansion for progressively closed coding. Each set needed in the proof appears in a set-sized initial segment. The corresponding forcing and reflection argument handles any specified finite list of set-theoretic axioms and formulas. The preparation is the high-start $\diamond^*$ version of [18, Appendix, Lemma 20]. Its closure and lifting checks are given in Lemma 12.3. The later GCH collapse and capped-history coding are both set forcing.

A.2. Uniformity of the high coding. The decoder uses three ordinal parameters: the presentation length τ , an initial protection bound, and the envelope rank η . The site recursion takes its size bounds from all of 2^τ , so the candidate forcing Q_e and cap cardinal ρ are defined before the spectrum is recovered. The final predicate asserts that e admits the two generic filters over the class $B = D(V_\eta)$. This is first-order because the posets and their dense subsets belong to the specified definable inner models.

Homogeneity requires the family of all presentation codes. Its common pointed collapse makes A OD without making any particular wellordering of $\text{TC}(\{A\})$ OD. The final HOD can therefore remain choiceless.

APPENDIX B. THE OBSTRUCTION WITHOUT MEASURABLES

The measurable hypothesis cannot simply be omitted, even from the comparison between the weakest external theory and the theory with dependent choice. Put

$$A_L \equiv \mathcal{P}(\omega) = \mathcal{P}^L(\omega) \wedge \omega_2 = \omega_2^L.$$

Lemma B.1. *In any model of $\text{ZF} + A_L$, every countable ordinal sequence belongs to L . Consequently*

$$\text{ZF} + A_L + \text{KW}_1^{*,\omega} \vdash \text{AC}.$$

If $U \models \text{ZFC}$ and $\mathcal{H}^U \models A_L$, then $\mathcal{H}^U = \text{HOD}^U$ and this inner model satisfies Choice.

Proof. Fix an ordinal sequence s in the given universe W , and apply Jensen covering in the choice inner model $L[s]$; see [8, Theorem 1.1]. This use of covering does not assume Choice in W . The models L and $L[s]$ have the same reals, hence the same ω_1 , and ω_2^L remains a cardinal in $L[s]$ because it remains a cardinal in W . Also $0^\#$ does not exist in $L[s]$, since it would be a nonconstructible real.

Covering gives a constructible set y containing $\text{ran}(s) \cup \omega_1$ whose cardinality in $L[s]$ is ω_1 . Its cardinality in L is at most ω_1 . Otherwise an L -injection of ω_2^L into y would collapse that cardinal in $L[s]$. Fix an L -injection $e : y \rightarrow \omega_1$. In $L[s]$, the sequence $e \circ s$ is bounded in ω_1 . An L -enumeration of a countable bound codes this sequence by a real. That real is constructible, so $s \in L$.

Thus each whole sequence space μ^ω is its constructible counterpart and is wellorderable. Under $\text{KW}_1^{*,\omega}$, least preimages wellorder every nonempty set. For the last assertion, $\mathcal{H}^U$ contains every ordinal ω -sequence of U . Applying the first assertion inside $\mathcal{H}^U$ makes every allowed sequence parameter constructible. Every constructible set is ordinal-definable in U , using its index in a canonical constructible wellordering. Replacing the sequence parameters by these ordinal definitions gives $\mathcal{H}^U = \text{HOD}^U$. $\square$

Proposition B.2. *There is a first-order sentence σ_0 such that*

$$\text{ZFC} \vdash \mathcal{H} \models \neg \exists \theta (V_\theta \models \sigma_0),$$

whereas, relative to $\text{Con}(\text{ZFC})$,

$$\text{Con}(\text{ZF} + \text{DC} + \exists \theta (V_\theta \models \sigma_0)).$$

The countermodel can also satisfy $A_L + \neg \text{DC}_{\omega_1}$.

