

UNDER MEASURABLES, $\text{HOD}_{\text{Ord}^\omega}$ IS LOCALLY ARBITRARY

ABSTRACT. For the inner model $\text{HOD}_{\text{Ord}^\omega}$, we derive an analogue for Roguski’s classical result that without nontrivial assumptions on V , HOD is an arbitrary model of ZFC . For any sentence σ , letting CM denote “class of measurables,” the following theories are equiconsistent:

- (1) $\text{ZFC} + \text{CM} + [\text{HOD}_{\text{Ord}^\omega} \models (\exists \kappa V_\kappa \models \sigma)]$;
- (2) $\text{ZFC} + \text{GCH} + \text{CM} + [\text{HOD}_{\text{Ord}^\omega} \models (\text{SVC} \wedge \text{CM} \wedge \exists \kappa V_\kappa \models \sigma)]$;
- (3) $\text{ZF} + \text{DC} + \text{CM} + \exists \kappa V_\kappa \models \sigma$.

Thus, under measurables we have that $\text{HOD}_{\text{Ord}^\omega}$ is “locally arbitrary.”

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1. EXTERNAL HOD AND LOCAL ARBITRARINESS

Roguski’s work on the theory of HOD is the starting point for this paper [1]. In the countable transitive formulation of his realization theorem, every countable transitive model M of ZFC has a larger model N of ZFC such that

$$M = \text{HOD}^N.$$

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Ordinary HOD can thus realize a prescribed ZFC universe. We seek an analogue when definitions may use countable sequences of ordinals as parameters. Our result concerns the theories of rank segments of the resulting inner model, under the assumption of a class of measurable cardinals.

For $U \models \text{ZFC}$, write

$$\mathcal{H}^U = \text{HOD}_{\text{Ord}^\omega}^U.$$

An allowed parameter is one function from ω to the ordinals, together with finitely many ordinal parameters; the hereditary requirement is retained. This inner model contains every ordinal ω -sequence of U and satisfies $\text{ZF} + \text{DC}$ (Lemma 10.1). The local analogue of Roguski's theorem will therefore compare its rank theory with that of choiceless models satisfying dependent choice.

Write CM for the assertion that measurable cardinals are unbounded in the ordinals, abbreviated as “class of measurables.” A measurable cardinal is uncountable and carries a nonprincipal, cardinal-complete ultrafilter. We require normality only in choice models. Put

$$T_0 = \text{ZF} + \text{DC} + \text{CM}.$$

For each first-order sentence σ , consider the theories

$$\begin{aligned} E_1(\sigma) &= \text{ZFC} + \text{CM} + [\mathcal{H} \models \exists \kappa (V_\kappa \models \sigma)], \\ E_2(\sigma) &= \text{ZFC} + \text{GCH} + \text{CM} + [\mathcal{H} \models \text{SVC} \wedge \text{CM} \wedge \exists \kappa (V_\kappa \models \sigma)], \\ E_3(\sigma) &= T_0 + \exists \kappa (V_\kappa \models \sigma). \end{aligned}$$

In the first two theories, the witnessing rank is computed inside $\mathcal{H}$. Theory E_2 requires GCH in the ambient universe and SVC together with a class of measurables inside $\mathcal{H}$. Our main result is the following.

Theorem 1.1 (External rank equiconsistency). *For every σ ,*

$$\text{Con}(E_1(\sigma)) \longleftrightarrow \text{Con}(E_2(\sigma)) \longleftrightarrow \text{Con}(E_3(\sigma)).$$

This is the sense in which $\text{HOD}_{\text{Ord}^\omega}$ is locally arbitrary under measurables. Every rank-sentence assertion consistent with $\text{ZF} + \text{DC} + \text{CM}$ can hold inside the sequence-parameter HOD of a $\text{ZFC} + \text{GCH} + \text{CM}$ universe, even when that HOD is itself required to satisfy $\text{SVC} + \text{CM}$. GCH is imposed on the ambient universe, so it does not restrict the rank theory being realized inside $\mathcal{H}$. The comparison concerns consistency of rank-sentence theories. Exact realization of an entire starting model is available under the prepared-ground hypotheses stated below.

1.1. Realization and rank preservation. The construction first preserves a specified rank while passing to a model generated by a set of ordinal sequences. It then realizes that rank inside external HOD. Two presentation principles enter the first step. Write $\text{SVC}(S)$ when every nonempty set is a surjective image of $\eta \times S$ for some ordinal η ; SVC asserts that such a set S exists. The assertion $\text{KW}_1^{*,\omega}$ says that every nonempty set is a surjective image of μ^ω for some ordinal μ , with the sequence power computed in the model under discussion. Write $T_s = T_0 + \text{SVC}$.

The external realization step takes the following form.

Theorem 1.2 (External realization). *Suppose $M \models T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$. For every ordinal α there is an outer model $U \models \text{ZFC} + \text{GCH} + \text{CM}$ of M such that*

$$\mathcal{H}^U \models \text{SVC} + \text{CM}, \quad V_\alpha^{\mathcal{H}^U} = V_\alpha^M.$$

If M has a presentation as in Proposition 13.1 whose choice envelope satisfies eventual GCH, one may require $\mathcal{H}^U = M$.

The secure-ground case gives equality with the whole generated model by further set forcing over its fresh choice envelope. The general preparation uses class forcing, so the outer model in Theorem 1.2 is not asserted to be a set-forcing extension of M .

The next theorem supplies the required presentation while retaining any chosen rank and its full power set. In particular, SVC is not an assumption on the original model. Here $D(A)$ denotes the model generated over a transitive base. It contains D and the transitive closure of $\{A\}$.

Theorem 1.3 (Rank transfer without SVC). *Suppose $V \models T_0$ and $\alpha \in \text{Ord}$. In a set-generic extension of V there is a model M such that*

$$M \models T_0 + \text{SVC} + \text{KW}_1^{*,\omega}, \quad V_\alpha^M = V_\alpha^V, \quad \mathcal{P}(V_\alpha^V)^M = \mathcal{P}(V_\alpha^V)^V.$$

Moreover, $M = D(A)$ for a choice symmetric ground $D \models \text{ZFC} + \text{CM}$ and a set $A \subseteq \mu^\omega$.

The model M contains a rank-preserving SVC inner model of V , but need not contain all of V . Once the starting model satisfies SVC, the local construction also gives containment.

Theorem 1.4 (Local rank-to-sequence transfer). *Suppose $N \models T_s$ and $\alpha \in \text{Ord}$. In a set-generic extension of N there is an inner model M containing N such that*

$$M \models T_s + \text{KW}_1^{*,\omega}, \quad V_\alpha^M = V_\alpha^N.$$

More precisely, one can choose $\lambda > \alpha$ and $X = V_\lambda^N$ so that

$$\mathcal{P}(X)^M = \mathcal{P}(X)^N, \quad V_\lambda^M = X.$$

There are a choice symmetric ground $D \models \text{ZFC} + \text{CM}$, an ordinal μ , and $A \subseteq \mu^\omega$ in M such that $M = D(A)$.

The two transfer statements also give the following equiconsistency.

Corollary 1.5 (Rank-sentence equiconsistency). *For every first-order sentence σ , the theories*

$$T_0 + \exists \alpha (V_\alpha \models \sigma) \quad \text{and} \quad T_0 + \text{SVC} + \text{KW}_1^{*,\omega} + \exists \alpha (V_\alpha \models \sigma)$$

are equiconsistent.

Proof. Apply Theorem 1.3 to a witnessing rank. The converse is immediate. $\square$

Combining rank transfer with external realization gives the implication from $E_3(\sigma)$ to $E_2(\sigma)$ at the level of consistency. The implication from $E_2(\sigma)$ to $E_1(\sigma)$ is immediate. For the remaining implication, Proposition 10.3 uses core-model preparation and homogeneous descent to obtain an inner model of the given sequence-parameter HOD while retaining its witnessing rank. This step does not assume that the original HOD already has a class of measurables.

1.2. Structural conservativity. The rank-preserving construction also yields a structural conservativity theorem. This result concerns full second-order truth in arbitrary set-sized structures and allows a starting theory without Foundation. Put

$$T_- = (\text{ZF} - \text{Foundation}) + \text{DC} + \text{CM}.$$

Definition 1.6 (Structural Π_2 sentences). A *structural Π_2 sentence* is a set-theoretic assertion of the form

$$\forall \mathcal{M} (\mathcal{M} \text{ is a set-sized } \mathcal{L}\text{-structure} \implies \mathcal{M} \models_{\text{full}} \varphi),$$

where φ is a fixed second-order sentence in a finite signature $\mathcal{L}$. Satisfaction uses full second-order semantics: if A is the domain of $\mathcal{M}$, each n -ary second-order relation variable ranges over all of $\mathcal{P}(A^n)$ in the ambient universe. This assertion has a Π_2 formulation in the Lévy hierarchy. The second-order quantifier complexity of φ is unrestricted.

Theorem 1.7 (Structural conservativity). *The theory $T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$ is conservative over T_- for structural Π_2 assertions. That is, for every fixed full second-order sentence φ , the assertion that φ holds in every set-sized structure has the same provability in the two theories.*

Foundation removal preserves structural truth. Theorem 1.7 does not identify the non-well-founded membership structure of an arbitrary starting universe with a rank of a well-founded model. The argument represents a full second-order counterexample by a structure on atoms. After the local forcing construction, fibers of ordinal-sequence surjections give a pure copy of that structure and of an SVC witness. The full power sorts retain the truth of the original sentence.

Remark 1.8 (Role of the earlier preparation). The core-model argument gives a passage to SVC that preserves each specified rank. We use this rank-preserving result directly. The proof uses a core model over the coding real inside relative HOD. Its pure universe is invariant under finite recodings and has a set-forcing presentation over its own HOD. The proof does not require $V = \text{HOD}(r)$. The earlier $\text{SVC} + \text{KW}_1^{*,1}$ conservativity result is compatible with this preparation and is not an additional hypothesis of the structural conservativity theorem.

1.3. Limits of the conclusions. The large-cardinal assumption cannot simply be deleted from Theorem 1.1. Put $\mathbb{R} = \mathcal{P}(\omega)$ and

$$A_L \equiv \mathbb{R} = \mathbb{R}^L \wedge \omega_2 = \omega_2^L.$$

Over ZF, this agreement implies that every countable ordinal sequence belongs to L . To see this, fix such a sequence s and apply Jensen covering in the choice model $L[s]$. A constructible cover of its range has size at most ω_1 there. Agreement at ω_2 makes its constructible size at most ω_1 as well; agreement on reals then returns s to L . Thus

$$\text{ZF} + A_L + \text{KW}_1^{*,\omega} \vdash \text{AC}.$$

Likewise, if $\mathcal{H}^U \models A_L$ for $U \models \text{ZFC}$, all its allowed sequence parameters are constructible, and $\mathcal{H}^U = \text{HOD}^U$. A symmetric extension of L using countable conditions and countable supports satisfies $\text{ZF} + \text{DC} + A_L$ while $\mathcal{P}(\omega_1)$ is not wellorderable and DC_{ω_1} fails. Appendix B gives the construction and a single guarded rank sentence σ that can hold in this model but in no rank of any external $\mathcal{H}^U$. Hence the rank-equiconsistency conclusion itself fails when CM is removed.

There is also an obstruction below measurables to the structural conservativity theorem. Let CR assert a class of Ramsey cardinals, with Ramseyness understood as $\kappa \rightarrow (\kappa)_2^{<\omega}$. There is a parameter-free Π_3^2 sentence Φ of third-order arithmetic such that, for $T_R = \text{ZF} + \text{DC} + \text{CR}$,

$$T_R + \text{KW}_1^{*,\omega} \vdash \Phi, \quad \text{Con}(\text{ZFC} + \text{CR}) \implies \text{Con}(T_R + \neg\Phi).$$

Its semantic form over this base is

$$[\neg\exists 0^\dagger \wedge \mathbb{R} = \mathbb{R}^K \wedge \omega_2 = \omega_2^K] \implies \text{Fodor}(\omega_1),$$

where K is the classical Dodd–Jensen core computed in HOD. The Ramsey class and the absence of $0^\dagger$ exclude an inner model with a measurable. Strong covering then returns every countable ordinal sequence to K , so $\text{KW}_1^{*,\omega}$ implies Choice under the displayed agreement. A countably supported symmetric extension of the classical core retains the agreement, DC, and a class of Ramseys, but has a stationary regressive map with no stationary fiber. Appendix C proves the claim and gives its third-order formulation. The Π_3^2 bound counts quantifiers over sets of reals; it is not a claim of minimal complexity. The same sentence has a structural formulation as in Definition 1.6. Thus CR cannot replace CM in Theorem 1.7.

The choice of inner model matters as well. Let $C = L[\text{Ord}^\omega]$ denote the Chang model, the least transitive inner model of ZF containing every countable ordinal sequence. Under the same ambient measurables, its failure of SVC prevents the strengthened conclusion of Theorem 1.1.

Proposition 1.9 (The Chang obstruction). *If $U \models \text{ZFC} + \text{CM}$, then $C^U \models \neg\text{SVC}$.*

Proof. Every member of C is uniquely definable over C from a countable ordinal sequence, with finite ordinal parameters absorbed into that sequence [25, Section 1.2]. For a normal ultrapower

$j : U \rightarrow M$, countable closure gives $C^M = C^U$; hence $j \upharpoonright C : C \rightarrow C$ is elementary. We also use Kunen’s finite-movement lemma: after choosing normal measures at distinct measurable cardinals, each ordinal is moved by only finitely many of the corresponding ultrapowers [24, Lemma 4.10]. The stated form follows from the countable-sequence version by selecting infinitely many exceptions if they exist.

Suppose $S \in C$ witnesses SVC. Choose an uncountable set I of ambient measurables above $\text{rank}(S)$, with a normal measure at each, and let $\lambda > \sup I$. There is in C a surjection $f : \eta \times S \rightarrow \lambda^\omega$; Chang contains the full ambient sequence space. Let $p \in \text{Ord}^\omega$ define f uniquely over C . Only countably many of the selected ultrapowers move a coordinate of p . Choose distinct remaining measurables $\langle \kappa_n : n < \omega \rangle$ and put $z = \langle \kappa_n : n < \omega \rangle$. Take one preimage $f(\xi, s) = z$. Some selected ultrapower j with critical point κ_n fixes ξ . It fixes s by rank and fixes f by its definition from p over the unchanged Chang model. Therefore

$$j(z) = j(f)(j(\xi), j(s)) = f(\xi, s) = z, \quad j(z)(n) = j(\kappa_n) > \kappa_n = z(n),$$

a contradiction. All choices here are made in U . $\square$

Replacing $\mathcal{H}$ by C in $E_2(\sigma)$ therefore makes that theory inconsistent for every σ . This already rules out the three-way Chang replacement. The proposition uses internal definability over Chang and its invariance under ambient normal ultrapowers. It does not assert invariance of the ambiently computed $\text{HOD}_{\text{Ord}^\omega}$, nor does global failure of SVC exclude SVC from each individual Chang rank.

Even the comparison of theories (1) and (3) can fail after replacing $\mathcal{H}$ by C . Relative to ZFC with a proper class of strong cardinals and an inaccessible limit of strong cardinals, there is a sentence σ for which theory (3) is consistent but the Chang version of theory (1) is inconsistent. Appendix D gives the statement and proof. This explains why the choice of $\text{HOD}_{\text{Ord}^\omega}$ matters even without the SVC requirement in theory (2). Whether an anti-inner-model hypothesis restores the two-way comparison remains open; Question D.5 considers excluding inner models with a measurable of Mitchell order at least two.

1.4. Proof outline and formalization. Sections 2–7 prove the local rank-to-sequence theorem. A high Prikry sequence and a countably closed labeling forcing produce $Q \subseteq \kappa^\omega$ and $F : Q \rightarrow X$. Paired comparison removes the labeling coordinate from low-rank names while keeping their values fixed. External countable domination gives DC in $N(Q, F)$. Section 8 supplies the initial passage to SVC without changing the protected rank. The core-model and atomization reductions originate in the research archive [2]; Section 9 gives the atomization and fiber purification needed for structural conservativity.

Sections 10–14 complete the external-HOD argument. Sparse Cohen rows with evaluated terminal caps code the sequence generator over a recognizable choice envelope. The code makes the generator ordinal-definable, while homogeneous descent bounds the resulting HOD by the intended model. Repeated $\diamond^*$ codes provide a secure ground with eventual GCH. A countably closed collapse gives GCH in the choice envelope, and the final coding preserves it. The final consistency implications combine this realization with the rank-preserving preparation. Appendix D gives the separate strong-cardinal obstruction to the Chang replacement.

We use transitive-model notation for the forcing and inner-model interpretations. This notation does not assume that consistency supplies a transitive model. The set-forcing arguments use the forcing theorem. Generated models use the definable transitive-base construction, and the Foundation-free construction uses the interpreted cumulative hierarchy over a set of atoms. These constructions can equivalently be carried out over countable models, using their internal notions of well-foundedness and satisfaction. The KW and Foundation reductions in Sections 2–9 use

only set forcing. The external realization uses the progressively closed $\diamond^*$ coding of Section 12. Each of its set arguments takes place in a set-sized initial segment.

2. SMALL FORCING AND GENERATED-MODEL PRESENTATIONS

Lemma 2.1 (Small forcing in ZF). *Let $V \models \text{ZF}$, let $\mathbb{B} \in V$ be a complete Boolean algebra, and let $\mathcal{U}$ be a nonprincipal κ -complete ultrafilter on κ , where $\kappa > h(\mathbb{B})$. For every V -generic $G \subseteq \mathbb{B}$, the filter*

$$\mathcal{U}^G = \{a \in \mathcal{P}(\kappa)^{V[G]} : \exists b \in \mathcal{U} (b \subseteq a)\}$$

is a nonprincipal κ -complete ultrafilter in $V[G]$. In particular, arbitrary set forcing over ZF preserves a proper-class tail of measurable cardinals.

Proof. For a name $\dot{a} \subseteq \check{\kappa}$, the range of

$$\zeta \mapsto \|\dot{\zeta} \in \dot{a}\|_{\mathbb{B}}$$

is canonically well-orderable by least preimages. Its cardinality is below $h(\mathbb{B})$, hence below κ . The partition into Boolean-value fibers therefore has a unique $\mathcal{U}$ -positive fiber $b_{\dot{a}}$, with associated Boolean value $v_{\dot{a}}$. In the generic extension, this old fiber is contained in a if $v_{\dot{a}} \in G$, and is contained in $\kappa \setminus a$ otherwise. Thus the generated filter decides every subset of κ .

To prove completeness, take a name for a sequence $\langle a_i : i < \delta \rangle$, where $\delta < \kappa$. Localize and give the name a fixed default value outside the localization. For every i , form the unique positive Boolean-value fiber b_i in V . Uniqueness makes this construction uniform, so the sequence $\langle b_i : i < \delta \rangle$ belongs to V without a choice of antichain witnesses. If each evaluated $a_i \in \mathcal{U}^G$, its associated Boolean value belongs to G . Hence

$$b = \bigcap_{i < \delta} b_i \in \mathcal{U}, \quad b \subseteq \bigcap_{i < \delta} a_i.$$

Nonprincipality is immediate. Completeness also prevents κ from acquiring cardinality below κ . A short enumeration would express κ as a union of fewer than κ null singletons. $\square$

For a normal measure in a ZFC ground and forcing of size below its measurable cardinal, we also use the usual normal generated lift and the downward Lévy–Solovay theorem; see [7]. If $D \subseteq W \subseteq D[G]$ and the forcing is small relative to a normal D -measure $\mathcal{U}$ on κ , its restriction to W is the set

$$\mathcal{U}^W = \{a \in \mathcal{P}(\kappa)^W : \exists b \in \mathcal{U} (b \subseteq a)\}. \quad (2.1)$$

This definition belongs to W when $D \subseteq W$ and $W \models \text{ZF}$. Restricting the ambient lift gives normality and completeness. The argument requires this restriction construction; measurability need not be downward absolute to an arbitrary inner model.