Proof. Over L , force with the countable partial functions

$$P = \text{Fn}_{<\omega_1}(\omega_1 \times \omega_1, 2),$$

ordered by reverse inclusion. Permute the first coordinate, and use the normal filter generated by pointwise stabilizers of countable sets of coordinates. Let N be the hereditarily symmetric model. The forcing is countably closed and, by CH in L , has size ω_1 . It therefore adds no reals and preserves all cardinals. The same reals, ω_1 , and ω_2 occur in L , N , and the full extension. The countably complete symmetry filter and countable closure give $N \models \text{DC}$ by [27, Lemma 3.1].

Let g_i be the generic subset of ω_1 at coordinate i . The set $X = \{g_i : i < \omega_1\}$ belongs to N . Every ordinal-indexed map into X in N has range contained in its countable family of supported generics. Indeed, if a supported name for f satisfied $f(\xi) = g_i$ outside its support, transpose i with a coordinate absent from both the support and a condition deciding this equality. The condition and its image are compatible and force two distinct values at ξ . This is impossible. The generics on each countable support have a countable enumeration in N . Hence X is not wellorderable, and neither is $\mathcal{P}(\omega_1)$. The tree of injections from countable ordinals into X is closed under increasing countable sequences and has no maximal node. It has no ω_1 -branch, since

such a branch would give an injection of ω_1 into X . Thus N also fails transfinite DC_{ω_1} , in the tree formulation of [27, Theorem 2.1].

Choose σ_0 to assert that there is no greatest ordinal; that $\lambda = \omega_1$ and $\mu = \omega_2$ exist, with a limit ordinal $\nu > \mu$; that every real is constructible; that $\lambda = \omega_1^L$ and $\mu = \omega_2^L$; and that $\mathcal{P}(\lambda)$ is not wellorderable. These are combined into one first-order sentence, with λ, μ, ν existentially quantified.

Suppose $V_\theta^W \models \sigma_0$, where $W \models \text{ZF}$. The first clause makes θ a limit, and the guard gives $\theta > \mu + \omega$. All functions between ordinals at most μ , and all subsets and binary relations on $\mathcal{P}(\lambda)$, belong to this rank whenever they belong to W . Thus its ω_1 and ω_2 are those of W , and its nonwellorderability assertion remains true in W . The local constructible hierarchy agrees with the actual one at every level below θ . To verify this, code finite assignments by finite function graphs: the satisfaction relation for a set structure and a hierarchy sequence through a fixed level require only a fixed finite increase in rank. All such witnesses remain below the limit θ . In particular, every real of W is constructible.

It remains to check the second constructible cardinal. Equality of reals gives $\lambda = \omega_1^L$ globally, and inner-model cardinal comparison gives $\gamma = \omega_2^L \leq \mu = \omega_2^W$. If $\gamma < \mu$, then γ is also a cardinal of the rank's local L : a local constructible bijection collapsing it would be a global constructible bijection. This puts an L -cardinal strictly between the asserted first and second uncountable L -cardinals, a contradiction. Therefore $W \models A_L$.

If $W = \mathcal{H}^U$ for a choice universe U , Lemma B.1 gives Choice in W , contradicting the retained nonwellorderability assertion. On the other hand, the symmetric model N satisfies σ_0 globally, so reflection gives a rank satisfying it. This is a formal relative-consistency construction; no transitive-model assumption is required. $\square$

The same sentence gives a rank-conservativity obstruction: $\text{ZF} + \text{KW}_1^{*,\omega}$ proves that no rank satisfies σ_0 , while $\text{ZF} + \text{DC}$ does not, assuming consistency. This proves that CM cannot be deleted from the stated theorem. It does not identify the weakest sufficient large-cardinal hypothesis.

APPENDIX C. A RAMSEY OBSTRUCTION

Let CR assert that the cardinals ρ satisfying $\rho \rightarrow (\rho)_2^{<\omega}$ are unbounded in the ordinals, and put $T_R = \text{ZF} + \text{DC} + \text{CR}$. In this section $\mathbb{R} = \mathcal{P}(\omega)$. The three sorts of arithmetic are ω , $\mathbb{R}$, and $\mathcal{P}(\mathbb{R})$. A Π_3^2 sentence has three alternating blocks of third-sort quantifiers, beginning universally, followed by a matrix with only number and real quantifiers. Finite tuples of third-sort variables may be combined by fixed recursive tags.