Lemma 2.2 (Bounded presentations). *Suppose $N \models \text{ZF}$ and $M = N(Q, F)$, where $Q \subseteq \kappa^\omega$ and $F : Q \rightarrow X \in N$. Each set in M is the value of a bounded, set-sized evaluator whose inputs are the whole sets Q, F , finitely many members of Q , parameters in N , and ordinal codes. The evaluator is absolute between transitive outer models containing the same inputs. For every nonempty $Y \in M$, there are an ordinal η , a nonempty set $Z \in N$, and a surjection in M*

$$\pi : \eta \times Z \times Q^{<\omega} \longrightarrow Y.$$

The whole sets Q, F remain fixed parameters of this surjection; they are not replaced by finitely many of their members.

Proof. We use transitive-base constructibility. For a sufficiently large ordinal θ , put

$$B_\theta = \text{TC}(\{V_\theta^N, Q, F\}), \quad J_0(B_\theta) = B_\theta, \quad J_{\xi+1}(B_\theta) = \text{Def}(J_\xi(B_\theta)),$$

and take unions at limits. The transitive base makes these levels transitive and increasing. Taking the union of these hierarchies over ground ranks and construction stages gives $N(Q, F)$. Thus each $Y \in M$ is contained in a set-sized level $J_\beta(B_\theta)$. This transitive-base construction includes the members of the adjoined sets, which predicate constructibility may omit; see [4, 8].

At a fixed level, code each element by a finite expression. A base element is a leaf. An element first produced at a successor stage is specified by a formula over the preceding set-sized level and finitely many parameters from that level. Substitute descriptions of those parameters. Transfinite induction gives a finite expression for each element, with leaves in B_θ and ordinal stage labels. At a limit stage use an earlier stage. At a successor stage combine the finitely many earlier descriptions. A description of a member of $J_\beta(B_\theta)$ uses stages bounded by β , allowing a finite increase in β if needed.

To evaluate an expression, use satisfaction for the indicated set structures $J_\xi(B_\theta)$. Set satisfaction, the definable-subset operation, and the recursion through a bounded list of levels are absolute between transitive outer models with the same base and ordinals. This proves the evaluator's absoluteness without a satisfaction predicate for the proper class M or N .

We next reduce its leaves. Apart from the whole sets Q, F , an element of $\text{TC}(\{Q, F\})$ is obtained by finitely many pairing and projection operations from individual members of Q and ground elements. Indeed, values of F and their hereditary members belong to N , and the members occurring in a sequence $q \in Q$ are built from natural numbers and ordinals below κ . The same statement holds for the ordered pairs coding the graph of F . Leaves from $\text{TC}(\{V_\theta^N\})$ are already ground elements. Consequently every expression can be converted to one using the fixed whole parameters Q, F , a finite tuple from Q , and a finite tuple from one set V_ρ^N , where ρ may be chosen uniformly from θ, X, κ .

Let $Z = (V_\rho^N)^{<\omega}$, increasing ρ so that Z is nonempty and contains all the required fixed ground data. Finite syntactic expressions, finite ordinal labels below the chosen bound, and finite type information form a wellorderable set. Index it by an ordinal η . Evaluate the codes with all parameters in Z and all finite tuples from Q . Keep an output exactly when the code is valid and its value belongs to Y . This defines a set-sized partial function in M with range Y . Fix one $y_0 \in Y$ and send every unused argument to y_0 , obtaining the total map π . Using all codes avoids choosing a definition for each $y \in Y$ or a wellordering of the ground parameter set Z . $\square$

3. THE CHOICE SYMMETRIC GROUND FROM SVC

Proposition 3.1 (A fixed small presentation with a measurable tail). *If $N \models \text{ZF} + \text{SVC} + \text{CM}$, there is a choice inner model D with*

$$D \models \text{ZFC} + \text{CM}, \quad N = D(z)$$

for some $z \in N$. There are fixed set forcings $P \in N$ and $T \in D$ and a common extension

$$E = N[H] = D[G]$$

such that N is a symmetric submodel of $D[G]$ for a set-sized symmetric system in D .

Proof. By [8, Proposition 5.4], SVC gives a choice symmetric ground D and a set generator z with $N = D(z)$. Use [8, Lemma 4.2] to fix a common-cover presentation in which N is a symmetric submodel. An arbitrary pair of forcings witnessing a common extension need not give this symmetric system; see [8, Note 4.6]. A sufficiently large pair of collapses gives the required presentation.

It remains to show that this same D has a proper class of measurables. Put

$$\theta = h^N(\text{RO}^N(P)), \quad \tau = |T|^D.$$

These are fixed ordinals. For every N -measurable $\kappa > \max\{\theta, \tau, \omega\}$, Lemma 2.1 lifts its measure to $N[H]$. Thus κ is measurable in the choice model $D[G]$. Downward Lévy–Solovay for the D -forcing T , whose D -size is $\tau < \kappa$, puts a measure on κ in D . The bounds and the ground were fixed independently of κ , so $D \models \text{CM}$. No absoluteness of the Hartogs calculation between N and D is used. $\square$

Now assume also $N \models \text{DC}$. Fix an SVC witness $S \in N$, the generator z , the protected rank, and the ground symmetric-system data. Choose an uncountable regular D -cardinal ρ above the size of T and the rank bounds for this presentation. Then choose a sufficiently high beth fixed point λ of D with $\text{cf}^D(\lambda) = \rho$. One obtains such a point as the supremum of a sufficiently high, strictly increasing continuous sequence of beth fixed points of length ρ . The forcing T preserves this cofinality. Since $D \subseteq N \subseteq D[G]$, it follows that $\text{cf}^N(\lambda) = \rho > \omega$.

Choose λ large enough that the transitive set $X = V_\lambda^N$ contains z , S , and the protected data. Then

$$N = D(X), \quad N \models \text{SVC}(X).$$

For the first equality, $D(X) \subseteq N$ and $z \in X \subseteq D(X)$; for the second, extend each given surjection on $\eta \times S$ by a default value on the rest of $\eta \times X$. Increase the bounds once more if necessary, and choose a measurable κ of D above them. The fixed forcing T is small below κ , so κ remains inaccessible in $D[G]$. All rank-bounded data below λ , including X and the old countable-condition posets introduced below, have size below κ there.

This construction preserves N and its protected ranks. We use the common choice cover to choose subsequent generics and make the size calculations.

4. THE PRIKRY AMBIENT UNIVERSE

For the next four sections fix a symmetric presentation

$$D \subseteq N \subseteq D[G], \quad D \models \text{ZFC} + \text{CM}, \quad N \models \text{ZF} + \text{DC} + \text{SVC}.$$

Write $\mathcal{S} = (\mathbb{A}, \mathcal{G}, \mathcal{F}) \in D$ for its symmetric system, and take G to be $\mathbb{A}$ -generic over D .

Choose a transitive rank $X = V_\lambda^N$ containing a generator for N over D , an SVC witness, and the structure to be protected. Choose λ sufficiently closed, with uncountable cofinality in the full choice cover; one may take a ground beth fixed point of cofinality greater than the size of the fixed small presentation. Then choose a measurable κ of D above $\lambda + \omega$ and all presentation bounds. In particular, arrange

$$N = D(X), \quad N \models \text{SVC}(X), \quad |X|^{D[G]} < \kappa, \quad \mathcal{S}, \text{RO}(\mathbb{A}) \in V_\kappa^D. \quad (4.1)$$

The forcing $\mathbb{A}$ and all subsequent choice-envelope forcing below κ have size less than κ . We can impose these bounds before choosing κ . The presentation is fixed, and a small forcing has fewer than κ bounded-rank names below a sufficiently large ground inaccessible. The small cover preserves the chosen ground cofinality of λ , so $\text{cf}^N(\lambda) > \omega$.

Let $\mathcal{U} \in D$ be a normal measure on κ , and let $\mathbb{P}_{\mathcal{U}}$ be its *old* Prikry forcing in D . Its ground wellordering belongs to N . The old measure base generates the normal lift in every small choice cover under consideration; therefore $\mathbb{P}_{\mathcal{U}}$ is dense in the Prikry forcing computed there from that lift.

Put $\nu = \omega_1^N$ and let

$$\mathbb{C} = \{k : k \text{ is an } N\text{-countable partial function } \nu \longrightarrow X\}^N,$$

ordered by reverse inclusion. We keep the poset computed in N fixed. Every condition belongs to X , since its domain is bounded in $\nu < \lambda$ and its countable range has bounded rank below λ . We may use the dense initial-segment presentation $\bigcup_{\gamma < \nu} \gamma X$ computed in N . In the full choice cover $|\mathbb{C}| < \kappa$, after increasing the bound on the small presentation if necessary.

Choose (c, K) generic for the actual product $\mathbb{P}_{\mathcal{U}} \times \mathbb{C}$ over the *full choice cover* $D[G]$, and set

$$W = N[c][K] = N[K][c].$$

The product uses these fixed old posets. Genericity over the full cover gives the displayed genericities over N and the stronger genericity needed in part (3) below. We do not assume that the old collapse remains countably closed after adjoining c .

In N , DC makes $\mathbb{C}$ countably closed. Its generic union enumerates X with domain ν . By $\text{SVC}(X)$, $N[K]$ can wellorder every old set, and the same is then true of every set of names and its evaluated range. Thus $N[K] \models \text{ZFC}$. The full cover $D[G][K]$ is a small forcing extension of D , so its normal lift of $\mathcal{U}$ restricts to the normal generated measure in $N[K]$. Product genericity makes c Prikry generic over this choice model. Consequently $W \models \text{ZFC}$.

Lemma 4.1. *In this setup:*

- (1) ${}^\omega X \cap W = {}^\omega X \cap N$;
- (2) every W -countable family of elements of N is contained in an N -wellorderable set;
- (3) $V_\kappa^{N[c]} = V_\kappa^N$.

Proof. For (1), first force with $\mathbb{C}$. The deciding-sequence argument uses DC in N and the countable closure of $\mathbb{C}$. It shows that $N[K]/N$ adds no countable sequence of elements of any fixed old set. In $N[K]$, X has size at most $\nu < \kappa$. The Prikry property for the lifted normal measure implies that the subsequent forcing adds no bounded subset of κ . Use the enumeration of X to code a countable sequence from X in W . The sequence therefore already lies in $N[K]$, and countable closure returns it to N . The same argument gives equality of the reals. We use the Prikry property and infinite-subsequence criterion as stated in [3, Section 3].

For (2), bound the ranks of the countably many ground elements by an ordinal and place them in one set $Z \in N$. Let $\langle z_n : n < \omega \rangle \in W$ be the resulting sequence. View it as the value of a $\mathbb{C}$ -name $\dot{z} \in N[c]$. In W , for each n choose $k_n \in K$ which forces

$$\dot{z}(n) = \check{z}_n$$

over $N[c]$. Such a condition exists by the truth lemma; a fixed localization can be included in all the k_n . Since $\mathbb{C} \subseteq X$, (1) gives $\langle k_n : n < \omega \rangle \in N$. Inside $N[c]$, that array, Z , and $\dot{z}$ reconstruct $\langle z_n \rangle$ by the forcing relation: z_n is the unique $z \in Z$ for which $k_n \Vdash_{\mathbb{C}} \dot{z}(n) = \check{z}$. Hence the whole sequence belongs to $N[c]$.

Take a $\mathbb{P}_{\mathcal{U}}$ -name $\dot{v} \in N$ for it and let

$$B = \{z \in Z : \exists n < \omega \exists p \in \mathbb{P}_{\mathcal{U}} (p \Vdash \dot{v}(n) = \check{z})\}.$$

This is an N -set containing every z_n . Fix the old wellordering of $\omega \times \mathbb{P}_{\mathcal{U}}$. Send each $z \in B$ to its least deciding pair. Distinct values have distinct pairs, so the map induces an N -wellordering of B without wellordering Z .

For (3), use the following generic-intersection fact, valid without Choice:

$$N_0 \subseteq E_0, \quad \mathbb{P} \in N_0, \quad h \text{ generic over } E_0 \implies N_0[h] \cap E_0 = N_0,$$

where the transitive models satisfy ZF and have the same ordinals, and the forcing is the same old poset in both models.

We give the proof. First, if h_0, h_1 are mutually $\mathbb{P}$ -generic over N_0 , then

$$N_0[h_0] \cap N_0[h_1] = N_0.$$

Induct on the rank of a common set x . Its members belong to N_0 by induction, so $x \subseteq V_\gamma^{N_0}$ for some ordinal γ . Choose names $\tau_0, \tau_1 \in N_0$ for x and a product condition (p_0, p_1) forcing their equality and that common value to be a subset of $V_\gamma^{N_0}$. For each $y \in V_\gamma^{N_0}$, p_0 must decide $y \in \tau_0$:

otherwise two extensions giving opposite decisions, paired with one extension of p_1 deciding $y \in \tau_1$, contradict the forced equality. Hence

$$x = \{y \in V_\gamma^{N_0} : p_0 \Vdash \check{y} \in \tau_0\} \in N_0.$$

Separation and the forcing theorem suffice; no choice of deciding conditions for all y is needed.

Now let $x = \tau^h \in N_0[h] \cap E_0$, where $\tau \in N_0$ is a forcing name. The forcing theorem over E_0 gives $p \in h$ with

$$p \Vdash_{E_0, \mathbb{P}} \tau = \check{x}.$$

In a further auxiliary extension, choose h' generic over $E_0[h]$ with $p \in h'$. The filters h, h' are mutually generic over E_0 , hence over N_0 , and $\tau^{h'} = x$. The preceding intersection argument gives $x \in N_0$. This assertion concerns membership in the original ground, so the auxiliary extension does not change it. The forcing theorem for the further product gives an equivalent formulation that does not infer transitive models from consistency.

Apply this fact with $E_0 = D[G]$, $N_0 = N$, and $\mathbb{P} = \mathbb{P}_U$. The old forcing is dense in normal Prikry forcing for the lifted measure over the choice model E_0 . Small forcing preserves the inaccessibility of κ , and the Prikry property gives

$$V_\kappa^{E_0[c]} = V_\kappa^{E_0}.$$

The absence of new bounded subsets suffices because E_0 satisfies Choice and each old rank below the inaccessible κ has size below κ . Induction on rank proves the displayed equality. If $x \in V_\kappa^{N[c]}$, the inclusion $N[c] \subseteq E_0[c]$ puts x in E_0 . Generic intersection then puts x in N . The converse inclusion is immediate. This proves (3) using the fixed old posets, without reducing to bounded symmetric names or recomputing the collapse after Prikry forcing. $\square$

5. TAGGED CAPS AND THE LABELING FORCING

All sequences, carriers, records, and diagram witnesses in this section belong to N . The word *old* means “in N ,” even when we work in an auxiliary extension. Changing finitely many initial blocks does not affect any asymptotic requirement. Fix $m_0 \geq 4$ and, for $m \geq m_0$, put

$$I_m = [2^m, 2^{m+1}), \quad w_m = 2^{\lfloor \sqrt{m} \rfloor}, \quad G_m = 2^{\lfloor m^{1/4} \rfloor}, \quad J_m = [2^m + w_m, 2^m + 2w_m].$$

For $m < m_0$ use $I_0 = \{0, 1\}$ and $I_m = [2^m, 2^{m+1})$ for $1 \leq m < m_0$, put $J_m = \{2^m\}$ and $G_m = w_m = 1$, and give every row the value 2^m there. Thus the blocks partition all of ω , including the index 0. Include $E_* = [0, m_0)$ in every cap and diagram exceptional set. The point 0 and all data in the finite initial interval $[0, 2^{m_0})$ contribute only finite carrier data. Every core is therefore a subset of its block. All separation assertions concern the eventual tail. An eventually positive sequence d_m is *super-G* if

$$\frac{d_m}{G_m^k} \longrightarrow \infty \quad \text{for every fixed } k < \omega.$$

By changing an initial segment, we may take every witness reserve below to be a positive integer sequence. The core width w_m is *super-G*. A *row* is an old function f with $f(m) \in J_m$ and *super-G* distances from the two core endpoints. Write

$$A_f = \{f(m) : m < \omega\}, \quad f \prec g \iff g(m) - f(m) \text{ is super-G}.$$

A *row chain* is a set of rows linearly ordered by $\prec$. Its pairwise order and separation requirements hold on every sufficiently late block, including exceptional blocks.

The ideal of label errors is

$$\mathcal{J} = \{a \subseteq \omega : |a \cap n| = o(\log n)\}^N. \quad (5.1)$$

If $a \in \mathcal{J}$, its occupied block set $\{m : a \cap I_m \neq \emptyset\}$ has asymptotic density zero, and $|a \cap I_m| = o(m)$. Indeed, the number of occupied blocks below m is bounded by $|a \cap 2^{m+1}| = o(m)$. This also bounds the load of the m th block. Since $m/G_m \rightarrow 0$, this load is $o(G_m)$.

Definition 5.1 (Cap records and diagrams). A *cap record* consists of an old carrier $a \subseteq \omega$, a countable row support L_a , an old density-zero set $Z_a \subseteq \omega$, and, for every $m \notin Z_a$, a labeling of the *entire* finite set $a \cap I_m$ by members of L_a . It satisfies:

- (1) $|a \cap I_m| = o(G_m)$ on all blocks, and $a \cap I_m \subseteq J_m$ whenever $m \notin Z_a$;
- (2) for each fixed $f \in L_a$, on all sufficiently late blocks outside Z_a , the canonical point $f(m)$ belongs to a and has tag f .

A *finite diagram* consists of finitely many *whole cap records* and finitely many specified canonical row points. A witness for the diagram is an old density-zero exceptional set, containing the exceptions of its cap records, and a super- G reserve d_m . Outside that exceptional set, equal points have equal tags; the convex envelopes of distinct represented tags occur in row order; the gaps between consecutive distinct-tag envelopes and the two core margins are at least d_m . A canonical point $f(m)$ has tag f .

A single tag may occupy several widely separated pieces, with no bound on its diameter. The finite-diagram requirement controls *all* tags in each selected cap block, including tags whose canonical-point requirements have not yet stabilized. Clause (2) stabilizes separately for each cap–row pair, not uniformly over all its tags. On a record’s exceptional blocks there is no tag or canonical-point requirement; only its all-block load bound remains.

Definition 5.2 (The forcing $\mathbb{R}$ and designated tops). A condition consists of a countable row chain L , a coloring $\ell : L \rightarrow X$, and a countable nonempty family of immutable cap records supported on L . Fix once and for all the distinguished empty cap with empty support, exception set E_* , and empty labelings, together with the default witness $(E_*, \langle w_m : m < \omega \rangle)$ for its one-cap diagram and for the empty diagram. Require every condition to contain this record and these same default witnesses. This family is *finitely directed modulo finite inclusion*: for any finitely many of its carriers $a_1, \dots, a_k$, it contains a carrier b with $a_i \subseteq^* b$ for every i . For every finite diagram of its caps and canonical rows, the condition includes a witness as in Definition 5.1. Extension retains the rows, their colors, all cap records, and all previously assigned diagram witnesses.

A listed cap b is a *designated top* of this condition if its support is the entire current row chain and

$$a \subseteq^* b \quad \text{for every listed cap } a, \quad A_f \subseteq^* b \quad \text{for every } f \in L.$$

These containments hold modulo a finite set of *points*, including on exceptional blocks. They are additional properties of a designated top, not requirements on every cap record.