Theorem C.1 (Ramsey obstruction). *There is a parameter-free Π_3^2 sentence Φ such that*

$$T_R + \text{KW}_1^{*,\omega} \vdash \Phi, \quad \text{Con}(\text{ZFC} + \text{CR}) \implies \text{Con}(T_R + \neg\Phi).$$

Consequently, relative to $\text{Con}(\text{ZFC} + \text{CR})$, adding $\text{KW}_1^{,\omega}$ is not conservative over T_R for third-order arithmetic. The same example gives a structural Π_2 assertion separating these theories.*

We use the classical Dodd–Jensen core model. In a choiceless universe W , set $K = (K^{\text{DJ}})^{\text{HOD}^W}$. The semantic form of the separating implication is

$$[\neg\exists 0^\dagger \wedge \mathbb{R} = \mathbb{R}^K \wedge \omega_2 = \omega_2^K] \longrightarrow \text{Fodor}(\omega_1). \quad (\text{C.1})$$

We first prove this implication with $\text{KW}_1^{*,\omega}$, then give its third-order formulation and a counter-model.

Proof. The covering implication. Under CR, absence of $0^\dagger$ excludes an inner model with a measurable. Indeed, from such an inner model obtain its canonical $L[U]$, with measurable κ . Its wellordering gives an ordinal code $A \subseteq \alpha$ for the transitive closure of U , where $\alpha > \kappa$, $A \in L[U]$,

and $L[A] = L[U]$. A Ramsey above $\alpha + \omega$ supplies indiscernibles for $L[A]$. Shifting above α gives a nontrivial self-embedding of $L[U]$ with critical point above κ , which implies $0^\dagger$. Here the relativized indiscernibles argument is applied to canonical structures with definable Skolem functions; it is available in ZF [29, Propositions 7–8] [28, p. 262]. The last implication is the standard nonrigidity characterization of $0^\dagger$ [30]. The possibly uncountable parameter sharp is not itself being identified with a real.

Assume the antecedent of (C.1), and fix $s \in (\text{Ord}^\omega)^W$. Code its graph by a set of ordinals $\bar{s}$. Apter–Koepeke’s adjoining construction gives

$$C = (\text{HOD}^W)[\bar{s}] \models \text{ZFC}, \quad K \subseteq C \subseteq W, \quad K^{\text{DJ}, C} = K$$

with $s \in C$ [29, Proposition 9]. The endpoint agreements give common ω_1 and ω_2 in these three models. Apply strong Dodd–Jensen covering inside C to $\text{ran}(s) \cup \omega_1$ [8, Theorem 1.2]. It has a cover $y \in K$ with $|y|^C = \omega_1$. Correct ω_2 implies $|y|^K \leq \omega_1$. For a K -surjection $e : \omega_1 \rightarrow y$, the sequence of least e -indices of the values of s is bounded in ω_1 in C . A K -enumeration of a countable bound codes that sequence by a real. The real belongs to K , so $s \in K$.

Thus $(\xi^\omega)^W = (\xi^\omega)^K$ for every ordinal ξ . Each of these spaces has a K -wellordering. Under $\text{KW}_1^{*, \omega}$, least preimages under a sequence-space surjection wellorder every nonempty set. Hence AC, and therefore Fodor’s lemma, follows.

The third-order formula. Use the classical one-measure lower-part presentation of K [33, Section 3]. Countable mousehood has a projective definition, and the corresponding countable-hull tests use only real codes [8, Lemma 3.32 and p. 92]. For $A \subseteq \mathbb{R}$, a certificate for $\text{KSet}(A)$ consists of a wellordered real domain, a well-founded extensional premouse relation on it, its measure predicate and satisfaction relation, actual-real labels, and a distinguished lower-part node whose elements are exactly A . Require every real-coded countable elementary hull to collapse to an iterable classical premouse. Also allow a constructible level certificate.