With the fixed default witnesses, the condition with no rows and one empty cap is a largest condition. All countable families and their assigned witness functions are actual old sets included in the condition. Conditions admit uniform codes in X . The countably many colors have ranks bounded below λ , since $\text{cf}^N(\lambda) > \omega$. The remaining countable data and the finite rank overhead of coding also stay below λ . A countable decreasing chain of conditions has its literal union as a lower bound. Every pair of rows and every finite diagram occurs at an earlier stage, so its requirements and witness persist. Finite directedness modulo finite inclusion persists for the same reason. We use DC only to show that the union of the old countable families is countable. The diagram witnesses are already assigned.

Lemma 5.3 (Dense tops and insertions). *The following extensions are dense in $\mathbb{R}$.*

- (1) *A condition can be extended by a designated top containing every previous cap and every current canonical row modulo finite sets.*
- (2) *A new row of any prescribed color can be inserted at any old cut in the countable current row chain.*
- (3) *Every old $a \in \mathcal{J}$ can be included in a cap, and every fixed integer translate of a listed cap can be included in a cap.*

Proof. Tops. Enumerate the caps as a_i and the rows as f_i ; when a list is finite, omit nonexistent entries in the formulas below. Let $\mathcal{D}_n$ be the diagram of the first n whole caps and first n canonical rows. Take increasing old density-zero sets U_n containing their assigned diagram exceptions and cap exceptions. Choose increasing thresholds M_n , $n \geq 1$, such that, for $m \geq M_n$,

$$\sum_{i < n} |a_i \cap I_m| + n \leq G_m/n,$$

and, off U_n , the diagram $\mathcal{D}_n$ has reserve at least $4G_m^{n+4}$. We may also require $w_m \geq 4G_m^{n+4}$ and

$$|U_n \cap t|/t < 1/n \quad (t \geq M_n).$$

Each step imposes only finitely many eventual requirements. On $M_n \leq m < M_{n+1}$ define

$$b \cap I_m = \bigcup_{i < n} (a_i \cap I_m) \cup \{f_i(m) : i < n\}.$$

Take arbitrary finite initial data, for instance empty blocks. Let E contain the initial block indices and consist thereafter of the m which belong to U_n on the n th band. If t is in that band, the exceptional indices from all earlier bands lie in U_n ; consequently

$$|E \cap t|/t \leq 1/n + O(1/t),$$

so E is old and density zero.

On a block outside E , use the labels of $\mathcal{D}_n$. This copies every point of each selected cap, including points with transient high tags. Equal points receive consistent labels. The load is at most G_m/n . Every fixed old cap and row is included on *all* sufficiently late blocks. Give b support L and exception set E . It is a valid record. A super- G reserve is given, for example, by G_m^{n+2} on the n th band.

For a finite diagram involving b and specified old caps and rows, take the band index large enough to include all those old items. Their points are then already in b , with the same tags outside E , and their union diagram is a subconfiguration of $\mathcal{D}_n$. Thus E , enlarged by a finite initial segment, and the same bandwise reserve give a *new* witness. No old witness is changed. The enlarged cap family is directed modulo finite inclusion because b dominates all its members. The same construction applies to any countable compatible family of records and diagram witnesses, even if it is not directed. Adjoining the constructed top makes it directed.

Insertion at a cut. First choose a designated top b . Enlarge the construction's exceptional set to include the exception of b and the witness exception for the diagram consisting of the whole cap b . On a good block, partition its finitely many active tag packets according to the given cut. Choose the new point in the gap between the last left packet and the first right packet, using a core endpoint if a side is empty. If both sides are empty, use the core itself. The gap or margin has a super- G reserve. Its integer midpoint is separated from both bounding packets or endpoints by half that reserve minus one.

On exceptional blocks, use a separate slow interpolation of canonical rows. Enumerate the old rows and activate the first n only after their finite all-block order and margin bounds exceed, say, $4G_m^{n+4}$. Between the last active row on the left and first on the right, with core endpoints at empty sides, choose an integer midpoint. The activation thresholds tend to infinity, and every fixed row

is eventually active. This defines an old row with the required cut and super- G margins on the exceptional blocks. On good blocks, every fixed old row eventually has its canonical point in b with the correct tag; the packet-gap construction gives the same required order and separation there. Together these estimates give the eventual row inequalities on all blocks.

Give the new row the requested color. For each new finite diagram, eventually its old cap points and canonical anchors lie in b and agree with b 's tags outside the corresponding old diagram exceptions. The chosen gap separates the new row from every packet. Finite unions of these exceptions and the midpoint reserve supply new witnesses. This extends the row chain without changing any old record. Apply the top construction again to obtain a new designated top.

Errors and translates. For $a \in \mathcal{J}$, use a record with carrier a , empty support, and exception set its occupied block set. It has no good-block points requiring tags and has load $o(G_m)$. It is compatible with every old finite diagram after adjoining this density-zero exception. Apply the compatible-family version of the top construction.

For a fixed $j \in \mathbb{Z}$ and a listed cap a , take the new carrier

$$a \cup (a + j), \quad a + j = \{k + j : k \in a, k + j \geq 0\},$$

with the same support, retaining the original tags and giving translated points their source tags. On good sufficiently late blocks, core margins and distinct-tag gaps exceed $|j|$ by a super- G amount. Thus the translated points remain in the core and inconsistent distinct-tag overlaps cannot occur. The original copy retains each fixed supported canonical point. Adjoin the old exceptional block set, its adjacent block indices, and a finite initial segment to the new exception set. This covers all boundary crossings. The new load at block m is bounded by a fixed sum of the old loads at $m - 1, m, m + 1$. It is therefore $o(G_m)$, since the ratios of these neighboring G -values are bounded. For a new finite diagram, use the old diagram including a and its other constituents; subtracting $|j|$ from its eventual gaps and margins leaves a super- G reserve. Use these reserves to assign consistent new witnesses without changing any old witness. Finally apply the compatible-family top construction. $\square$

The dense initial-segment presentation $X^{<\omega_1^N}$ of $\mathbb{C}$ projects onto $\mathbb{R}$. Fix the uniform coding of conditions by elements of X and start at the largest condition. At a successor entry, adopt the decoded condition if it is a valid extension of the current one; otherwise keep the current one. At a countable limit take the literal union. This recursion is set-defined in N and requires no global choice of lower bounds. Every extension of the current decoded condition is reached by appending its code, which proves the projection lifting property. By Lemma 4.1(1), the relevant countable X -strings and condition codes are the same in $N[c]$. The old projection therefore works there as well. Write g for the projected generic.

5.1. The finite matching lemma. For an increasing bijection between finite frozen subsets of two integer intervals, including their endpoints, say it is L -balanced if every pair of corresponding consecutive frozen gaps d, d' satisfies

$$\min(d, L) = \min(d', L).$$

Give the endpoints two distinct formal sentinel tags, one below and one above every genuine tag. These tags occur nowhere else.

Lemma 5.4 (Finite tagged interpolation). *Let A_i be finite prescribed subsets of integer intervals $[u_i, v_i]$, including the endpoints, for $i = 0, 1$. Let $h : F_0 \rightarrow F_1$ be an increasing frozen bijection, with $F_i \subseteq A_i$, matching the endpoints. Choose a common finite linear tag order T amalgamating the two tag orders over the frozen tags, and suppose:*

- (1) *the tag maps on A_i are weakly increasing, and frozen matched points have equal tags;*

- (2) *distinct-tag envelope gaps and the sentinel margins in each prescribed configuration are at least the positive integer D ;*
- (3) *h is L -balanced, where $L \geq D$.*

Take T to consist of the tags occurring on at least one side, including the sentinels. Let $n \geq |A_0| + |A_1| + |T|$ and suppose $D \geq 100(n+1)^3$. Then there are finite extensions $A'_i \supseteq A_i$ and an increasing tag-preserving bijection $h' : A'_0 \rightarrow A'_1$ extending h such that, with

$$\varepsilon = \left\lfloor \frac{D}{100(n+1)^3} \right\rfloor,$$

the map h' is ε -balanced and every distinct-tag envelope gap and sentinel margin in the final configurations is at least ε . Every tag occurs on both sides, and the total number of added final points is at most $|A_0| + |A_1|$, in particular $O(n)$. For tags initially present on only one side, the construction supplies the partner packets. Partners are not prescribed independently.

There is also an untagged version. Under the balance hypothesis, the same bound on D and the prescribed point count give an increasing ε -balanced extension retaining all points. The same linear addition bound holds, but there is no tag-separation conclusion.

Proof. We first give a two-pass untyped procedure. Suppose matched endpoint intervals have lengths either equal and less than R , or both at least R , with at most n prescribed interior points in total and $R \geq (n+1)^2$. If the lengths are equal and short, translate the two patterns into one interval, take their union, and translate back. Otherwise put $d = \lfloor R/(n+1) \rfloor$ and insert the prescribed source points, with successive positive gap lengths $g_0, \dots, g_k$, where $k+1 \leq n+1$. Assign target gaps the preliminary lengths $\min(g_j, d)$. Their sum is at most $(n+1)d \leq R$, no greater than the target interval length. At least one source gap is at least d . Otherwise the source interval length, which is their sum, would be less than R . Put all leftover target length in the first such gap. The map copies short gaps exactly. Both members of every other gap pair have length at least d .

Now insert the originally prescribed target points, working backwards in each resulting gap. Translate in equal gaps shorter than d ; in long gaps repeat the allocation with threshold $\lfloor d/(n+1) \rfloor$. The resulting map is balanced at

$$q = \left\lfloor \frac{R}{(n+1)^2} \right\rfloor.$$

Every allocation is positive, so new interior points cannot collide with old matched endpoints. At most one partner is added for each originally prescribed point. This proves the untyped procedure.

Lower the frozen balance threshold from L to D and work separately in consecutive frozen gaps. If the two endpoint tags of such a gap are equal, tag monotonicity forces every prescribed point in the gap to have that tag. Use the untyped procedure with $R = D$ and give all added points that tag. This handles every short frozen gap. Distinct endpoint tags, including a genuine tag next to a sentinel, would force the gap to have length at least D .

It remains to treat a frozen gap with distinct endpoint tags $s < t$. Both corresponding gaps have length at least D . The frozen tags determine the same list of required tags between s and t on both sides. On one side, call the closed envelope of the prescribed points of a represented tag its *packet*. The packets are ordered, with gaps of length at least D between consecutive packets. Put

$$R = \left\lfloor \frac{D}{4(n+1)} \right\rfloor.$$

For consecutive nonempty packets with last and first points $u < v$, let $h_0 \leq n$ be the number of intervening tags missing on this side. Extend the left packet's zone to $u + R$, assign the j th

missing tag the zone

$$[u + 2jR, u + (2j + 1)R] \quad (1 \leq j \leq h_0),$$

and start the right packet's zone at $v - R$. Every empty guard between successive zones has length at least R , because

$$(2h_0 + 3)R \leq D \leq v - u.$$

At a frozen endpoint, begin or end the corresponding endpoint-tag zone at that endpoint. Thus every zone contains its entire prescribed packet and has length at least R ; an absent tag gets a zone of length R . No packet is compressed, regardless of its diameter. Apply this on both sides with the same ordered tag list and match the two endpoints of corresponding zones and guards. The resulting skeleton extends the frozen endpoint map and is R -balanced. Sentinel zones contain only the original outer endpoint and a temporary artificial inner endpoint.

In each matched same-tag zone apply the untyped procedure to the original prescriptions and label each added point by that tag. Our bound on D gives $R \geq (n + 1)^2$. Each active tag occurs on at least one side. If it is missing on the other side, the procedure adds a partner point there. Leave the guards empty. The refined skeleton is balanced at

$$\left\lfloor \frac{R}{(n + 1)^2} \right\rfloor = \left\lfloor \frac{D}{4(n + 1)^3} \right\rfloor \geq \varepsilon.$$

Every artificial zone endpoint lies in a free interpacket gap and has an artificial partner. Delete these endpoints in matched pairs, retaining the original frozen endpoints and all original prescriptions. A resulting consecutive gap is a sum of corresponding skeleton gaps. If a constituent pair is long at the displayed threshold, both sums are long; otherwise every constituent pair is equal, so the sums are equal. Balance therefore survives deletion. An entire empty guard remains between distinct final tags, including each sentinel and its neighboring genuine tag. Thus the final separation and margins satisfy the same bound.

After removing the artificial skeleton points, every added final point is the partner of an original prescribed point. This gives the asserted linear bound. Adjacent frozen gaps agree at their shared frozen endpoint, so their maps glue. For the untagged version, apply the untyped procedure with $R = D$ separately in each frozen gap and join the resulting maps. No zones are needed. $\square$

Remark 5.5. Translation is used only in short equal same-tag gaps at the *current* threshold. After lowering the threshold from the original L to D , requiring translation in every gap below the original threshold is unnecessary and may be incorrect. The tag-preserving conclusion does not identify two separately prescribed same-tag canonical points unless the frozen map already contains that pair.

To apply the lemma to caps, adjoin the core endpoints as sentinels. Take D_m no larger than the diagram reserve, the frozen map's balance reserve, and w_m . Each selected finite diagram uses entire prescribed cap blocks and only finitely activated canonical anchors. The number of its active tags is bounded by its point count, so these data have $n_m = o(G_m)$ after sufficiently slow activation. If D_m is super- G , then so is

$$\left\lfloor \frac{D_m}{100(n_m + 1)^3} \right\rfloor : \quad \frac{D_m}{(n_m + 1)^3 G_m^k} \geq \frac{D_m}{G_m^{k+3}} \rightarrow \infty$$

for each fixed k , on all sufficiently late blocks. Thus the linear load increase remains $o(G_m)$ and the polynomial reserve loss remains super- G . A growing collection of diagram requirements must be activated by an old slow diagonal. The argument does not assert that an arbitrary growing minimum of reserves is super- G . The finite construction is uniform. It uses the displayed leftmost allocations and assigns leftover length to the first long gap, so old finite input arrays produce old output arrays. On exceptional blocks use the untagged version. Any finitely many initial blocks

can be handled separately when map extensions are required only eventually, while retaining the prescribed root points.

6. PAIRED FORCING AND LITERAL COMPARISON

Throughout this section, all conditions, rows, cap records, and block maps are old, meaning that they belong to N . A designated top of a condition is a listed cap whose support is the entire row chain and which contains each listed cap and each canonical row carrier modulo finite sets. The last containment holds on all sufficiently late blocks, including exceptional blocks. Cap families are finitely directed under modulo-finite inclusion.

Fix a root condition r_* with a designated top b . To fix finitely many old carriers, first strengthen the root and pad a new top to contain them literally. No existing cap record or assigned witness is changed.

Definition 6.1. A paired condition over r_* consists of conditions $r_0, r_1 \leq r_*$, an order- and color-preserving bijection $\pi : L_0 \rightarrow L_1$ between their row chains fixing the root rows, designated tops B_0, B_1 containing b literally, and old increasing bijections

$$h_m : B_0 \cap I_m \rightarrow B_1 \cap I_m.$$

The maps fix b pointwise. Outside the core J_m the two tops agree and the maps are the identity. With core endpoints adjoined as sentinels, the maps are L_m -balanced on every block, for an old super- G sequence $\langle L_m : m < \omega \rangle$.

There is one old density-zero set E containing the exceptional sets of the two tops such that h_m transports all top tags by π whenever $m \notin E$. Moreover, for every fixed $f \in L_0$ there is $M_f < \omega$ such that

$$h_m(f(m)) = \pi(f)(m) \quad (m \geq M_f, m \notin E). \quad (6.1)$$

The designated-top requirement puts these canonical values in their respective tops on all sufficiently late blocks. There is no canonical-match requirement on blocks in E . Each side's row chain still has super- G margins and row inequalities on all sufficiently late blocks.

An extension extends both side conditions and the row bijection, retaining all their immutable data. For some $M < \omega$, its block map extends the earlier map on its entire domain whenever $m \geq M$. The extension may use a new comparison exception set and balance reserve.

The exception in (6.1) is essential. Let $Z \subseteq [m_0, \omega)$ be an infinite old density-zero set, let $p(m)$ be the midpoint of J_m for $m \geq m_0$, and set $p(m) = 2^m$ on the initial blocks. Give $b = \{p(m) - 1, p(m) + 1 : m \in Z\}$ empty support and exception set $Z \cup E_*$. This is a valid root cap. Insert the row p on both sides with different colors. Matching its canonical point on every sufficiently late block, while fixing b , would force its partner to equal $p(m)$ on all sufficiently late $m \in Z$. The differently colored partner must be distinct from the native row p on that side, contradicting their super- G separation. Requiring matches only off E avoids this obstruction.

Lemma 6.2 (Paired-cap lifting). *The paired forcing is countably closed in N . Both coordinate projections have the lifting property for arbitrary one-sided extensions. Any two extensions of r_* can be strengthened to the two coordinates of a paired condition.*

Proof. We first specify a schedule and the treatment of canonical anchors, then construct the extensions and check their records.

Old diagonal scheduling. Consider countably many old density-zero sets, super- G reserves, eventual equalities and containments, and $o(G_m)$ load functions. At stage $n \geq 1$, select finitely many requirements. Increase the selected families so that every fixed requirement is eventually selected. Requirements concerning a particular old top or map selected at stage n may also be included. Let U_n be the union of all exceptions selected up to stage n . Choose strictly increasing M_n such

that, for $m \geq M_n$, all selected eventual assertions hold, the selected total point load and number of active anchors are at most G_m/n , and every selected reserve and w_m are at least CG_m^{n+10} . Here C is a fixed sufficiently large constant for Lemma 5.4. Also require

$$|U_n \cap t|/t < 1/n \quad (t \geq M_n).$$

Each choice is possible because it involves only finitely many asymptotic requirements. Set $s(m) = n$ on $[M_n, M_{n+1})$ and

$$E = \{m \geq M_1 : m \in U_{s(m)}\},$$

with finitely many initial blocks added if necessary. If t is in the n th band, all earlier exceptional blocks after M_1 belong to U_n , so $|E \cap t|/t \leq 1/n + M_1/t$. Thus E is old and density zero; each fixed selected exception is contained in E modulo finite.

The bandwise load is $o(G_m)$. The point-and-tag count is bounded by a fixed multiple of the total whole-cap load, the active anchors on both sides, and the sentinel count. The G_m/n schedule therefore gives $\nu_m \leq G_m$ eventually. For this count and input reserve $D_m \geq CG_m^{s(m)+10}$, enlarge the thresholds to accommodate rounding and obtain

$$\left\lfloor \frac{D_m}{100(\nu_m + 1)^3} \right\rfloor \geq G_m^{s(m)+5}.$$

This is super- G and absorbs the polynomial loss in Lemma 5.4. Only finite minima are used at each stage. The exception set is a diagonal pseudounion, not a countable union.

The row order and canonical anchors. For one-sided lifting use the stronger source row order as an abstract order T , identifying old target rows by the existing row bijection. For initial two-sided amalgamation, amalgamate the native orders over the root by comparing their cuts in the root order. Within a single cut, put the foreign left rows before the foreign right rows and preserve each native order. Opposite non-root rows remain distinct abstract labels even when their native old functions coincide. Retain their inherited colors.

Every label is native on at least one side. Labels native on both sides are already identified. Choose increasing finite H_n exhausting T . Include their native canonical-anchor diagrams, and the eventual containments and matches of identified anchors, in the schedule. On a good block in the n th band prescribe the *entire* input top blocks and the native anchors from H_n . Previously identified anchors are frozen matched pairs after their stabilization bounds and off their selected exceptions. Apply the tagged finite lemma. Define a missing canonical value to be the image or inverse image of its native anchor, with the intended tag.

On an exceptional block, retain each side's native anchors from H_n and interpolate the missing labels independently in the order T , using core endpoints as sentinels. Finite native chains have all-block super- G gaps and margins. Divide each gap among its finitely many missing labels; the schedule makes the resulting integer gaps and margins at least G_m^{n+5} . Then include *all these canonical values on both sides* in the prescribed point sets for the untyped finite interpolation. This puts the canonical points in their tops on exceptional blocks too, although their images need not be their canonical partners there. The additional cost is linear in $|H_n|$.

Before a missing row becomes active, assign it any fixed interior value on blocks $m \geq m_0$, and the required value 2^m on the initial blocks. Each fixed row, and each fixed pair, is permanently active eventually. The two block constructions therefore give old row functions with all-block super- G margins and pairwise inequalities in the order T . A high-index tag in an input block receives its partner *packet* immediately because the whole block is prescribed. Its canonical anchor is activated later.

One-sided lifting. Given a paired condition with tops B_0, B_1 and map h , strengthen its first side arbitrarily and extend it to have a designated top A_0 . On each sufficiently late block prescribe

the entire A_0 block, the entire B_0 block if not already included, the entire B_1 block, and the active native anchors. Freeze h and the core sentinels. Outside the core add the union of the prescribed points on both sides and use the identity.

The source diagrams involving A_0, B_0 and active native anchors, and the target diagrams involving B_1 and its native anchors, supply old exceptions and reserves. Include the old paired exception and the selected canonical stabilization bounds. The old map's balance reserve, these diagram reserves, the native-row gaps, and w_m give the finite reserve minimum for the schedule. The prescribed loads are $o(G_m)$, so the constructions for good and exceptional blocks apply. The new maps eventually extend the entire old maps, fix b , and have all-block super- G balance. They define the extended target chain and color-preserving row bijection without changing any native row.

Initial two-sided amalgamation. Extend the two given conditions to have tops A_0, A_1 containing b literally. A previous top contains b modulo finite, so padding a new top by the finite remainder suffices. Amalgamate the abstract row orders as above, freeze the identity on b and the sentinels, and apply the same construction to the two entire top blocks. Include the finite diagrams comparing b and the native input tops. On exceptional blocks interpolate the active anchors separately, insert all their values into the untyped point sets, and retain the identity at b .

In both constructions use the identity on the union of all prescribed points on finitely many initial blocks if necessary. Non-root map extension is only eventual; the root remains fixed throughout. Sentinels are auxiliary. For each matched pair of core endpoints, retain both copies if either was a prescribed carrier point; otherwise remove both copies from the new tops. This preserves the bijection. The margin requirement keeps prescribed good-block points away from the endpoints. Measure balance with the sentinels adjoined. The numerical reserve requirements are asymptotic.

Records and immutable witnesses. Let C_i be the new top carriers, with full new row supports, the good-block tags just constructed, and exception set E . Their loads are $o(G_m)$ and their good-block distinct-tag and margin reserves are super- G . Every fixed canonical row belongs to its top on all sufficiently late blocks, including exceptional ones, and has its own tag on good late blocks. Thus the new records are valid designated tops.

Keep every native old record, labeling, exception set, and assigned finite-diagram witness literally. Each native cap a lies modulo finite in its input top and hence in C_i . A transported record on the other side, if included, has the exact carrier

$$a' = \bigcup_m h_m[a \cap C_i \cap I_m]$$

(or the corresponding inverse image), and transported support. Its exceptions include E , those of a , the old diagram exception comparing a to its input top, and a finite initial containment error. Outside that finite union, its labels agree with the top's labels, and each fixed supported row eventually has its canonical point. Its load is at most that of a . Do not add canonical corrections on exceptional blocks.

For a new finite diagram, unite the relevant finitely many exceptions and enlarge its finite initial segment. Beyond it, all prescribed points and correctly tagged canonical anchors lie in C_i , and their tag envelopes are sub-envelopes of the new top's packets. The top reserve witnesses the diagram. Assign the new witnesses without changing any old one. Each new top dominates all native and transported records modulo finite, so the cap family is finitely directed. The constructions therefore give side extensions in the stated forcing.

Countable closure. Let $t_0 \geq t_1 \geq \dots$ be an old decreasing sequence of paired conditions. Take the unions of the side conditions, row bijections, and assigned witnesses. Each finite diagram occurred

at a stage, so the side unions are conditions. Enumerate their caps, rows, finite diagrams, earlier maps, and exceptions.

At stage n choose a later $t_{k(n)}$ containing the first n enumerated items, with $k(n)$ increasing, and use its *whole* designated tops and map. Include in the schedule their eventual containment of the first n caps and canonical rows on all blocks; their agreement with the first n earlier maps; the first n canonical matches off their comparison exception; the diagrams of the *selected tops* together with the first n old caps and canonical anchors; their top and diagram exceptions; their map and diagram reserves; and their two top loads. Agreement comes from the extension order. Choose M_n by the schedule above.

On the n th band, copy the selected whole top blocks and map. This eventually extends every earlier map and contains every old cap and canonical row. Each fixed canonical match holds eventually off the diagonal E . The root and the identity off the core persist. Loads are $o(G_m)$, and all-block balance reserves are super- G . Copy the selected top labels. For every finite diagram involving a new top and specified old records, the corresponding diagrams with the selected stage top have been scheduled from some stage onwards. Their copied blocks and reserves witness the new-top diagram off E and a finite initial segment. Add only the two new tops and their new witnesses. They restore directedness and give a paired lower bound. Each block copies a single selected top, rather than an infinite pointwise union of cap loads. $\square$

6.1. The objects retained in the inner model. Let $g \subseteq \mathbb{R}$ be generic and let ℓ_g be its union coloring. Define

$$\mathcal{I}_g = \{a \in \mathcal{P}(\omega)^N : a \subseteq^* b \text{ for some cap } b \text{ appearing in } g\}.$$

By finite directedness and Lemma 5.3, this ideal is hereditary under old subsets, is closed under finite unions and integer translates, and contains $\mathcal{J}$. Let $Q(c, g)$ consist of the increasing enumerations of finite modifications, using arbitrary ordinals below κ , of $c[a]$ for infinite old $a \in \mathcal{I}_g$.

Fix $x_0 \in X$. For $q \in Q(c, g)$ set

$$\iota_c(q) = \{n : c(n) \in \text{ran}(q)\}.$$

If a generic row f and an integer j satisfy $\iota_c(q) \triangle (A_f + j) \in \mathcal{J}$, put $F(c, g)(q) = \ell_g(f)$. If there is no such row and translate, put $F(c, g)(q) = x_0$. The index carrier differs finitely from an old carrier, so is old. Distinct rows have disjoint translated $\mathcal{J}$ -clouds: their super- G separation gives distinct canonical points on every sufficiently late block. One point on each such block is not a $\mathcal{J}$ -error, so the definition is unambiguous. Lemma 5.3(2) makes F onto X .

A carrier already supported by a cap cannot acquire the cloud of a later row outside its recorded support. The finite diagram of the whole cap and that row separates the canonical point from all cap points on a density-one set of blocks. The reserve eventually exceeds any fixed translating integer. The difference from the translated row therefore has a point on a density-one set of blocks and is not in $\mathcal{J}$. Hence localized label decisions remain unchanged by later extensions.

Every member of Q is Prikry generic over N . To verify this for the ambient generic, work in the collapse-first choice cover $N[K]$, which contains every old carrier. There c is generic for the lifted measure, so Mathias's criterion applies, together with its infinite-subsequence consequence and finite-change invariance. The old measure base gives genericity for $\mathbb{P}_U$ and hence over N ; see [3, Section 3]. Only the evaluations of the fixed N -names Q, F are retained. The row order, tags, cap records, and master sequence are not separately adjoined.

Lemma 6.3 (Literal common-model comparison). *Let $r_0, r_1 \leq r_*$ and let $p_0, p_1 \in \mathbb{P}_U$ be compatible. Fix finitely many supported Q -parameters whose old carriers lie literally in the root top b , with their finite ordinal decorations. In an auxiliary extension there are genuine $\mathbb{P}_U \times \mathbb{R}$ -generics*

$(c_0, g_0), (c_1, g_1)$ over N , meeting the respective conditions, such that

$$Q(c_0, g_0) = Q(c_1, g_1), \quad F(c_0, g_0) = F(c_1, g_1),$$

and the supported parameters are literally equal. In particular the two models $N(Q, F)$ are literally the same inner model there.

Proof. Work in the full choice cover of Section 4. Lemma 6.2 gives a paired condition whose coordinates strengthen r_0, r_1 . Force below it with the *old* paired forcing. Its lifting assertions involve fixed old condition sets, so they remain valid in this cover. Each coordinate is therefore a projection below that condition. Given a dense open subset of the old $\mathbb{R}$ in the cover, strengthen the selected coordinate into it and lift that extension. The resulting marginal filters g_i are genuinely $\mathbb{R}$ -generic over the cover and over N .

The old paired forcing and its data have size below κ in the cover by (4.1). A further forcing of size below κ makes its generic filter countable. Enumerate that filter and use its directedness to take a cofinal decreasing sequence. The sequence need not belong to N . Its maps eventually extend each fixed earlier map. Copy one whole map per block beyond selected agreement thresholds to obtain an increasing block-preserving partial bijection h of ω , fixing b , which eventually agrees with each old top map in the filter.

Old localization of the carrier action. For $a \in \mathcal{I}_{g_0}$ choose an old paired condition t in the filter whose designated top contains a modulo finite. Directedness of the filter and the top property give such a t . Then

$$a' = h^t[a \cap B_0^t] \in N, \quad a' \subseteq B_1^t, \quad h[a \cap \text{dom}(h)] =^* a',$$

and $a \setminus \text{dom}(h)$ is finite. Thus a' is an old allowed carrier on the other side. Apply the same argument to the inverse map. Although the real h may be new, its action on each specified carrier is determined modulo finite by one old map.

For an old $d \in \mathcal{J}$ its localized image is again in $\mathcal{J}$. If $n \in I_m$, then

$$|h^t[d \cap B_0^t] \cap n| \leq |d \cap 2^{m+1}| = o(m) = o(\log n).$$

For a fixed generic row f and integer j , choose an old paired condition t containing f whose first top contains both A_f and $A_f + j$ modulo finite. Translate absorption, dense tops, and lifting make this a dense requirement. Write $q = \pi^t(f)$. Once the balance reserve exceeds $|j|$,

$$h_m^t(f(m) + j) = h_m^t(f(m)) + j.$$

Indeed, each consecutive domain-top gap between these points is shorter than the reserve and has its length copied exactly. The core margins keep both endpoints in the same block eventually. The fixed row's canonical match holds eventually off the old E_t . Its image and $A_q + j$ therefore differ, modulo finite, by at most two points in each block in E_t . Up to block m there are at most $2|E_t \cap (m+1)| + O(1) = o(m)$ such points. This is an old $\mathcal{J}$ -error. Combined with the error-image estimate, it shows that the map transports the entire translated cloud and preserves its color. For a particular carrier in that cloud, choose one paired condition whose top covers that carrier, the canonical translate, and the specified error set. Directedness and dense tops meet these finitely many requirements, so the same old map handles the whole comparison. Apply the argument to the inverse map as well. Only the error for the fixed row is used, not a countable union of errors. Whole exceptional cap blocks are transported literally, not discarded.

Genuine Prikrý and product genericity. Choose a common strengthening $p_* = (s, A)$ of p_0, p_1 , including a common reservoir as well as a common stem. Normalize finitely many initial blocks of h to the identity on the whole block, including all integers below $|s|$. The resulting map still eventually agrees with each old map and fixes b .

Amalgamate two ordered copies of ω , identifying n on the first with $h(n)$ on the second. Between identified points, interleave the finite remainders in any fixed order preserving the order of each side; do likewise in an empty block. Since h preserves finite blocks, the resulting order has type ω . Its increasing cofinal embeddings $e_i : \omega \rightarrow \omega$ satisfy

$$e_0(n) = e_1(h(n)) \quad (n \in \text{dom}(h)), \quad e_0(k) = e_1(k) = k \quad (k < |s|).$$

The only cross-side identifications are the specified h -pairs. Thus $e_0(n) = e_1(m)$ if and only if $n \in \text{dom}(h)$ and $h(n) = m$.

All preparation precedes the Prikry forcing and has total forcing size below κ over the original choice ground. The old normal measure generates its normal lift to this prepared choice universe, with the old measure-one sets as a base. Choose a master sequence z Prikry generic over that entire universe through (s, A) , and put $c_i = z \circ e_i$. The maps e_i are in the prepared universe, so Mathias's criterion makes each c_i Prikry generic there. The dense old presentation gives genericity for the old $\mathbb{P}_U$ as well. Each c_i contains s and has all subsequent entries in A , so meets p_i .

Thus c_i is $\mathbb{P}_U$ -generic over $N[g_i]$. Since g_i is $\mathbb{R}$ -generic over N and these are fixed old posets, the product lemma makes (c_i, g_i) $\mathbb{P}_U \times \mathbb{R}$ -generic over N . No Prikry generic was chosen before the comparison preparation.

Literal retained equality. For corresponding carriers, $c_0[a] =^* c_1[a']$. Closure under arbitrary finite ordinal modifications in the fixed definitions gives equality of the sets Q . For any q in this common set, the cross-side identification gives

$$\iota_{c_1}(q) =^* h[\iota_{c_0}(q) \cap \text{dom}(h)].$$

Indeed, all but finitely many values of q lie on its allowed carrier in the first master, and that carrier is almost contained in $\text{dom}(h)$; the only common master values are the identified pairs. Both index carriers are old. Applying the cloud correspondence in both directions gives equality of the two functions F on every such q , including their default-label regions.

Because h fixes b , the two subsequences on every protected carrier $a \subseteq b$ agree literally. The same finite ordinal decorations therefore give the same actual parameters. These are equal evaluations of the fixed N -names Q, F in one ambient universe, so their generated models $N(Q, F)$ are literally equal. The carrier and error ideals and the definition of Q are never recomputed using new auxiliary reals. $\square$

7. RANK PRESERVATION AND DEPENDENT CHOICE

Put

$$M = N(Q, F), \quad Q = Q(c, g), \quad F = F(c, g),$$

using the literal evaluations of the fixed names from the preceding section. Thus $N \subseteq M \subseteq W$, and $M \models \text{ZF}$ by transitive-base constructibility.

Lemma 7.1 (Eliminating the labeling coordinate). *If $Y \in N$, $a \subseteq Y$, and $a \in M$, then $a \in N[c]$.*

Proof. By Lemma 2.2, fix a bounded set-sized evaluator for a . Its inputs are the whole sets Q, F , finitely many actual members $q_0, \dots, q_{d-1}$ of Q , fixed ground parameters, and fixed ordinal codes. Each q_i is the increasing enumeration of a finite ordinal modification of $c[a_i]$, for an old carrier a_i supported by a generic cap. Choose a product condition (p_*, r_*) in the actual generic that verifies these descriptions and the presentation.

By dense tops and finite padding, strengthen this condition *within the actual generic* so that its designated top b contains every a_i literally. First cover the finitely many carriers modulo finite, then add their remaining finite points to the new top. Rename the strengthened condition (p_*, r_*) . Retain the old records and pad only the new top. Keep all finite ordinal decorations as fixed parameters.

Let $\dot{a}$ be the name obtained from this evaluator and the fixed names for its inputs, with a default value outside the chosen localization. For $y \in Y$, define in N

$$\begin{aligned} E_y^+ &= \{p \leq p_* : \exists r \leq r_* (p, r) \Vdash \check{y} \in \dot{a}\}, \\ E_y^- &= \{p \leq p_* : \exists r \leq r_* (p, r) \Vdash \check{y} \notin \dot{a}\}. \end{aligned}$$

Their union is dense below p_* . To see this, strengthen (p, r_*) to decide the indicated statement. No member of E_y^+ is compatible in the old Prikry order with a member of E_y^- . Otherwise Lemma 6.3, applied to their labeling witnesses, would give two genuine product-generic realizations with the same literal N, Q, F , the same supported q_i , and the same ground and ordinal parameters. The evaluator in Lemma 2.2 uses satisfaction only for these same set-sized constructibility levels. It therefore has the same value in both realizations, contradicting the opposite membership decisions.

Form the old Prikry name

$$\dot{b} = \{(\check{y}, p) : y \in Y, p \in E_y^+\}.$$

Its actual value is a . For every y , the truth lemma supplies a deciding product condition in the actual generic below the localization. Its Prikry coordinate belongs to the appropriate $E_y^\pm$. The incompatibility just proved excludes the opposite decision from the Prikry filter. Thus $a = \dot{b}^c \in N[c]$. This argument selects no family of maximal antichains. $\square$

Corollary 7.2. $\mathcal{P}(X)^M = \mathcal{P}(X)^N$ and $V_\lambda^M = X$.

Proof. Every M -subset of X belongs to $N[c]$ by Lemma 7.1. Its rank is at most $\lambda + 1 < \kappa$, so it belongs to N by Lemma 4.1(3). This proves the power-set equality. If there were a new member of V_λ^M , choose one of least rank. All its members would belong to $V_\lambda^N = X$. It would then be a new subset of X , contrary to the power-set equality. $\square$

Lemma 7.3 (External countable domination). *For every $\langle q_n : n < \omega \rangle \in W$ with $q_n \in Q$, there is an increasing $q^* \in Q$ such that*

$$\text{ran}(q_n) \subseteq^* \text{ran}(q^*) \quad (n < \omega).$$

Proof. In W , choose old carriers witnessing the q_n and conditions of g containing their supporting caps. Since g is a filter and $W \models \text{ZFC}$, choose a decreasing sequence $\langle r_n : n < \omega \rangle$ in g , with each term below the witnesses chosen so far. The labeling conditions have codes in X , as verified in Definition 5.2. By Lemma 4.1(1), the whole code array belongs to N . Decode it there to obtain the old decreasing sequence of conditions of $\mathbb{R}$.

For any old decreasing sequence $\langle r_n \rangle$, the set

$$D_{\vec{r}} = \{r \in \mathbb{R} : (\exists n \ r \perp r_n) \text{ or } (\forall n \ r \leq r_n)\}$$

is old and dense. Starting with any condition, recursively take common refinements with the successive r_n as long as possible. If a step fails, the current condition is incompatible with that r_n . Otherwise DC in N gives a descending sequence, and countable closure gives a lower bound below all its terms. Since all r_n belong to g , a member of $g \cap D_{\vec{r}}$ must be below every r_n .

The dense-top lemma gives a further member of g with a top b containing all the carrier witnesses modulo finite. Let q^* be the increasing enumeration of $c[b]$. It belongs to Q , and its range contains each $\text{ran}(q_n)$ modulo the finite carrier errors and finite ordinal decorations. Genericity is used only after the descending sequence has been shown to be old. $\square$

Lemma 7.4. $M \models \text{DC}$.

Proof. We first prove that every W -countable family from Q has an M -wellorderable cover contained in Q . For $\langle q_n : n < \omega \rangle$ choose q^* as in Lemma 7.3, and put

$$u_n = \{k < \omega : q^*(k) \in \text{ran}(q_n)\}.$$

The array $\langle u_n : n < \omega \rangle$ is coded by a single real of W , so it belongs to N by Lemma 4.1(1). Each u_n is infinite. Let v_n be the increasing enumeration of $q^* \restriction u_n$. Equivalently, restrict q^* to u_n and reindex by the increasing enumeration of u_n . The sequence $\langle v_n \rangle$ is definable in M from q^* and the old mask array, and therefore belongs to M .