The domain wellorder makes the entire labelled certificate ordinal-codable. In the choice model obtained by adjoining that code to HOD , the countable-hull copying criterion gives iterability, and Proposition 9 of [29] identifies the core with K . This proves soundness. For completeness under $\mathbb{R} = \mathbb{R}^K$, choose the witness inside K , with both $\mathbb{R}^K$ and A in its lower part. A canonical hull containing every real and these two nodes has size at most ω_1 and its collapse fixes A . The small-mouse argument in the proof of [33, Lemma 3.1], applied inside K at ω_2^K , also gives this witness directly. CH in K supplies a real-domain code. Equality of reals preserves its countable-hull tests. Consequently

$$\mathbb{R} = \mathbb{R}^K \implies (\text{KSet}(A) \longleftrightarrow A \in K).$$

Under DC, a relation on a real domain is well-founded exactly when it has no real-coded descending omega-sequence. All the certificate conditions therefore have a second-order matrix after one existential third-sort quantifier: $\text{KSet}(A) \equiv \exists P k(A, P)$. Let CopyK say that every wellorder on a subset of the reals has an isomorphic real-domain wellorder with such a core certificate. Bundling the copy, certificate and isomorphism gives

$$\text{CopyK} \equiv \forall E \exists P c(E, P),$$

where c is second-order; malformed inputs permit a dummy certificate. On $\mathbb{R} = \mathbb{R}^K$, this is equivalent to $\omega_2 = \omega_2^K$. For if the cardinals agree, every real-domain order type is below ω_2^K and has a core copy. If $\omega_2^K < \omega_2^W$, an ambient real-domain wellorder of type ω_2^K has no such copy. The old CH bijection excludes the reverse cardinal inequality.

Write B for the second-order guard $\neg \exists 0^\dagger \wedge \forall x \in \mathbb{R} (x \in K)$. Here $0^\dagger$ uses its usual real-coded two-block indiscernible theory and countable well-foundedness tests. Let U code a wellorder w of all reals with countable proper initial segments, a set S of its points, and a regressive function on S . Such a w has type ω_1 . Fodor’s assertion says either that a club avoids S , or that one fiber

meets every club. An existential third-sort witness V stores a tag and either that first club or the single real naming the fiber. Thus its real-domain version is $\forall U \exists V \forall D q(U, V, D)$, where D tests clubs and q is second-order. Invalid U is accepted automatically. No family of clubs is selected.

The required Φ is the following normal form:

$$\forall U \exists (E, V) \forall (P, D) [\neg B \vee \neg c(E, P) \vee q(U, V, D)]. \quad (\text{C.2})$$

All prefix variables have the third sort. If CopyK fails, a single E with no certificate makes the matrix true. If it holds, a certificate for the selected E reduces the matrix to the Fodor clause. Thus (C.2) is equivalent, over $\text{ZF} + \text{DC}$, to (C.1). This proves the advertised Π_3^2 upper bound, without claiming minimality.

The countermodel. Start with $\text{ZFC} + \text{CR}$ and pass to its classical core D . Mitchell's Ramsey preservation theorem gives $D \models \text{ZFC} + \text{GCH} + \text{CR}$ [28, Theorem 1]. Also $K^D = D$ and D has no $0^\dagger$. Apply Karagila's local symmetric construction at $\kappa = \mu = \omega_1^D$ [31, Section 4]. First add $g : \kappa \rightarrow \kappa$ by bounded conditions, and set $S_\beta = \{\xi < \kappa : g(\xi) \geq \beta\}$. Then use the countable-support product of the club-shooting orders on the S_β , with the normal filter fixing old countable sets of coordinates. Let N be the symmetric model.