Take the increasing enumerations of all finite modifications of the ranges of the v_n , allowing arbitrary finite ordinal additions and deletions below κ . Intersect this family with Q and call the resulting set C . It contains every q_n , since only finitely many values of q_n lie outside $\text{ran}(q^*)$. The set C is an image of a wellorderable parameter set consisting of $n < \omega$, finitely many ordinals below κ , and finite operation codes. Least preimages give a wellordering of C in M . This does not imply that the original external sequence $\langle q_n \rangle$ belongs to M .

Let $R \in M$ be a serial relation on a nonempty set $Y \in M$, and prescribe $y_0 \in Y$. Choose a presentation in M

$$\pi : \eta \times Z \times Q^{<\omega} \rightarrow Y, \quad Z \in N,$$

as in Lemma 2.2. Seriality is absolute to W because it quantifies only over the fixed sets Y, R . In $W \models \text{ZFC}$, choose an R -chain starting at y_0 and an argument of π for each entry. By Lemma 4.1(2), an N -wellorderable set $B \subseteq Z$ covers the resulting countably many ground coordinates. Intersect the cover with Z if necessary. Combine the finite Q -tuples into one countable family and apply the preceding paragraph to obtain an M -wellorderable cover $C \subseteq Q$. Then

$$Y_0 = \pi[\eta \times B \times C^{<\omega}]$$

belongs to M , is wellorderable there by least preimages, and contains the chosen external chain.

Inside M , form the tree T of finite R -chains in Y_0 starting at y_0 . Write $t \triangleleft s$ when t properly extends s . A rank function for this relation must assign smaller values to longer chains. The tree is wellorderable. If the extension relation were not well founded, a nonempty subset of T would have no terminal member. Using the fixed wellordering, recursively choose the least proper extension within that subset. This gives an infinite branch in M without any choice principle.

Thus, if there were no internal branch, the extension relation would be well founded and would have an ordinal rank function by well-founded recursion in ZF . The same function would remain a rank function in W , contradicting the chosen external branch. Hence an R -chain starting at y_0 exists in M , proving DC. $\square$

Proposition 7.5 (The sequence seed). *The model M satisfies $\text{ZF} + \text{DC} + \text{CM} + \text{SVC} + \text{KW}_1^{*,\omega}$. There are an ordinal μ and a set $A \subseteq \mu^\omega$ with $M = D(A)$. Moreover, D is a symmetric ground of M .*

Proof. Because $N \models \text{SVC}(X)$ and $F : Q \rightarrow X$, the presentation in Lemma 2.2 gives, for every nonempty $Y \in M$, a surjection

$$\eta \times Q^{<\omega} \rightarrow Y$$

after increasing the ordinal factor. To obtain it, cover the ground parameter set Z by $\xi \times X$, replace the X -coordinate by $F(q)$, and combine the two ordinal factors and the finite Q -coordinates. Thus $Q^{<\omega}$ is an SVC witness in M .

For each such Y , choose an ordinal μ large enough to contain η, κ and the finite coding symbols. To inject an ordinal coordinate and a finite tuple of members of $Q \subseteq \kappa^\omega$ into μ^ω , code the tuple length and ordinal in the initial positions, then interleave the finitely many sequences. Transfer the displayed surjection to the image of this injection. Extend it to all of μ^ω by assigning one fixed element of Y outside that image. This proves $\text{KW}_1^{*,\omega}$ with the power μ^ω computed internally in M .

To obtain one sequence seed for the whole model, define

$$R(p, q) \iff F(p) \in F(q) \quad (p, q \in Q).$$

Transitivity of X and surjectivity of F give

$$F(p) = F(q) \iff \forall t \in Q [R(t, p) \leftrightarrow R(t, q)].$$

Indeed, distinct sets in the transitive set X have a distinguishing member in X , and that member has an F -preimage. Equality of predecessor sets is therefore an equivalence relation. Its quotient relation is well founded and extensional, and the map $[q] \mapsto F(q)$ is an isomorphism to membership on X .

Use different finite tags for domain and relation codes, and interleave pairs of sequences. The resulting set $A \subseteq \mu^\omega$ codes the whole domain Q and binary relation R , for one sufficiently large ordinal μ . Both sets can be decoded in $D(A)$. Their quotient relation remains well founded and extensional. Its unique Mostowski collapse therefore recovers the same literal X and the map F . Hence $D(A)$ contains $N = D(X)$ and then $M = N(Q, F)$. Conversely, $A \in M$ and $D \subseteq M$, so minimality of transitive-base constructibility gives $D(A) \subseteq M$. Therefore $M = D(A)$.

The full choice cover $D[G][c][K]$ is a set-forcing extension of D . Above the size of its forcing, every normal measure in D has its normal generated lift in this cover. For each such ground measure $\mathcal{U}$ on δ , its restriction to M is the set

$$\{a \in \mathcal{P}(\delta)^M : \exists b \in \mathcal{U} (b \subseteq a)\},$$

which belongs to M by Separation. It is a nonprincipal δ -complete ultrafilter. Every M -sequence of fewer than δ members is also a sequence in the full cover, where the generated lift is complete. Its intersection belongs to M and satisfies the same ground-witness definition. Hence $M \models \text{CM}$. Only a proper-class tail is needed, although the selected κ was singularized.

Finally, $D \subseteq M \subseteq D[G][c][K]$ and $M \models \text{SVC}$. By the intermediate symmetric-model theorem, in the form of [8, Corollary 5.10], D is a symmetric ground of M . This uses the intermediate-model theorem; the notation $D(A)$ alone would not give the conclusion. $\square$

Proof of Theorem 1.4. Start with the $\text{ZF} + \text{DC} + \text{SVC} + \text{CM}$ model N in the local statement. By Proposition 3.1, this same N has a symmetric-ground presentation over a choice ground $D \models \text{ZFC} + \text{CM}$. Choose the protected rank $X = V_\lambda^N$ and the measurable κ as in (4.1), with λ above the required rank. The construction gives $N \subseteq M$. Corollary 7.2 proves $\mathcal{P}(X)^M = \mathcal{P}(X)^N$ and $V_\lambda^M = X$, while Lemma 7.4 and Proposition 7.5 give the required theory and the presentation $M = D(A)$ with D a symmetric ground. This proves the local theorem. Sections 8–9 remove SVC and Foundation from the starting assumptions of the structural conservativity theorem. $\square$

8. THE CORE-MODEL PREPARATION REMOVES SVC

We prove the preparation used in the earlier research [2] by specifying the relative construction and its uses of classical mouse theory. The proof assumes neither SVC nor a stronger unrelativized anti-large-cardinal hypothesis.

8.1. Classical mice over a fixed real. We use the fine coherent-measure mice of [9, Definition 3.24], over a real r . All iteration maps fix the bottom predicate, and measure critical points lie above its rank. We compute relative soundness, projecta, and hulls in this expanded language. Throughout, we use fixed-base initial-segment comparison. Sound iterable mice projecting to their common base are initial-segment comparable. This is the relative form of [9, Lemma 3.39]; see also [11, Section 1.1]. A *selected r -sword* is the minimal sound, base-projecting iterable r -mouse with a coherent upper measure of order one. Its distinguished lower and upper measures are both total over the passive sword universe; the lower is an element and the upper is the active predicate. No proper initial segment has the order-one obstruction. We use the usual least-segment and sound-core normalization of this obstruction. The selection specifies the transitive mouse and its predicates; it does not select a least real coding that mouse.

Lemma 8.1 (Noncreation of the selected sword). *Set forcing over a choice universe cannot create a selected r -sword for a ground real r .*

Proof. Two selected witnesses over r are initial-segment comparable, so minimality makes them literally equal. Suppose a condition forces that a witness exists, and take mutually generic filters G_0, G_1 through that condition. Each extension supplies a countable code for its witness. Iterability of a supplied classical mouse code is Π_2^1 [10, Proposition 2], so both witnesses remain iterable in the product extension. The structural requirements are absolute for these set structures. Uniqueness identifies their transitive universes and predicates there. The generic-intersection theorem returns the common structure Q to the ground.

We must also establish countability in the ground. In the product extension, sound projection supplies a finite degree n and a finite ordinal tuple p with

$$Q = \text{Hull}_n^Q(\omega \cup \{r, p\}).$$

Both n and p are ground objects. The ground computes the same satisfaction relation for the fixed set structure Q and hence the same Skolem map on the countable set of formula codes and integer parameters. Surjectivity onto Q is absolute. The ground therefore has a countable code for Q , whose iterability returns by the same Π_2^1 absoluteness. This contradicts nonexistence in the ground. This use of absoluteness concerns the supplied code, not the existential assertion that some mouse code exists. $\square$

In the no-sword case, $K^c(r)$ and $K(r)$ denote the classical coherent-measure construction with the bottom real retained. Replace the mice in [9, Definition 4.1] by mice over r . Small-mouse adjunction uses the longest good extension of the current sequence; initial-segment comparison makes it unique. The full-measure step uses the coherent-measure test, with countable completeness and the external cofinality condition for $K^c(r)$, and the universal-ultrapower test for the true core. The bicephalus comparison gives at most one such full measure. At limit stages take unions. In the thick-hull description of the true core use

$$\bigcap_{\Gamma \text{ thick}} \text{Hull}^{K^c(r)}(\Gamma \cup \{r\})$$

and take its transitive collapse. The construction is relative at every stage. Adjoining r to an already constructed core is a different operation [12].

Lemma 8.2 (Fixed-base classical core). *Let $B \models \text{ZFC}$ and $r \in B$. If there is no selected r -sword, the construction just specified gives a choice inner model $K(r)^B$ containing r . It has the classical weak-covering and universal-core comparison properties in the category of r -premise, and*

$$K(r)^B = K(r)^{B[g]}$$

for set forcing over B .

Proof. We check the fixed-base requirements of the classical proof. All hulls contain r and the integers. Their collapses and all copying and iteration maps fix the bottom real. Relative condensation and initial-segment comparison therefore give the same candidate uniqueness and copying arguments. In the countably complete construction, evaluate countably many representatives at one point in their measure-one intersection. This also preserves the predicate. These operations establish iterability and extract the true core in [9, Section 4.1].

Adding the fixed countable base to the hulls in the weak-covering proof leaves its cardinal-tail bounds unchanged. A failure configuration is now an iterable premouse over that same base. The high Mitchell-order obstruction excluded in the classical theorem would contain an initial segment with a coherent order-one upper measure. Its minimal sound core over r is a selected r -sword. Thus the required obstruction is absent in this relative category. The argument makes no assertion

about an arbitrary unrelativized inner model that might omit r . The covering diagram and its derived measures use collapses of hulls containing the base; the associated comparison maps are maps of r -premise. The order bound used in [9, Lemma 4.5 and Claims 4.23–4.24] is a bound on the measures of these constructed premise. The relative cutoff supplies that bound. The singular-strong-limit hull-size calculation and the hull construction of cofinality ω_1 are unchanged by including the countable base.

We include r when computing thick classes and their hull intersections. The classical universality and true-core comparison proofs then apply with the same fixed parameter on both sides. Apply the classical set-generic-invariance proof, including covering and thick-hull comparison, with this base in both the ground and the extension. The proof allows forcing to add codes for old mice. A fixed set forcing preserves the relevant successor-cardinal tail, so the old core remains universal. The forcing and thick-class comparison identifies the true-core hull. All these hulls contain r , and the forcing interpretation uses its ground check name. This argument does not assert clausewise absoluteness of K^c or of its countable-completeness test. Lemma 8.1 ensures that the relative obstruction remains absent in the extension. This gives literal equality of the two cores. The unrelativized application of this classical generic-invariance theorem is recorded explicitly in [5, Section 4, Lemma 4.2]. $\square$

8.2. The measure-restriction step. Classical ultrafilter maximality requires weak amenability. We verify this requirement for the restriction used here.

Lemma 8.3 (Amenability of the normalized restriction). *Work in a choice universe C with the no-sword core $K = K(r)^C$. Suppose $F \in C$ is a normal measure on $\mathcal{P}(\kappa)^K$ (external to K), countably complete for C -countable families, and complete for K -sequences of length less than κ . Suppose also that κ is inaccessible in K . Then F is weakly amenable to K , and F belongs to the coherent sequence of K .*

Proof. Form the full class ultrapower $j : K \rightarrow Q = \text{Ult}(K, F)$. Countable completeness makes it well founded. Normality identifies the seed $[\text{id}]$ with κ . We obtain the fine presentation from pure full-class Łoś and image set-segment codes. Each sufficiently long coded structure $(K \restriction \theta, \in, \vec{U} \restriction \theta, r)$ is a set of K by amenability of its existing sequence. Its image has the same local premouse axioms, and the image structures cohere. Replacement in K , applied to a representative, places each ultrapower element in some image structure. Their union therefore supplies the fine class presentation of Q . This step uses all functions in K . It does not invoke a degree ultrapower theorem that presupposes weak amenability of F .

To verify iterability, take a countable hull of a sufficiently long set initial segment of Q . Choose representatives for its elements and take the countably many measure-one truth sets for the relevant set-structure formulas. Simultaneous evaluation at one point in their intersection embeds the collapsed hull into an initial segment of K . Copying supplies iterability of the hull, and the usual countable-hull reduction for classical iterations gives iterability of Q . Only set satisfaction is used here.

Let λ be a sufficiently large singular strong-limit cardinal of C with $\text{cf}^C(\lambda) > \kappa$. Weak covering gives $(\lambda^+)^K = \lambda^+$. For $\xi < \lambda$ the set of K -functions $\kappa \rightarrow \xi$ has C -size less than λ ; hence $j(\xi) < \lambda$. All κ -functions into λ are bounded, so $j(\lambda) = \lambda$. Also $\lambda^\kappa = \lambda$ in C . The same counting below λ^+ and continuity at that regular cardinal give $j(\lambda^+) = \lambda^+$. Therefore

$$(\lambda^+)^Q = j((\lambda^+)^K) = \lambda^+.$$

Such λ form a stationary class. In any class club take a sufficiently high closure point of fixed regular cofinality above κ , also in the club of strong-limit cardinals. The classical successor-correctness criterion makes Q universal [9, Proposition 4.4].

The universal-core comparison theorem and its proof show that the Q side of the comparison with K does not move. Thus Q is the nondropping comparison iterate of K [9, Theorem 4.12]. These premice agree below κ , since j fixes the K -sets of rank below κ . For partial measures, the classical indexing rule is $\gamma = (\delta^{++})^{L[r, \vec{U} \upharpoonright \gamma]}$ at critical point δ [9, Definition 3.24]. Its truncated-sequence model is an inner model of K , so this index is bounded by the corresponding double Hartogs bound in K . If $\delta < \kappa$, inaccessibility puts it below κ even when δ later becomes countable in the full core. Comparison therefore uses only measures with critical point at least κ . These internal amenable measures preserve the power set of κ , as do their nondropping direct limits. Consequently

$$\mathcal{P}(\kappa)^Q = \mathcal{P}(\kappa)^K.$$

For a K -sequence $a = \langle a_i : i < \kappa \rangle$ of subsets of κ , Separation in Q forms

$$\{i < \kappa : \kappa \in j(a)(i)\} = \{i < \kappa : a_i \in F\}.$$

Power-set equality puts this trace in K . A K -enumeration of any K -set of K -size κ now gives weak amenability. Classical ultrafilter maximality now applies, so F is a sequence measure [9, Corollary 4.13]. All maps in this argument fix r , as required by Lemma 8.2. $\square$

Lemma 8.4 (Measurables in one fixed relative core). *In a ZF universe with unbounded measurables, put $B = \text{HOD}(r)$. If B has no selected r -sword, then $K(r)^B \models \text{CM}$.*

Proof. Fix an ambient measurable κ above the bottom real and an ambient κ -complete non-principal ultrafilter U . No normality in the choiceless universe is assumed. The restriction $u = U \cap \mathcal{P}(\kappa)^B$ is coded by a set of ordinals using a B -wellordering of $\mathcal{P}(\kappa)^B$. Relative Vopěnka gives a choice set-generic extension C of B , inside the ambient universe, containing that code. The core is unchanged by Lemma 8.2.

In C take the well-founded ultrapower of B by u and derive the normal B -measure D at the ordinal seed κ . Choose a representing function $f \in B$ for this seed. Then, for $a \in \mathcal{P}(\kappa)^B$,

$$a \in D \iff f^{-1}[a] \in u.$$

Thus D is a set of C . Its restriction F to the unchanged core is normal, countably complete for C -families, and complete for core sequences of length below κ . These facts follow from U and the displayed pushforward description, since C is an inner model of the original universe.

Ambient measurability also makes κ inaccessible in the choice inner core. Regularity is downward absolute. If a smaller ordinal ξ admitted κ distinct subsets in the core, intersecting the ξ many sides selected by U would force two of them to have the same incidence pattern. This contradicts their distinctness. Core Choice therefore gives the strong-limit assertion. Lemma 8.3 puts F in the one fixed core $K(r)^B$.

Apply this argument separately to unboundedly many ambient measurables. It requires neither a class choice of their measures nor a single cover containing all the restrictions. Normalization in this form also occurs in [15, Theorem 3.11]. $\square$

8.3. Recursive changes of the bottom real.

Lemma 8.5 (Recoding covariance). *Let r, s be mutually recursive reals in a common choice universe. The classical recipe above, and the selected sword followed by its upper-measure iteration when that sword exists, give the same pure membership universe after changing r to s .*

Proof. We have $\text{HYP}(r) = \text{HYP}(s)$ because every transitive admissible set containing either real contains the other by recursive comprehension. This common passive level lies below the first critical point κ . Indeed, in the passive premeasure structure P , κ is inaccessible; at its uncountable regular $\eta = \omega_1^P < \kappa$, the usual regular-cardinal Collection argument makes $L_\eta[r]$ admissible. The least r -admissible height is therefore below κ . Here we use the premeasure structure. A partial

measure's critical point can be countable in the eventual full mouse. Above the common level, rudimentary successor closures with the same measure sequence and limit unions give the same pure levels.

The two predicate languages translate into one another using finite oracle computations. With ω and the hereditarily finite sets named as bounds, the translations are bounded. Their parameter-free Δ_1 definitions on the developed levels eliminate these bounds, preserving Σ_n and Π_n for $n \geq 1$. Hence

$$\text{Hull}_n(\Gamma \cup \{r\}) = \text{Hull}_n(\Gamma \cup \{s\}) \quad (n \geq 1).$$

For each ordinal tuple, theory and master-code predicates translate on formula codes without changing ordinal coordinates. Thus projecta, the least ordinal standard parameters, and pure sound-core collapses agree. This agreement does not assert equality of the literal syntax of master codes. Degree-zero ultrapowers use the same whole function sets, and higher-degree ultrapowers use the corresponding definable functions. Their equivalence relations, membership relations, maps, and limit systems agree. Iterability and comparison are consequently preserved. This real-premouse rearrangement is used in [13, p. 45] and [11, Lemma 3.5.2].

Rearrangement takes minimal sound swords to minimal sound swords. Initial-segment uniqueness identifies the selected structures, and the identical upper-measure iterations give identical stabilized pure universes.

For the core, apply the recursion specified above. Candidate small mice correspond, and their initial-segment chains and longest good extensions agree. Candidate full measures and their coherence tests agree. We compute countable completeness and the cofinality test in the same universe. The universal-ultrapower test agrees because iterable extensions correspond. Intrinsic uniqueness therefore gives the same new measure. At limit stages take the same unions. Small-mouse adjunction may skip passive stages, but all measure entries below the common passive level are empty. A candidate with a nonempty sequence lies above that level, and a shorter passive candidate contributes no different measure entry. Thus no first discrepancy in the sequence can arise. Thick hulls containing the bottom real agree by the displayed hull equality, so their pure collapses agree as well. $\square$

8.4. The choice model and its own HOD.

Lemma 8.6 (Stabilized sword model). *If a choice universe B has a selected r -sword, its upper-measure iteration gives a choice inner model C_r containing r , with unbounded measurables and an externally iterable fine-structural coherent-measure presentation.*

Proof. Use the active presentation of the sword P . Its upper measure is total over the passive universe, weakly amenable, and of order one; the lower measure belongs to the mouse and is also total over that passive universe. Iterate the upper measure and its images, taking well-founded direct limits. These total top-measure iterations have no model or degree drops. Their critical points κ_α increase continuously and are unbounded in Ord^B . For each fixed η , the fine segment through η is unchanged once $\eta < \kappa_\alpha$. Let E_∞ be the stabilized sequence and put

$$C_r = J_{\text{Ord}}[r, E_\infty].$$

Induction on the hierarchy index shows that every set-sized segment equals the corresponding segment of a sufficiently late iterate of P . Thus E_∞ is coherent and amenable, and r is retained. This is the standard good Ord-iteration of the order-one mouse [16, Theorem 14].

At κ_α , coherence leaves the lower measure in the next iterate, while weak amenability of the upper measure preserves its entire power set. All later critical points exceed κ_α , and a sufficiently late stage also fixes the lower measure as a set. Both the power set and the measure stabilize. The surviving measure is therefore total in C_r , and the unbounded critical sequence gives CM^{C_r} .

Full Replacement in the original sound mouse is not needed to obtain ZFC in the final class. The set-stage iteration, the stabilized sequence, and its entire hierarchy are definable in B from P . For any fixed finite list of formulas relativized to this class, B -Replacement bounds witness stages for tuples from each level. Closing under these bounds produces arbitrarily high reflecting levels. Separation follows by defining the desired subset over such a level. For Power Set, the collection of all C_r -subsets of a given set is first a B -set; its entry stages are bounded by B -Replacement. A later level contains and defines the whole collection. The same bounds and finite reflection argument give Replacement for C_r -definable functions. The other pure axioms follow from the full hierarchy. Canonical wellorders of set levels, using their truncated amenable sequence data, belong to later levels and give Choice.

Finally, any set-sized segment is a segment of an iterable set iterate of P . A proposed failure of iterability reduces to a countable hull and copies to such an iterate. This supplies the external iterability of the stabilized fine presentation. $\square$

Lemma 8.7 (Internal HOD ground). *Each of the two models C_r just constructed is a set-forcing extension of HOD^{C_r} .*

Proof. Both final coherent sequences have order zero. In the core branch, a positive-order entry would yield a selected sword by the least-segment and sound-core normalization. In the sword branch, the same minimality makes its top upper measure the only positive-order entry; its critical point escapes to Ord . The stabilized entries therefore have order zero.

Use the usual normal-measure coding of an already fine coherent-measure premouse by short type-1 extenders. For a measure U over its designated segment, derive E from its ultrapower j_U . The component at the seed κ recovers U , and the other components are its pullbacks by representing functions. The map

$$[b, f]_E \mapsto j_U(f)(b)$$

is onto the measure ultrapower because the seed κ generates it, and it respects membership and embeddings. Applying this coding to the local fine-structural codes transports permissible definable functions, sound cores, and degree ultrapowers; initial segments, drops, and direct limits correspond. This is the standard short-measure presentation equivalence, with one fixed indexing and trivial-completion convention throughout. It applies to the fine mice with which we began, without upgrading a coarse structure.

No additional tree-strategy hypothesis is needed. These short measures do not overlap at distinct critical points. The supports of lower measures lie below the next measurable critical point in the relevant premouse segment. An order-zero measure leaves no measure at its own critical point after it is used. A first genuine return of a normal tree to an earlier model would require a new critical point below an earlier short support. Distinct critical points are excluded by nonoverlap and prefix agreement; the same critical point is excluded by order-zero coherence. After removing padding the tree is therefore a classical normal linear iteration, with its usual possible drops. Classical iterability supplies the well-founded unique limit branch. This is the nonoverlap regime described in [14, Lecture 5]. Thus the fine presentation M_r is externally $(0, \omega_1^B + 1)$ -iterable in its choice background B , and its pure universe satisfies Power Set.

Apply extender definability [6, Theorem 1.1] in its real-relative form. The relative condensation-stack argument is used explicitly in [17, proof of Lemma 5.4]. The resulting parameter may be taken as

$$a = (r, M_r \restriction \omega_1^{C_r}) \in C_r.$$

This ordinal is the model's own ω_1 . The theorem requires external iterability; it does not require a strategy belonging to C_r . The definable hierarchy and its wellorders give $C_r \models V = \text{HOD}(a)$. By Choice in C_r , code $\text{TC}(\{a\})$, its membership relation, and its marked root a by a set of ordinals $t \in C_r$. Then $C_r \models V = \text{HOD}(t)$ as well.

Vopěnka's theorem inside C_r supplies a set forcing in $D_r = \text{HOD}^{C_r}$ and a D_r -generic g such that

$$C_r = D_r[g].$$

The relative HOD recovered by the theorem is all of C_r , by set-parameter definability as above. See [18, Theorem 6.2.7]. This does not identify D_r with the original background $\text{HOD}(r)$. $\square$

Lemma 8.8 (Core-model preparation). *In a ZF universe with a proper class of measurables there is a uniform assignment, for every real r , of a choice inner model C_r such that:*

- (1) $r \in C_r$ and $C_r \models \text{ZFC} + \text{CM}$;
- (2) the pure universe of C_r is unchanged by the finite recursive recodings induced by permuting a collapse enumeration;
- (3) C_r is a set-forcing extension of $D_r = \text{HOD}^{C_r}$.

Proof. Work in $B_r = \text{HOD}(r)$. If there is a selected r -sword, use Lemma 8.6; otherwise use Lemmas 8.2 and 8.4. Finite recursive recodings s satisfy $B_s = B_r$. Lemma 8.5 identifies the branch, the construction, and the resulting pure universe in this common background. Lemma 8.7 gives the final clause. These recipes require no additional choice of a real code or a background wellorder. $\square$

8.5. Rank and dependent choice.

Proposition 8.9 (Rank-preserving passage to SVC). *Suppose $V \models T_0$. For every ordinal β there is an inner model $N \subseteq V$ satisfying $T_0 + \text{SVC}$ such that*

$$V_\beta^N = V_\beta^V, \quad \mathcal{P}(V_\beta^V)^N = \mathcal{P}(V_\beta^V)^V.$$

One can arrange a choice ground $D \models \text{ZFC} + \text{CM}$ and a choice set-forcing cover $C = D[g]$ with $D \subseteq N \subseteq C$.

Proof. Choose $\lambda > \beta + \omega$ with $\text{cf}^V(\lambda) > \omega$, and let $B = V_\lambda^V$. DC supplies such a limit ordinal. For example, start sufficiently high and add ω_1^V . Collapse B to be countable, and let r code membership and equality of entries in its generic enumeration. By Lemma 2.1, this fixed collapse preserves a proper-class tail of measurables. Apply Lemma 8.8 in the collapse extension and put $C = C_r$.

Inside C , first quotient the coded enumeration by its equality relation and then take its well-founded extensional collapse. This recovers the same transitive set B . Define

$$N = \text{HOD}_{B \cup \{B\}}^C,$$

allowing individual elements of B as parameters as well as the whole set B . This relative hereditary-definability construction satisfies ZF and contains B and all its elements.

Finite permutations of the ω -domain witness weak homogeneity of the collapse by moving the finite domain of one condition away from that of another. They replace r by exactly the recodings covered by Lemma 8.8(2) and fix the actual ground set B and its members. Thus the group preserves every fixed ordinal-and- B definition over C . The usual hereditary rank induction gives $N \subseteq V$. Consequently

$$V_\lambda^N = B = V_\lambda^V.$$

In particular, both the rank V_β and all its subsets have been retained.

For every nonempty $Y \in N$, the relative-HOD definition gives a surjection in N from $\eta \times B^{<\omega}$ onto Y , for some ordinal η . The surjection enumerates all bounded definition codes and finite parameter lists, without choosing a definition for each element of Y . Finite lists from the limit rank B can be coded into B , so $N \models \text{SVC}(B)$.

For DC, let $R \in N$ be serial on Y , fix $y_0 \in Y$, and take $f : \eta \times B \rightarrow Y$ in N . At a current y put

$$\xi(y) = \min\{\xi < \eta : \exists b \in B (y R f(\xi, b))\}.$$

Use DC in V to choose successive b_n at these least ordinal coordinates. The array $s = \langle b_n : n < \omega \rangle$ belongs to B because $\text{cf}^V(\lambda) > \omega$ and λ is a limit. Hence it belongs to N . Within N , recursion from s , y_0 , and the least-coordinate rule reconstructs the entire chain. Thus $N \models \text{DC}$.

Finally, $D = \text{HOD}^C \subseteq N \subseteq C = D[g]$. Since the forcing is fixed, downward Lévy–Solovay gives a measurable tail in D from $C \models \text{CM}$. For every sufficiently high ground measure U , its generated lift in C restricts to the set

$$\{a \in \mathcal{P}(\delta)^N : \exists b \in U (b \subseteq a)\}$$

in N . This is the complete nonprincipal ultrafilter required for CM^N . We fix the ground D and cover C before taking this tail. If desired, the intermediate symmetric-model theorem and SVC^N make D a symmetric ground of N . $\square$

Proof of Theorem 1.3. Apply Proposition 8.9 above $\alpha + 1$ to obtain $N \subseteq V$. Then apply the local Theorem 1.4 to N , protecting the same rank and its power set. The resulting M has the displayed rank and power-set equalities with V , together with the stated sequence-seed presentation. Choose all required generics over the original collapse extension. The construction then takes place in a set-generic extension of V . The inclusion asserted is $N \subseteq M$; the argument does not assert $V \subseteq M$. $\square$

9. REMOVING FOUNDATION BY ATOMIZATION AND FIBERS

We now reduce the Foundation-free starting theory to the preceding case. The atomization and fiber purification follow the earlier argument of [2, 23 September, 06:07 UTC]. We first prove the version of the local forcing theorem needed here. In this section, ZFA has Foundation above a fixed set $\mathcal{A}$ of atoms, and Ker denotes the pure kernel.

9.1. The local construction with atoms.

Lemma 9.1 (The choice-envelope version). *Let $N \subseteq E$ be ZFA models with the same ordinals and the same set $\mathcal{A}$ of atoms. Suppose:*

- (1) $N \models \text{DC} + \text{SVC}(X)$, where $X = V_\lambda^N(\mathcal{A})$ is a sufficiently high limit rank, $\text{cf}^N(\lambda) > \omega$, and $\mathcal{A} \subseteq X$;
- (2) E satisfies Choice, $\mathcal{A}$ is countable in E , and $\text{Ker}(E) = D[g]$ for a pure $D \subseteq N$ satisfying $\text{ZFC} + \text{CM}$ and fixed set forcing;
- (3) a measurable $\kappa > \lambda + \omega$ of D is chosen above the size of this fixed forcing and the sizes in E of X , the old collapse, and the cap and comparison posets.

Then the local Prikry–labeling construction produces $N \subseteq U$ with

$$U \models \text{ZFA} + \text{DC} + \text{CM} + \text{SVC} + \text{KW}_1^{*,\omega}, \quad \mathcal{P}(X)^U = \mathcal{P}(X)^N, \quad V_\lambda^U(\mathcal{A}) = X.$$

Proof. This version requires only the pure D and the choice envelope E . Use the old pure Prikry poset from D with its old wellordering. The old normal measure base generates the normal lift in E and its small choice extensions. Compute the countably closed collapse in N and choose the product generic over E , as in Section 4. Internal countable closure and condition coding use $\text{cf}^N(\lambda) > \omega$. No requirement is imposed on $\text{cf}^E(\lambda)$.

The small-forcing assertions with atoms reduce to their pure counterparts. Fix an enumeration of the atoms in E . Recursively code each atom by its tagged index and each set by the tagged set of its members' codes. Every resulting code is pure and belongs to $\text{Ker}(E)$. Choice in E codes any set forcing by an ordinal condition set and a pure order. Translate names using the same

atom indices. Decoding a name for a pure value uses no atoms. The pure kernel after forcing is therefore the corresponding set-forcing extension of the old pure kernel. We use this coding only in the envelope and do not adjoin its atom enumeration to N or U .

The collapse-first proof of Lemma 4.1(1)–(2) therefore applies to atomful X and arbitrary ground values. SVC makes $N[K]$ a choice model with atoms. Since Prikry forcing adds no bounded subsets, we obtain the same preservation of countable sequences from X and wellorderable covers of ground families.

For part (3), we need preservation of atom-ranks. Since $|\mathcal{A}|^E < \kappa$ and κ remains inaccessible in the choice envelope, every old atom-rank below κ has size below κ . Prikry forcing preserves these ranks because it adds no bounded subsets. A new set of least rank would be a subset of a small old rank; its pullback by an old wellordering would be a new bounded subset of κ . The atoms are unchanged. The generic-intersection proof proceeds by rank over the common atom set, starting with the atoms, which already belong to N . Thus

$$V_\kappa^{N[c]}(\mathcal{A}) = V_\kappa^N(\mathcal{A}).$$

Use transitive-base constructibility with atoms retained at every successor stage, for example

$$J_{\xi+1} = \mathcal{A} \cup \text{Def}(J_\xi; \in, \text{Atom}),$$

so that taking definable subsets does not discard the atoms. Lemma 2.2 still gives finite expressions. Atoms are ground leaves, and the atom predicate has the same interpretation in the compared universes. The presentation lemma does not use an equality $N = D(X)$.

All cap and paired constructions concern integer carriers and their literal colors in X . A color may be an atom. Since the constructions never recover a color from its members, the proofs in Sections 5–6 are unchanged. The common atom predicate also gives bounded-evaluator absoluteness. Labeling elimination and the atom-rank equality just proved yield $\mathcal{P}(X)$ preservation.

The old condition-array argument, the real-coded masks, and the wellorderable-tree argument prove DC as before. Bounded presentations and $F : Q \twoheadrightarrow X$ give SVC and KW for every nonempty set of U , including sets containing atoms. Finally the full choice envelope after these set forcings has pure kernel a set-forcing extension of D , by the coding paragraph above. A sufficiently high tail of ground measures has generated lifts there, and the old-base definition used in Proposition 7.5 restricts those lifts to measures belonging to U . This proves CM^U .

The atom lemma does not include the pure sequence-seed and symmetric-ground conclusions of Proposition 7.5. The predecessor-set quotient does not distinguish distinct atoms, and a pure $D(A')$ cannot contain them. Neither conclusion is needed below. $\square$

9.2. Preparation of the protected atom structure. Start with $V \models T_-$ and a full second-order counterexample $\mathcal{M}$ to a fixed sentence. Take its relational multisorted presentation, with a first-order sort S_0 and, for each arity k in the sentence, a sort representing all V -subsets of S_0^k with the incidence relation. The two-sorted presentation is the unary or tuple-coded case. Replace the combined domain by a set $\mathcal{A}$ of formal atoms, with distinct tags for the sorts.

Form the full interpreted cumulative hierarchy $\mathcal{H}(\mathcal{A})$ using the original universe's power sets. Treat its atoms as opaque labels whose original members do not become members of the atoms. Non-well-founded membership in V therefore introduces no cycles in the interpreted hierarchy. The hierarchy satisfies ZFA. It also satisfies DC, since an ambient chain through a set of its coded objects has bounded atom-rank and belongs to the hierarchy. Its pure kernel has the original ordinals and their full power sets. It therefore retains the original measures and the ordinal-indexed families testing their completeness. Hence

$$\mathcal{H}(\mathcal{A}) \models \text{ZFA} + \text{DC} + \text{CM}.$$

The atomized structure remains full because inverse images under the atom-label bijection identify its subsets and finite-arity relations with all the original ones.

Choose a limit λ of uncountable cofinality in this hierarchy, high enough to contain the structure and the full power sorts, and set

$$B = V_\lambda^{\mathcal{H}(\mathcal{A})}(\mathcal{A}).$$

Collapse B to be countable. Its real code r records membership, equality of repeated enumeration entries, and which entries are atoms. The small-forcing proof of Lemma 2.1 works with fixed atoms, so a measurable tail remains. Apply the earlier core-model Lemma 8.8 in the pure kernel of this collapse extension, obtaining $C_r = D[g]$.

Construct the choice companion E as follows. Let I be the least enumeration indices of the distinct atoms, as determined by r . Inside C_r , form the hierarchy over the formal atom set indexed by I . Use the collapse enumeration to relabel those formal atoms by the actual atoms. The resulting E satisfies Choice, has pure kernel C_r , and contains the literal B . To prove the last assertion, quotient the code by its equality predicate and take its well-founded collapse with the specified atom labels. These labels distinguish the atoms from one another and from the empty set. Uniqueness of the collapse gives the same B .

The finite permutation group used in Proposition 8.9 preserves this companion. The pure C_r is unchanged, and the induced bijection between the two atom-index sets belongs to the same pure model. Transporting the hierarchy along this bijection gives the identical actual companion. Therefore

$$N = \text{HOD}_{B \cup \{B\}}^E,$$

with the atom predicate retained, descends to $\mathcal{H}(\mathcal{A})$ by the same hereditary homogeneity argument. It contains B and has $V_\lambda^N(\mathcal{A}) = B$. Relative-HOD presentations give $\text{SVC}(B)$, and the least-ordinal-coordinate DC proof in Proposition 8.9 applies using the countable closure of B in $\mathcal{H}(\mathcal{A})$.

The pure $D = \text{HOD}^{C_r}$ is contained in N , because definitions over the definable pure kernel of E are available over E . Ground measure lifts restrict to N , giving CM^N . We have a pure ground inside N and a choice envelope above it. This does not assert a choice ground with atoms inside N , and it does not apply the pure SVC symmetric-ground theorem to an atom model.

Lemma 9.1 now applies with $X = B$. In particular, $\text{cf}^N(\lambda) > \omega$ because $N \subseteq \mathcal{H}(\mathcal{A})$. The envelope may see B and λ as countable, which the lemma permits. The old condition sets have codes in B and are small in E . Once these set-sized bounds are fixed, choose a sufficiently high D -measurable κ . The lemma yields

$$U \models \text{ZFA} + \text{DC} + \text{CM} + \text{SVC} + \text{KW}_1^{*,\omega}$$

with the protected power sets unchanged. The original multisorted counterexample is consequently still full in U .

9.3. Fiber purification and the final theorem.

Lemma 9.2 (Fiber purification). *If $U \models \text{ZFA} + \text{DC} + \text{CM} + \text{SVC} + \text{KW}_1^{*,\omega}$, then its pure kernel P satisfies $T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$. Every full second-order structure in U has a full isomorphic copy in P .*

Proof. For the nonempty combined domain S of a relational presentation, take a surjection $f : \kappa^\omega \twoheadrightarrow S$ in U and put

$$j(s) = f^{-1}\{s\}.$$

The nonempty disjoint fibers make j injective. Each fiber is a set of ordinal sequences and is therefore pure. Thus the whole range $j[S]$, its sort predicates, and all transported finite-arity

relations are pure sets in P . The graph of j may contain atoms and need not belong to P . Empty domains can be treated directly.

To check fullness at each relevant arity, let $Y \in P$ with $Y \subseteq j[S_0]^k$. Its pullback under j^k is a U -subset of S_0^k , represented by an element of the full power sort in U . The image of that element represents exactly Y . Conversely, each represented subset in the transported structure is pure and belongs to P . In the application, preservation of the protected power sets identifies these U -relations with the relations of the original counterexample. The copied structure therefore has the same full second-order truth.

The pure kernel satisfies ZF, including Foundation. A DC chain through a pure set is itself pure. Measures on ordinals and all families testing their completeness are pure as well, so CM descends. The models have the same κ^ω ; a surjection from it onto a pure set has pure graph. Hence KW descends.