The combined forcing has a countably closed dense presentation of size ω_1 . A dense condition synchronizes the club maxima and chooses the value of g there above every active index. Countable unions have lower bounds of the same form, and CH bounds the number of codes. The reals and cardinals are unchanged, and the countably complete support filter gives DC in N . Every ordinal-coded object of N belongs to a bounded head $D[G_\alpha]$, and every such head is contained in N .

Put $T_* = \kappa \setminus \Delta_{\beta < \kappa} S_\beta$. Then $f = g \upharpoonright T_*$ is regressive and each fiber of value η misses the retained club through $S_{\eta+1}$. The set T_* is stationary in N . Indeed, a retained club belongs to a head $D[G_\alpha]$. Fusion in that head produces a limit point $\delta > \alpha + 1$ of the club at which one may set $g(\delta) = \alpha$. Appending δ to every active coordinate below α is legal, while $\delta \notin S_{\alpha+1}$. This is stationarity in N ; the full choice extension retains an additional diagonal club.

For any old Ramsey $\rho > \omega_1$, a coloring in N of $[\rho]^{<\omega}$ belongs to one bounded head. That head is a choice extension by a forcing of size $\omega_1 < \rho$. Small-forcing preservation supplies a homogeneous set there [32, Theorem 4.1], so that set also belongs to N . Hence $N \models \text{CR}$. Classical mouse absoluteness and set-forcing invariance identify $K^N = D$; unchanged reals preserve the $0^\dagger$ predicate. Thus the antecedent of (C.1) holds in N , while its conclusion fails. The old CH bijection transports this counterexample to the third-order formula. These forcing and inner-model constructions give the stated consistency implication. $\square$

For the structural conclusion, use a finite signature with number and real sorts partitioning its domain, the arithmetic operations on the number sort, and a membership relation between the two sorts. A full second-order sentence asserts the Peano axioms with second-order induction on the first sort and requires every subset of that sort to be coded by exactly one element of the second. Such a structure is isomorphic to the canonical structure on $\omega \sqcup \mathcal{P}(\omega)$. Translate third-sort variables in Φ into second-order subsets of its real sort. The universal assertion that every such frame satisfies this translation is equivalent to Φ in ZF. It has the structural Π_2 form of Definition 1.7. The Π_3^2 bound concerns third-order arithmetic, whereas this Π_2 classification concerns the Lévy hierarchy.

In particular, the Ramsey replacement of Theorem 1.8 fails even with Foundation retained in the starting theory, and hence also with Foundation omitted. This argument does not assert that $\text{KW}_1^{*,\omega}$ alone implies Fodor's lemma, nor does it supply a Ramsey replacement counterexample for the separate external rank-equiconsistency theorem without a further rank translation.

APPENDIX D. A STRONG-CARDINAL SEPARATION FOR THE CHANG MODEL

We give the separation used in the introduction to explain the choice of $\text{HOD}_{\text{Ord}^\omega}$ in Theorem 1.1. Throughout this appendix, $C^U = L(\text{Ord}^\omega)^U$ is computed in an ambient universe $U \models \text{ZFC}$.

Define

$$E_1^C(\sigma) = \text{ZFC} + \text{CM} + [C \models \exists \kappa (V_\kappa \models \sigma)].$$

Let **CS** assert that strong cardinals are unbounded, with strongness given by its usual first-order extender formulation, and put

$$B_{\text{str}} = \text{ZFC} + \text{CS} + \text{“there is an inaccessible limit of strong cardinals.”}$$

The following single first-order sentence will separate the theories:

$$\begin{aligned} \sigma_{\text{str}} \equiv & \exists \lambda \exists \nu [\lambda, \nu \in \text{Ord} \wedge \lambda < \nu \\ & \wedge \nu \text{ is a limit ordinal} \\ & \wedge V_\lambda \models \text{ZFC} + \text{CS}]. \end{aligned} \tag{D.1}$$

Here the rank and its satisfaction relation are expressed by set witnesses, including satisfaction of the recursive set of all ZFC axioms. The ordinal ν is a height guard: in a witnessing outer rank V_κ , it ensures $\lambda + \omega \leq \nu < \kappa$.

Theorem D.1 (Chang rank separation). *If $U \models \text{ZFC}$, no full rank $V_\lambda^{C^U}$ satisfies $\text{ZFC} + \text{CS}$. In particular, $E_1^C(\sigma_{\text{str}})$ is inconsistent, whereas*

$$\text{Con}(B_{\text{str}}) \implies \text{Con}(E_3(\sigma_{\text{str}})).$$

Thus, relative to B_{str} , substituting the Chang model for $\mathcal{H}$ destroys even the equiconsistency of theories (1) and (3). The prohibition on Chang ranks requires no ambient measurable cardinals. Strong cardinals enter the consistency bound for the separating instance of theory (3). For a tautological σ , the two Chang analogues remain consistent relative to $\text{ZFC} + \text{CM}$.

D.1. The imported mouse construction. We use Mitchell’s Chang-sharp construction. The relevant results are [24], Theorem 1.5, Sections 1.3 and 3, Definition 4.1, and Proposition 1.4.

The construction uses a set mouse over all the reals with a sufficiently long last extender. Section 3 reduces the hypothesis to set-sized data, and Definition 4.1 specifies the additional approximation system. The relevant mice use the linear iteration theory below a strong cardinal. Iterating the final extender gives a continuous class $I = \{\kappa_\alpha : \alpha \in \text{Ord}\}$. Definition 3.9 gives uniformly interpreted term codes from the mouse. These terms represent every element of Chang. Proposition 1.4 extends appropriate order-preserving maps of I to elementary embeddings of Chang.

The large cardinal existence input is that $\text{ZFC} + \text{CS}$ supplies such a mouse over its entire real set, with the required iterability. Gappo and Sargsyan explicitly identify this upper bound in the introduction to [25]. We call the full set data specified by Mitchell a *Mitchell datum*. This abbreviation includes the approximation system; a bare premouse with a long extender is not sufficient.

All structural requirements on a fixed datum are first-order assertions about set structures, their satisfaction relations, functions, and ordinals. Their interpretation is absolute when the data and the relevant countable sequences are shared. The remaining nonlocal requirement is iterability. The next section addresses that requirement rather than assuming it is absolute.

D.2. Countable hull iterability transfers from a Chang rank.

Lemma D.2. *Suppose $W = V_\lambda^{C^U} \models \text{ZFC}$. Then W and U have the same reals and the same ω_1 . If $a \in W$, every U -countable sequence of elements of a belongs to W .*

Proof. The Chang model is closed under countable sequences formed in U . In particular it contains every real and every countable ordinal sequence of U .

If $a \in W$ and $s : \omega \rightarrow a$ belongs to U , then $s \in C^U$. Its rank is bounded by $\max\{\text{rank}(a), \omega\}$ plus a fixed finite ordinal. This is below λ , since $W \models \text{ZFC}$. Hence $s \in W$.

Every real has bounded finite rank above ω , so the real sets agree. If an ordinal in W is countable in U , its enumeration belongs to W by the preceding argument. Applying this to ω_1^W , if necessary, proves equality of ω_1 . $\square$

Lemma D.3. *Let W be as above. Suppose $M \in W$ is a set premouse in the linear iteration setting of [24]. If W regards every countable elementary hull of M as iterable through all countable linear iterations, then the same assertion holds in U . Moreover M has no ill-founded set-length linear iteration in U .*

Proof. Include the extender predicates and the final extender in the structure being considered. Let $X \prec M$ be any elementary substructure countable in U . An enumeration of X is a countable sequence from the fixed set underlying M , so Lemma D.2 puts X and an enumeration in W . Set satisfaction is absolute, so W recognizes its elementarity. The two universes form the same transitive collapse $\bar{M}$.

A countable premouse, a countable putative linear iteration, and a descending sequence witnessing ill-foundedness can be coded by reals. The relevant code relations are statements of second-order arithmetic. Since W and U have exactly the same reals, they agree on these relations and on the assertion that every countable legal linear iteration of $\bar{M}$ is well-founded. Thus every countable hull remains countably iterable in U .