Finally, fix one SVC witness C in U and fiber-code it as above to obtain a pure copy $C^\flat$. For any nonempty pure Y , compose an SVC surjection $\eta \times C \twoheadrightarrow Y$ with the inverse of this fiber bijection. The resulting map $\eta \times C^\flat \twoheadrightarrow Y$ has pure graph and belongs to P . The same $C^\flat$ works for every Y , proving SVC in P . $\square$

Proof of Theorem 1.7. Given a full second-order counterexample in a model of T_- , atomize it, carry out the core-model preparation, and apply the local atom theorem as above. Lemma 9.2 gives a pure model of $T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$ containing a full counterexample to the same sentence. The countermodel contrapositive proves structural conservativity. The reverse implication of provability follows because the target theory extends T_- . All constructions are interpretations and set-forcing constructions over the starting model. No external transitive-model hypothesis is used. $\square$

10. EXTERNAL HOD AND HOMOGENEOUS DESCENT

Lemma 10.1. *If $U \models \text{ZFC}$, then $\mathcal{H}^U \models \text{ZF} + \text{DC}$. It contains every ordinal ω -sequence of U and is closed under U -countable sequences of its elements.*

Proof. The usual hereditary-definability proof of ZF applies to the definable class of allowed parameters. For example, the internal power set of $x \in \mathcal{H}^U$ is

$$\{y \subseteq x : y \in \mathcal{H}^U\},$$

which is definable from a definition of x and has only hereditary members of the class. The same argument gives Separation and Replacement.

Let $\langle x_n : n < \omega \rangle \in U$, with each $x_n \in \mathcal{H}^U$. In U , choose a definition of x_n from an ordinal sequence and ordinal parameters. Evaluate each definition at a rank level where its formula has the required unique value. Interleave the sequence parameters, the rank levels, the ordinal parameters, and the formula codes. This gives one $q \in (\text{Ord}^\omega)^U$ from which the entire sequence $\langle x_n \rangle$ is definable in U . Its hereditary members already belong to $\mathcal{H}^U$, so the sequence belongs to $\mathcal{H}^U$. Every $q \in (\text{Ord}^\omega)^U$ is itself hereditarily definable from q . To prove DC, choose a dependent-choice chain in U through a serial relation belonging to $\mathcal{H}^U$ and apply the countable closure just proved. $\square$

Lemma 10.2 (Homogeneous descent). *Let $M \models \text{ZF}$, and suppose $U = M[G]$ is a weakly homogeneous set-forcing extension with*

$$(\text{Ord}^\omega)^U = (\text{Ord}^\omega)^M.$$

Then $\mathcal{H}^U \subseteq M$. No ordinal-definability assumption on the forcing is needed.

Proof. Induct on hereditary rank. Suppose $x \in \mathcal{H}^U$ and every member of x is in M . Choose β with $x \subseteq V_\beta^M$. A definition of x in U uses an allowed parameter q belonging to M and ordinal parameters. By weak homogeneity, the largest condition decides the corresponding uniqueness assertion and, for each $y \in V_\beta^M$, whether y belongs to its unique value. The forcing relation therefore defines x as a subset of V_β^M inside M . $\square$

Proposition 10.3. *For every σ , $\text{Con}(E_1(\sigma))$ implies $\text{Con}(E_3(\sigma))$.*

Proof. Let $U \models E_1(\sigma)$, put $H = \mathcal{H}^U$, and choose κ with $V_\kappa^H \models \sigma$. Choose $\lambda > \kappa + \omega$ of uncountable cofinality in U , and set $X = V_\lambda^H$. This X is ordinal-definable in U , belongs to H , and is closed under U -countable sequences of its elements by Lemma 10.1.

Collapse X to be countable over U , coding both equality and membership and quotienting repetitions before collapse. Apply Lemma 8.8, and again set $W = \text{HOD}_{X \cup \{X\}}^{C_r}$. The collapse-homogeneity argument gives $W \subseteq U$ and controls the parameters in each descended definition. The collapse forcing is definable from the OD set X , so these definitions use X , finitely many elements of X , and ordinals. The set X is OD in U , and each of those finitely many elements is hereditarily definable from an ordinal ω -sequence. Combining their parameters gives $W \subseteq H$.

Since $X \in W$, we have $V_\lambda^W = X$. The presentation $\text{SVC}(X)$ and the ambient countable closure of X give DC in W . The same generated-measure argument gives CM. Therefore $W \models E_3(\sigma)$. $\square$

11. SPARSE EVALUATED COHEN CAPS

We work over a choice model B and code a *family of coherent rows*. The code records which bit positions belong to each row, rather than only their union.

We use the convention $\text{Add}(\gamma, 1) = 2^{<\gamma}$, ordered by extension; the dense partial-function presentation gives the same forcing. Throughout this section, the superscript specifies the model in which each poset is computed.

Theorem 11.1 (Fixed-base spectrum). *Let $B \models \text{ZFC}$ have a proper class of inaccessible cardinals, and let $\emptyset \neq F \subseteq 2^\tau$ belong to B , where τ is infinite. There are a schedule of inaccessible coding cardinals, uniformly defined forcings Q_e for $e \in 2^\tau$, and a cap cardinal ρ , all definable from B and ordinal parameters independently of F , such that the weakly homogeneous forcing P_F defined below has the following spectrum in $B[G]$:*

$$e \in F \iff \text{there is a } B\text{-generic } h_e \subseteq Q_e, \text{ and an } \text{Add}(\rho, 1)^{B[h_e]}\text{-generic over } B[h_e]. \quad (11.1)$$

The first coding cardinal can be placed above any prescribed bound, and the resulting forcing is correspondingly distributive.

11.1. The schedule and the actual forcing. Order the physical sites $s = (\xi, i)$, for $\xi < \tau$ and $i < 2$, lexicographically with ξ first. Distinct rows have independent copies at a shared physical site. Choose inaccessible cardinals γ_s recursively, with the first above $2^{|\tau|}$ and the prescribed protection bound, and with

$$|\text{RO}(S_{<s}^{\text{univ}})|^B < \gamma_s. \quad (11.2)$$

The universal prefix includes copies for every row in 2^τ , including those outside F . The schedule is therefore independent of the family being coded. At each step, choose the least inaccessible above the displayed bound.

For $e \in 2^\tau$, let

$$Q_e = \prod_{\xi < \tau}^{\text{full}} \text{Add}(\gamma_{(\xi, e(\xi))}, 1)^B.$$

Let ρ be an inaccessible above the Boolean completion of the entire universal lower product. Define

$$P_F = \prod_{d \in F}^{\text{full}} \left(Q_d * \text{Add}(\rho, 1)^{B[h_d]} \right). \quad (11.3)$$

Compute the terminal forcing in its *own row extension*. Form the actual extension from the evaluated terminal generics, without adjoining the larger raw term-space generics.

The full lower product

$$S_F = \prod_{d \in F}^{\text{full}} Q_d$$

can be grouped by physical sites. Denote the aggregate generic at s by G_s . Since $|F| < \gamma_0 \leq \gamma_s$, each site forcing is a product of fewer than γ_s copies of $\text{Add}(\gamma_s, 1)^B$. It is $< \gamma_s$ -closed in B and adds no shorter ordinal sequences over B .

For each d , let T_d be a full set of total names for cap conditions: the largest condition of Q_d forces each member to be a cap condition, and every total name has an equivalent member. Use all bounded Boolean-name codes for graphs on $\rho \times 2$ with this property, allowing equivalent codes. The resulting set is defined canonically from Q_d and ρ ; the forcing has no auxiliary wellorder as a parameter. Smallness of $\text{RO}(Q_d)$ and regularity of ρ bound the possible lengths of each name below ρ . Normalize forced unions and Boolean mixes to these Boolean-name codes. Lower automorphisms act on their Boolean values and preserve this full name system. The product

$$S_F \times \prod_{d \in F} T_d$$

with the ground term-space orders projects onto P_F and preserves the lower coordinate. To verify the lifting property, mix each desired stronger term on its strengthened lower cone with the old term elsewhere. Thus this projection commutes with the projection to S_F . Each ground term-space is $< \rho$ -closed. After the lower forcing, which has size below ρ , the full product is $< \rho$ -distributive by Easton's lemma. The actual cap quotient therefore adds no ordinal sequences of length below ρ over the lower extension. Here a closed *dominating* forcing proves preservation; the actual extension need not equal the larger term-space extension. See [20, Section 3] for the term-forcing facts.

This also proves preservation. If the first site is above a regular θ , the dominating forcing is $< \theta$ -closed in B , so the actual forcing is $< \theta$ -distributive. Taking $\theta > |V_\eta^B|$ preserves V_η^B by rank induction. The canonical lower-row automorphisms lift through the name for the recomputed Cohen forcing, which is forced homogeneous. Coordinatewise automorphisms then witness weak homogeneity of the full product.

11.2. Localization to physical sites.

Lemma 11.2. *Suppose a B -generic $h_e \subseteq Q_e$ belongs to the P_F -extension. Then*

$$B[h_e] \subseteq B[G_s : s \text{ is selected by } e].$$

Proof. The whole lower generic h_e can be coded by a subset of a ground ordinal below ρ . The cap quotient adds no such subsets, so h_e is in the lower extension. Let $k_s \subseteq \gamma_s$ be its component at an e -selected site.

If there is a site after s , its first cardinal exceeds the Boolean size of the prefix through s . The ground full tail is closed up to that cardinal. Easton's lemma over the smaller prefix therefore makes it add no subsets of γ_s ; at a final site there is no tail. Thus the required tail distributivity follows from (11.2), and

$$k_s \in B[G_s][G_{<s}].$$

The forcing producing $G_{<s}$ has size below γ_s . Consequently the pair

$$(B[G_s], B[G_s][G_{<s}])$$

has the $< \gamma_s$ -approximation property. To check that k_s is approximated over $B[G_s]$, take a $B[G_s]$ -set of ordinal positions of size below γ_s . It belongs to B , since G_s adds no short ordinal sequences. The restriction of the B -Cohen generic k_s to that set also belongs to B . Therefore

$$k_s \in B[G_s]. \quad (11.4)$$

For each site, fix in B a wellordering of a bounded set of names for subsets of γ_s . Choose the least name whose value is the actual k_s . The indices form an ordinal sequence of length $|\tau|$. The lower forcing adds no such sequence, so the whole vector of indices belongs to B . Hence the vector of all k_s belongs to the extension by the selected sites alone.

This vector determines the full product generic h_e . A ground product condition belongs to a generic filter exactly when it is compatible with the union of the generic bits at every coordinate. Otherwise, the filter's meeting the dense set deciding compatibility with that condition gives a contradiction. Hence h_e belongs to the selected-site extension. $\square$

11.3. The quotient and its term order. The spectrum depends on the following quotient calculation. Replacing this order by raw term forcing would destroy the spectrum.

Let $E = B[h_e]$. Use the old vector of names in Lemma 11.2 to name h_e over the selected sites. Restrict to an actual positive ground cone which forces that this name is Q_e -generic over B . This is a forcing statement: the ground dense subsets of Q_e form a set. The membership Boolean values of the named filter generate a complete subalgebra of that selected-site completion, on the cone, whose induced extension is exactly E . Its size is below ρ ; injectivity of the map from the whole Q_e completion is not required. The omitted sites remain independent product directions over this algebra. Quotient positivity depends only on the selected lower projection, since every terminal name has been totalized. Restricting to this cone leaves the actual extension unchanged. Let P be a dense old-condition presentation of the quotient. Let

$$T = \left(\prod_{d \in F} T_d \right)^B$$

be the old set of vectors of row terms. The quotient of the *actual* forcing (11.3) is represented by $P \times T$ with order

$$(s', t') \leq (s, t) \iff s' \leq s \text{ and } \forall d \in F \ s'_d \Vdash_{Q_d}^B t'_d \leq t_d. \quad (11.5)$$

The right side uses the original ground forcing relation. It does not quantify over all new E -names for terminal conditions.

We apply Jakob's criterion [19, Definition 3.5 and Theorem 3.8]: an iteration-like order with a square- ρ -cc base and a strongly $< \rho$ -distributive term order has the $< \rho$ -approximation property. Strong $< \rho$ -distributivity means that every ρ -long sequence of dense open sets has a descending sequence meeting them. A final lower bound is not required. We check these hypotheses for (11.5).

Lemma 11.3. *If $e \notin F$, the quotient order (11.5) has the $< \rho$ -approximation property over E .*

Proof. The quotient base and its square have size below ρ . For a fixed old lower condition s , the section order on T is $< \rho$ -closed in B . In each row, take a name for the union of the descending cap conditions below s_d and use a default value off that cone. The extension E/B is by forcing of size below ρ . By the preservation observation preceding Jakob's criterion, this chain-condition forcing preserves strong $< \rho$ -distributivity of the fixed ground section orders. Their disjoint union is the term order, so it is strongly $< \rho$ -distributive in E .

The order is based, and the forcing relation gives its refinement property. For the projection property, suppose $(s', t') \leq (s, t)$. In row d , mix the name t'_d below s'_d with t_d off that Boolean cone. The resulting old vector t'' is a section extension below s , and below s' it is equivalent to t' . All inputs to the mixing belong to the ground, and the mixing takes place in B .

To verify mixing, take two section extensions t^0, t^1 below t at a fixed s . For every $d \in F$, choose the least ξ_d with $d(\xi_d) \neq e(\xi_d)$. The site

$$s_d = (\xi_d, d(\xi_d))$$

is omitted by e and active in row d . Keep the selected sites unchanged, and refine s in two ways by assigning opposite values to a fresh bit in this d -copy, simultaneously for every $d \in F$. This gives legal old conditions because $|F|$ is below the first coding cardinal and the product has full support. Both refinements remain positive in the quotient because only omitted sites changed. Their projections are incompatible in every actual row.

In row d , Boolean-mix t_d^0 and t_d^1 below the incompatible row projections, retaining t_d elsewhere. The resulting old vector t^* is a section extension of t at s . Below the respective quotient refinements it specializes to t^0 and t^1 , as required for mixing. Jakob's theorem gives the approximation property. $\square$

Proof of Theorem 11.1. If $e \in F$, its planted row and cap witness the right side of (11.1). Conversely, suppose that side holds and $e \notin F$. Lemma 11.2 and Lemma 11.3 give the $< \rho$ -approximation property for $(B[h_e], B[G])$. A recomputed $\text{Add}(\rho, 1)^{B[h_e]}$ -generic would be a new subset of ρ whose every $B[h_e]$ -small restriction belongs to $B[h_e]$, contradicting approximation. Preservation and homogeneity were proved after (11.3). $\square$

Corollary 11.4 (Coding an orbit of presentations). *Let $x \in B$. There is an arbitrarily highly distributive, weakly homogeneous forcing, uniformly definable in B from x and ordinal parameters, which makes x OD relative to the class B . If B is ordinal-definable in the final extension, then x is OD there.*

Proof. Choose an infinite cardinal τ large enough to code $(\text{TC}(\{x\}), \in, x)$. Let $F_x \subseteq 2^\tau$ be the family of all pointed well-founded extensional presentation codes of this structure. The codes include domain and root data and allow unused positions. This nonempty family is defined from x, τ without choosing a presentation.

Apply Theorem 11.1. Relative to the class B , the first-order capped-history predicate recovers exactly F_x . Each member has pointed Mostowski collapse x , so x is OD relative to B . The schedules use the universal family 2^τ for their bounds. Thus the decoder can define the candidate forcings without knowing F_x . $\square$

Lemma 11.5 (GCH preservation). *If $B \models \text{GCH}$ in Theorem 11.1, then the actual evaluated forcing P_F preserves all cardinals and GCH. Consequently the coding forcing in Corollary 11.4 can be chosen to have these properties over a GCH base.*

Proof. Write $S = S_F$ for the lower product and let ν be the cardinality of the physical-site set. The schedule gives $\nu, |F| < \gamma_0$, where γ_0 is its first inaccessible. At an active site γ , the aggregate forcing has the dense partial-function presentation $\text{Add}(\gamma, m_\gamma)$ for some $m_\gamma < \gamma_0$. It has size γ and is $< \gamma$ -closed in B . Thus the full product S is $< \gamma_0$ -closed and adds no ordinal sequences of length less than γ_0 .

All arithmetic and cofinalities in the lower-product argument are computed in B . For an infinite cardinal λ , let $S_{\leq \lambda}$ contain the sites at most λ . If $\lambda \geq \gamma_0$ and $\text{cf}(\lambda) \geq \gamma_0$, then GCH gives

$$|S_{\leq \lambda}| \leq \lambda^\nu = \lambda.$$

This includes every regular $\lambda \geq \gamma_0$. If a tail remains, its first site η is greater than λ , and the old tail is $< \eta$ -closed. Easton's lemma makes it $< \eta$ -distributive over the head extension. Hence

every ordinal sequence of length λ in the full extension already belongs to the head extension. The same conclusion is immediate when the head is all of S .

We first prove cardinal preservation. Successors below γ_0 are preserved by closure. For $\lambda \geq \gamma_0$ with $\text{cf}(\lambda) \geq \gamma_0$, the head has size at most λ and preserves λ^+ ; the tail adds no λ -sequences and cannot collapse it. Now suppose that λ is singular and $\epsilon = \text{cf}(\lambda) < \gamma_0$. Fix a cofinal sequence $\langle \lambda_j : j < \epsilon \rangle$ of infinite regular cardinals between γ_0 and λ . If $f : \lambda \rightarrow \lambda^+$ belongs to the full extension, then $f \upharpoonright \lambda_j$ belongs to the $S_{\leq \lambda_j}$ -extension. A name for this restriction has at most λ_j possible values at each argument, so its range is bounded below the ground λ^+ . In the full extension, let $\beta_j < \lambda^+$ be the least strict bound on that range. The whole sequence $\langle \beta_j : j < \epsilon \rangle$ belongs to B by $< \gamma_0$ -closure. Ground regularity gives $\sup_{j < \epsilon} \beta_j < \lambda^+$, so f is bounded and cannot collapse λ^+ . Every ground successor cardinal is therefore preserved. A ground limit cardinal cannot collapse while all its smaller successor cardinals remain cardinals, so S preserves all cardinals.

When $|S_{\leq \lambda}| \leq \lambda$, Boolean-name counting gives at most $(2^\lambda)^\lambda = \lambda^+$ names for subsets of λ , and tail distributivity returns every such subset to the head extension. This proves GCH for the cases already covered by that size bound. Below γ_0 no subsets are added. For singular λ with $\epsilon < \gamma_0$, use the same regular sequence $\langle \lambda_j : j < \epsilon \rangle$. In B , fix for each j a list of at most λ_j^+ head names covering all subsets of λ_j in its head extension. For each actual $x \subseteq \lambda$, choose the least index representing $x \cap \lambda_j$ under the actual head generic. The resulting index vector has length $\epsilon < \gamma_0$, so the whole vector belongs to B . Its entries are below λ , and GCH and König's theorem give $\lambda^\epsilon = \lambda^+$ in B . A vector determines at most one x , by union of its evaluated restrictions. Cardinal preservation now gives $2^\lambda = \lambda^+$ in the full lower extension.