For completeness, suppose there were a bad set-length linear iteration of M in U . Retain the iteration through a first bad model, together with a descending-sequence witness for its formal ultrapower or direct-limit relation. In U , take a countable elementary hull of a sufficiently large set structure containing these data and the definitions of the iteration operations. Its transitive collapse gives a countable bad linear iteration of the collapse of a countable elementary substructure of M .

The collapse preserves the iteration rules. Formal ultrapowers use functions from the relevant source structure, and elementarity supplies representatives of their quotient elements in the hull. The same reasoning applies to the direct-limit construction. The retained descending sequence witnesses that the last relation is still ill-founded. This contradicts the preceding paragraph.

This argument concerns linear iteration. It does not assert preservation or existence of a strategy for arbitrary uncountable branching iteration trees. $\square$

D.3. Why the datum cannot belong to Chang.

Proposition D.4. *A Mitchell datum over all the reals of U , with the requisite external linear iterability, cannot belong to C^U .*

Proof. Suppose the datum belongs to $C = C^U$. Write (M, E) for its mouse and final extender. The canonical linear iteration by E can be computed inside C at every set-length stage. Its computation agrees with that in U : the source structures and their functions are the same, formal ultrapowers and direct limits are the same, and well-founded collapses are unique. Transfinite recursion in C supplies the set-length iteration systems. Thus the class

$$I = \{\kappa_\alpha : \alpha \in \text{Ord}\}$$

and the interpretation of Mitchell's term codes are definable over C from set parameters in the datum.

This last definability assertion can be read directly from Definition 3.9 of [24]. Its ingredients are the canonical iteration maps, countable sequences of term codes, composition, and satisfaction

in a set-sized Chang construction stage. In particular, the apparently class-length term $i_{\text{Ord}}(f)(a)$ is evaluated at the first set stage α for which all entries of the finite ordinal tuple a are below κ_α . The value $i_\alpha(f)(a)$ is below κ_α and is fixed by every subsequent iteration map. Thus evaluation is uniformly first-order definable from the set data. No truth predicate for the whole Chang model is required. The auxiliary approximation system and generic objects used in the proof of indiscernibility do not enter the interpretation of these terms.

Define an increasing map on ordinal indices by

$$f(\alpha) = \begin{cases} \alpha, & \alpha < \omega_1, \\ \omega_1 + \alpha, & \alpha \geq \omega_1, \end{cases} \quad s(\kappa_\alpha) = \kappa_{f(\alpha)}.$$

The map preserves successors and limit-index status. It is continuous at limit indices of countable cofinality. Its only initial discontinuity is at ω_1 , whose cofinality is uncountable in both universes. Therefore s satisfies the hypotheses of Mitchell's Proposition 1.4.

Using the term representation of Chang, define

$$j(\tau(B)) = \tau(s[B]).$$

The equality and formula preservation furnished by the sharp make this definition independent of the chosen representation and make $j : C \rightarrow C$ elementary. The relation defining j quantifies over set term codes and countable suitable sets B ; all its ingredients are definable over C from the datum. In particular j is a parameter-definable class of C . It is nontrivial because

$$j(\kappa_{\omega_1}) = \kappa_{\omega_1 \cdot 2} > \kappa_{\omega_1}.$$

Suzuki's theorem [26], which requires ZF rather than Choice, excludes such a parameter-definable nontrivial elementary self-embedding of C . This is a contradiction. $\square$

D.4. The separation.

Proof of Theorem D.1. Suppose $W = V_\lambda^{C^U} \models \text{ZFC} + \text{CS}$. Inside W , the imported mouse construction produces a Mitchell datum over $\mathbb{R}^W$. Lemma D.2 gives $\mathbb{R}^W = \mathbb{R}^U$ and the same countable sequences from each fixed set in the datum. The bijection witnessing $|M| = |\mathbb{R}|$ remains a bijection externally. All elementarity and internal cardinal computations in the fixed structures remain correct. The height of M still has cofinality ω_1 : its cofinal ω_1 -sequence remains cofinal, while any new countable cofinal sequence would belong to W by bounded closure. Thus the structural requirements hold externally. Lemma D.3 transfers the required linear iterability. The datum belongs to $W \subseteq C^U$, contradicting Proposition D.4.

If a rank $V_\kappa^{C^U}$ satisfies σ_{str} , the height guard puts $\lambda + \omega$ below κ . Induction on the witnessed cumulative hierarchy identifies its nested rank with the actual $V_\lambda^{C^U}$. Every subset required at an earlier hierarchy stage is present in the outer rank. Finite assignments and the full set satisfaction relation also lie below $\lambda + \omega$, so satisfaction agrees. This gives precisely the impossible W above, proving inconsistency of $E_1^C(\sigma_{\text{str}})$.

For the consistency direction, work in a model of B_{str} , and let δ be an inaccessible limit of ambient strong cardinals. Then $V_\delta \models \text{ZFC}$. Every ambient strong $\rho < \delta$ is strong in V_δ : for each target rank V_η below δ , choose the usual bounded extender E derived from an ambient strongness embedding. Inaccessibility of δ puts its support, the extender, and the required power sets below δ .

To check the target-rank containment, represent each $x \in V_\eta$ in the ambient extender ultrapower as $[a, f]_E$. Replace f by the function that keeps its values in $V_{\eta+1}$ and uses $\emptyset$ elsewhere. It represents the same x , since the ultrapower image of $\eta + 1$ exceeds η . The new function has rank below δ . Hence the natural factor map from the ultrapower of V_δ has every element of V_η in its

range. Its transitive collapse fixes these elements by induction on rank. Thus the ultrapower of V_δ contains V_η , as required. This proves $V_\delta \models \text{CS}$.

The ambient universe satisfies **CM**, since strong cardinals are measurable and are unbounded. It satisfies **ZFC**, hence **ZF + DC**. The rank $V_{\delta+\omega \cdot 2}$ satisfies σ_{str} , using $\lambda = \delta$ and $\nu = \delta + \omega$. This realizes $E_3(\sigma_{\text{str}})$. $\square$

The obstruction is a set-sized iterable datum that a full Chang rank satisfying **ZFC + CS** would contain. The height guard in σ_{str} ensures that the nested rank and its satisfaction relation have their intended meanings.

D.5. The anti-inner-model question. The separation above leaves open whether restricting the available inner models restores the comparison. Use the standard Mitchell-order convention, so a measurable cardinal has order at least one, and let

$$A_2 \quad \equiv \quad \text{“there is no inner model of ZFC with a measurable } \nu \text{ satisfying } o(\nu) \geq 2.”$$

The Mitchell order is computed in the indicated inner model. Equivalently, the excluded configuration consists there of two normal measures U_0, U_1 on the same ν with $U_0 \in \text{Ult}(M, U_1)$. Here M is the inner model, and the ultrapower uses its functions. We use the semantic inner-model formulation of A_2 ; any formal version of the following question must fix a corresponding mouse or scheme formulation on both sides.

Question D.5. Can an anti-inner-model hypothesis restore the equiconsistency of theories (1) and (3) after the Chang substitution? In particular, for every fixed first-order sentence σ , is

$$\text{Con}(E_1^C(\sigma) + A_2) \quad \longleftrightarrow \quad \text{Con}(E_3(\sigma) + A_2)?$$

In this question A_2 is appended to both ambient theories. The left-hand witnessing rank is still computed inside Chang, and the right-hand rank is computed in the **ZF + DC + CM** universe. The restriction permits order-one measurable cardinals and does not exclude measurable limits of measurable cardinals. No equiconsistency under this restriction is asserted here. Whether the Chang separation itself can be obtained from ordinary measurable-cardinal strength also remains open.

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