Finally, $|S| < \rho$ by the choice of the inaccessible cap cardinal, so ρ remains inaccessible in the lower extension. For each row d , the Boolean algebra $\text{RO}(Q_d)^B$ has size below ρ . The possible lengths of a total cap-condition name are therefore bounded below ρ . For each fixed length bound there are fewer than ρ bounded Boolean graph codes, by strong limitness. Taking all bounds gives $|T_d|^B \leq \rho$, and hence

$$\left| \left(\prod_{d \in F} T_d \right)^B \right| \leq \rho^{|F|} = \rho.$$

Together with $|S| < \rho$, this gives a dense old-condition presentation of the actual cap quotient of size at most ρ . The termspace projection established after (11.3) makes this quotient $< \rho$ -distributive over the lower extension. It preserves all smaller power sets and all cardinals at most ρ ; its ρ^+ -chain condition preserves higher cardinals. For $\lambda \geq \rho$, Boolean-name counting in the lower extension gives at most

$$(2^\rho)^\lambda = 2^\lambda = \lambda^+$$

names for subsets of λ . Thus the actual evaluated quotient also preserves GCH, completing the proof. $\square$

Remark 11.6. Theorem 11.1 uses no continuum-function comparison, large cap width, or singularization. Sparse site localization supplies the omitted quotient directions. A universal Cohen source is not retained. The ground termspaces prove preservation, and the quotient argument applies to the evaluated forcing (11.5).

12. A FRESH ENVELOPE AND A SECURE GROUND

Lemma 12.1 (A fresh choice envelope). *Suppose $M \models \text{ZF} + \text{DC} + \text{SVC}$. There is a weakly homogeneous set-forcing extension $B = M[K] \models \text{ZFC}$ with*

$$(\text{Ord}^\omega)^B = (\text{Ord}^\omega)^M.$$

Proof. Choose an SVC witness S , and force with countable partial functions from ω_1^M to S . Countable closure and ground DC imply that no countable sequences of ground elements are added. Permutations of ω_1 moving two countable domains apart witness weak homogeneity. The generic union maps ω_1 onto S , so least preimages wellorder S . The SVC surjections then wellorder every old set. Each new set has an old name whose collection of possible named elements is now wellorderable. Evaluating that collection gives a wellordering of the new set. Thus the extension satisfies Choice. $\square$

Definition 12.2 (Secure grounds). A choice model D is *secure* if, in every set-forcing extension U of D , we have $D \subseteq \text{HOD}^U$ and D is the mantle of U . The mantle is the intersection of the set-forcing grounds and is uniformly definable.

We obtain a secure ground by coding with $\diamond^*$. For an uncountable regular λ , this principle asserts the existence of a sequence $\langle \mathcal{D}_\xi : \xi < \lambda \rangle$ such that $\mathcal{D}_\xi \subseteq \mathcal{P}(\xi)$ has size at most $|\xi|$ and, for each $x \subseteq \lambda$, there is a club $C \subseteq \lambda$ on which $x \cap \xi \in \mathcal{D}_\xi$. Write *eventual GCH* for GCH at every cardinal above some bound.

Lemma 12.3 (Secure preparation with eventual GCH). *Suppose $D_0 \models \text{ZFC} + \text{CM}$. Given a regular cardinal χ and an ordinal α with $|V_\alpha^{D_0}| < \chi$, there is a progressively closed class-forcing extension D such that*

$$D \models \text{ZFC} + \text{CM} + \text{eventual GCH}, \quad V_\alpha^D = V_\alpha^{D_0},$$

and D is secure. Every set-sized initial segment of the preparation can be taken $< \chi$ -closed in D_0 .

Proof. Use the $\diamond^*$ option of [21, Appendix, Lemma 20], starting above χ . The cited iteration combines GCH forcing with coding on blocks of triple successors of both fixed points. It preserves measurables; using $\diamond^*$ for the codes permits GCH. We record the effect of omitting its bounded initial part.

Make every stage through χ trivial and retain the stated iteration above it, including its Cohen stages at inaccessible cardinals. All remaining factors are $< \chi$ -closed. A union of fewer than χ Easton supports still satisfies the support bounds, since all active locations exceed χ . Thus every initial segment is $< \chi$ -closed. The progressively closed set-name construction gives ZFC, with every set in an initial extension. It adds no ordinal sequences of length less than χ and hence preserves the indicated rank.

The GCH calculation applies above the omitted segment. For measurables $\kappa > \chi$, the cutoff is fixed by the normal ultrapower embedding. The iteration and its image agree below κ and have the same stage- κ Cohen preparation and closed upper tail used in the cited lift. Removing the bounded initial stages introduces no further lifting requirement. Consequently a class of measurables survives. These conclusions give eventual GCH and CM; they impose no GCH requirement on the protected rank.

The coding density argument applies above each prescribed bound. Every ordinal set appearing in an initial extension can be written on a fresh block by choosing the appropriate coding factors. Subsequent tails preserve that block's truth pattern. It follows that every ordinal set of D has arbitrarily high $\diamond^*$ codes. We next verify that these codes make D secure.

First, if $W[g]/W$ is set forcing and λ is regular above its size, then $\diamond_\lambda^*$ has the same truth value in the two models, by [21, Lemma 19]. The locations of sufficiently high codes also agree. Indeed, if the forcing has size ν , cardinal preservation and the bound of $(2^\kappa)^W$ on names for subsets of κ show that the cardinal-valued continuum functions agree for cardinals $\kappa > \nu$. Let b_ξ and b'_ξ be the two beth sequences. Above a fixed initial bound, monotonicity gives fixed ordinal shifts θ, θ' with

$$b_\xi \leq b'_{\theta+\xi}, \quad b'_\xi \leq b_{\theta'+\xi} \quad (\xi \in \text{Ord}).$$

For example, choose the shifted initial value beyond the disturbed continuum segment and induct using their common continuum function there. If δ is a beth fixed point in either model above these shifts, then δ is an infinite cardinal and $\theta + \xi, \theta' + \xi < \delta$ for every $\xi < \delta$. The inequalities put all values of the other beth sequence below δ . Continuity, together with $\beth_\delta \geq \delta$, makes it a fixed point there too. Thus the high beth fixed points and their triple successors agree. Describe a coding block by its actual first beth fixed point and its ordinal length. This avoids any change in the enumeration below the agreement bound.

Now let U be a set-forcing extension of D . For any ordinal set $x \in D$, choose one of its codes above the forcing size and the bound for agreement of locations. The unchanged truth pattern defines x in U from the starting point and length. Ordinal codes for transitive closures then give $D \subseteq \text{HOD}^U$.

The same reasoning gives the Ground Axiom in D . If $D = W[g]$ were a nontrivial set extension, a sufficiently high code for each ordinal set of D could be read in W . Separation in W would reconstruct every such set. Choice then gives $D = W$, a contradiction. Finally, let W be any ground of U . Downward directedness of grounds gives a common ground below D and W ; see [23, 8]. The intermediate-model theorem makes it a ground of D . The Ground Axiom makes this common ground equal to D . Since D itself is a ground of U , its mantle is exactly D . $\square$

The Continuum Coding Axiom also gives secure grounds by the same high-code and Ground Axiom argument [22]. Its continuum codes entail unbounded failures of GCH. The $\diamond^*$ preparation is what allows the later choice envelope to satisfy eventual GCH.

Lemma 12.4 (A fresh GCH envelope). *Suppose $M \models \text{ZF}$ and $B = M[K] \models \text{ZFC} + \text{CM}$ is a weakly homogeneous set-forcing extension with no new ordinal ω -sequences. If B satisfies eventual GCH, there is a further set-forcing extension $B' = B[H]$ such that*

$$B' \models \text{ZFC} + \text{GCH} + \text{CM}, \quad (\text{Ord}^\omega)^{B'} = (\text{Ord}^\omega)^M.$$

Moreover B'/M is a weakly homogeneous set-forcing extension.

Proof. In B , choose an inaccessible δ above all GCH exceptions. Use the following presentation of $L_\delta = \text{Col}(\omega_1, < \delta)$. A condition is a countable partial function p on $\delta \times \omega_1$ with $p(\beta, \xi) < \beta$, ordered by reverse inclusion. All ordinals below δ are coordinates; the definition does not use a model's list of cardinals.

In B this forcing is countably closed, has size δ , and is δ -cc. For the last assertion, the generalized Δ -system lemma gives a common root for δ many countable domains. Inaccessibility bounds the number of value assignments on that root below δ . Two conditions therefore agree on the root and are compatible. Each nonzero coordinate adds a surjection from ω_1 onto that ordinal. Hence ω_1 is preserved, δ becomes ω_2 , and all cardinals at least δ are preserved.

There are no new reals, and the old continuum has size less than δ , so CH holds. A nice name for a subset of ω_1 uses ω_1 antichains, each of size less than δ . The entire name has size less than δ , so $\delta^{<\delta} = \delta$ bounds their number by δ . Thus $2^{\omega_1} = \delta = \omega_2$. For each old cardinal $\lambda \geq \delta$, a name for a subset of λ is coded by a subset of $\lambda \times L_\delta$. There are at most $(2^\lambda)^B$ such codes, and that cardinal is preserved. The old subsets give the opposite bound, so the continuum cardinal at λ is unchanged. This proves global GCH. Small forcing preserves all B -measurables above δ .

The models M and B have the same countable ordinal sequences, so they agree on ω_1 and on this literal presentation of L_δ . An enumeration of a condition is coded by one ordinal ω -sequence and therefore belongs to M . The same observation applies to the countable graphs of the permutations used below. No inaccessibility assumption on δ in M is needed.

Permute the second coordinate ω_1 to move the countable domain of one condition away from that of another. A permutation with countable support suffices, and belongs to M . It witnesses weak homogeneity of L_δ there. If $\mathbb{C} \in M$ is the old forcing for K , the product lemma identifies

$B' = M[K, H]$ as an extension by the fixed product $\mathbb{C} \times L_\delta$. This product is weakly homogeneous. Countable closure is used in the choice model B , so it adds no ordinal ω -sequences. This proves all assertions. $\square$

Lemma 12.5 (Recognizing an envelope). *Suppose D is secure and $B = D(A, K)$ is a set-forcing extension of D satisfying ZFC. Choose η with $A, K \in V_\eta^B$. If U is a further set-forcing extension preserving V_η^B , then the actual class B is definable in U from the ordinal η by*

$$B = D(V_\eta^U).$$

Proof. The right-hand model is constructed inside B when $V_\eta^U = V_\eta^B$, and contains D, A, K . Thus it equals B . Security identifies D inside U as its mantle. $\square$

Remark 12.6. Definability of the class B does not imply $B \subseteq \text{HOD}^U$. Similarly, V_η^U is OD but need not belong to HOD. The decoder can therefore recognize the choice envelope without placing its new wellorderings in the target inner model.

13. EXACT EXTERNAL REALIZATION

Proposition 13.1 (Realization over a secure ground). *Suppose*

$$D \subseteq M = D(A) \subseteq B = M[K],$$

where $D \models \text{ZFC} + \text{CM}$ is secure, $A \subseteq \mu^\omega$, B is a set-forcing extension of D satisfying ZFC, and B/M is weakly homogeneous with no new ordinal ω -sequences. Then there is a further set-forcing extension U of B such that

$$U \models \text{ZFC} + \text{CM}, \quad \mathcal{H}^U = M, \quad M \models \text{SVC} + \text{CM}.$$

If B satisfies eventual GCH, one may also require $U \models \text{GCH}$.

Proof. If B satisfies eventual GCH, first apply Lemma 12.4 to obtain $B' = B[H]$ with GCH. Otherwise put $H = \emptyset$ and $B' = B$. In either case B'/M is weakly homogeneous and adds no ordinal ω -sequences. Since B is a set extension of D , so is B' , and we have $B' = D(A, K, H)$.

Choose η in this actual envelope B' so that $A, \langle K, H \rangle$ belong to $V_\eta^{B'}$. Apply Corollary 11.4 over B' , starting above $|V_\eta^{B'}|$, to code A , and call the extension U . Lemma 12.5, with the pair $\langle K, H \rangle$ as its second generator, gives

$$B' = D(V_\eta^U).$$

Thus the capped-history spectrum defines A in U . In the GCH case, Lemma 11.5 gives $U \models \text{GCH}$.

The combined forcing from M to U is weakly homogeneous. The name for the coding forcing is uniformly defined from the old A and ordinal parameters. Automorphisms of the first forcing carry it to a forced equal name and therefore lift to the iteration. The second forcing is forced weakly homogeneous. For a local name for a second-stage automorphism, use the identity off a condition forcing it to be an automorphism. This requires only localized Boolean mixing, without a choiceless maximum principle.

By hypothesis, the first extension adds no ordinal ω -sequences; the coding extension is highly distributive over the choice model B' . Lemma 10.2 gives $\mathcal{H}^U \subseteq M$. Conversely, $D \subseteq \text{HOD}^U \subseteq \mathcal{H}^U$ and $A \in \mathcal{H}^U$. The latter holds because it is OD in U and every member of A is an allowed ordinal-sequence parameter. Minimality of $D(A)$ gives $M \subseteq \mathcal{H}^U$.

The entire extension U/D is by set forcing. All sufficiently high normal D -measures lift to U , and their restrictions (2.1) belong to M . Thus both models have a proper class of measurables.

Finally, $B = M[K] \models \text{ZFC}$ is a set-forcing extension of M . The truth lemma gives a condition forcing Choice. Restricting the forcing to that condition shows that Choice is forceable over M . Blass's equivalence of this property with SVC, as stated in [8, Theorem 3.14], gives $M \models \text{SVC}$. $\square$

The next preparation allows us to apply this proposition without assuming that class forcing preserves DC in arbitrary choiceless intermediate models.

Proposition 13.2 (Installing the secure ground). *Suppose $M = D(A) \models \text{ZF} + \text{DC} + \text{CM} + \text{SVC}$, where $D \models \text{ZFC} + \text{CM}$ is a symmetric ground and $A \subseteq \mu^\omega$. Given an ordinal α , there are*

$$D^* \subseteq M^* = D^*(A) \subseteq B^* = M^*[K]$$

satisfying the hypotheses of Proposition 13.1, with B^ satisfying eventual GCH and*

$$V_\alpha^{M^*} = V_\alpha^M.$$

Proof. Take a fresh envelope $B = M[K]$ from Lemma 12.1. The symmetric-ground relation composes with set-forcing extensions. Since D and B satisfy Choice, D is a set-forcing ground of B by [8, Lemma 5.8]. Write $B = D[b]$ using a forcing presentation in D .

Let $\mathbb{P} \in D$ be the presentation with generic b . Choose in D a regular $\delta > \max(|\mathbb{P}|, \omega)$, above a uniform $\mathbb{P}$ -name bound for the cardinality of the protected B -rank. Such a bound is available before evaluating b , using a set of names for that rank. Choose the preparation $\mathbb{Q}$ of Lemma 12.3 in D , independently of b , with every set-sized initial segment $\mathbb{Q}_\beta < \delta$ -closed in D . Take a coherent class generic J over B for this fixed D -iteration in the set-name framework just specified. For every β , the ordinary product lemma gives

$$D[b][J_\beta] = D[J_\beta][b].$$

Since $\mathbb{P}$ is δ -cc, Easton's lemma makes the fixed $\mathbb{Q}_\beta < \delta$ -distributive in B . This does not assert that its ground closure persists in B . This commutation of small forcing with class preparation is used in [22, Appendix, Lemma 121].

Put $D^* = D[J]$. Every dense subset of the fixed $\mathbb{P}$ in D^* appears in some $D[J_\beta]$, where b is generic by the product lemma. Thus b is generic over D^* , and

$$B^* = D^*[b] = \bigcup_{\beta} B[J_\beta] \models \text{ZFC}, \quad M^* = D^*(A).$$

Every short ordinal sequence in B^* appears at a set stage. Distributivity at each such stage therefore shows that B^*/B adds none of length less than δ . It also preserves the protected rank: a first new set in that rank would be a new subset of an old set of cardinality less than δ , coded by its old wellorder as a short ordinal sequence. Thus, by freshness of B/M and the inclusions $M \subseteq M^* \subseteq B^*$,

$$(\text{Ord}^\omega)^{M^*} = (\text{Ord}^\omega)^{B^*} = (\text{Ord}^\omega)^M. \quad (13.1)$$

We claim

$$M^* \cap B = M. \quad (13.2)$$

Every set of $D^*(A)$ belongs to $M[J_\beta]$ for some set-sized initial segment of the class iteration. Indeed, take a bounded transitive-base construction witnessing membership, then a stage at which the required rank of D^* has stabilized. The old set forcings for K and J_β are mutually generic over M , so

$$M[J_\beta] \cap M[K] = M.$$

The usual product-intersection proof works over ZF : compare two names for a first new set of a given rank and use product conditions deciding membership. This proves (13.2). Together with preservation of the protected B -rank, it gives $V_\alpha^{M^*} = V_\alpha^M$.

The same filter K is generic over M^* for the *fixed* old collapse forcing. Each dense set in M^* appears in some $M[J_\beta]$, where mutual genericity makes K meet it. Also $B^* = M^*[K]$. One inclusion holds inside B^* . For the reverse inclusion, $D^* \subseteq M^*$ and $b \in B = M[K] \subseteq M^*[K]$ give $D^*[b] \subseteq M^*[K]$. The old collapse automorphisms still witness weak homogeneity, and (13.1) shows that there are no new allowed parameters.

By Lemma 12.3, D^* is secure and satisfies $\text{ZFC} + \text{CM}$ and eventual GCH. Its set-forcing extension B^* also satisfies eventual GCH: above the forcing size, the cardinals and continuum cardinal values are unchanged. Thus the GCH clause of Proposition 13.1 applies. The set-forcing identity $B^* = M^*[K]$ also yields $M^* \models \text{SVC}$ by Blass's equivalence. We have not assumed that the class preparation preserves DC in M^* . Once M^* is identified with $\mathcal{H}^U$, Lemma 10.1 gives DC. $\square$

Proof of Theorem 1.2. From SVC, take a choice symmetric ground D and a transitive generator T with $M = D(T)$; see [8, Proposition 5.4]. The choice ground may be taken to have a proper class of measurables: make a set-generic choice envelope, use Lemma 2.1, and apply downward Lévy–Solovay to the resulting choice-ground presentation. Alternatively, in the application from Theorem 1.3, the ground D already has this property.

By $\text{KW}_1^{*,\omega}$, choose a surjection $f : (\mu^\omega)^M \rightarrow T$. Code its domain and the relation $f(p) \in f(q)$ by a single set A of ordinal ω -sequences. Equality of predecessor sets and the well-founded extensional collapse, as in Proposition 7.5, recover T and f . Hence $M = D(A)$. Apply Proposition 13.2 and then Proposition 13.1. $\square$

14. THE THREE CONSISTENCY IMPLICATIONS

Proof of Theorem 1.1. The implication $\text{Con}(E_2(\sigma)) \Rightarrow \text{Con}(E_1(\sigma))$ follows directly from the definitions. If $U \models E_2(\sigma)$, Lemma 10.1 gives $\mathcal{H}^U$ as a model of $E_3(\sigma)$. Hence $\text{Con}(E_2(\sigma)) \Rightarrow \text{Con}(E_3(\sigma))$.

Proposition 10.3 gives $\text{Con}(E_1(\sigma)) \Rightarrow \text{Con}(E_3(\sigma))$ without assuming that the original external HOD already contains a proper class of measurables.

For the remaining direction, let $V \models E_3(\sigma)$ and choose κ with $V_\kappa^V \models \sigma$. Apply Theorem 1.3 above κ to obtain a sequence-generated model M with $V_\kappa^M = V_\kappa^V$. Apply Theorem 1.2 while protecting that rank. The resulting U satisfies $\text{ZFC} + \text{GCH} + \text{CM}$, and

$$\mathcal{H}^U \models \text{SVC} + \text{CM}, \quad V_\kappa^{\mathcal{H}^U} = V_\kappa^M = V_\kappa^V.$$

Hence $U \models E_2(\sigma)$, completing the cycle of implications. $\square$

Why this is an external theorem. The original Prikrý ambient universe contains the master sequence c . Every external $\mathcal{H}$ contains all ambient ordinal ω -sequences, including c . We therefore cannot simply take the final external HOD in that universe. The master can expose real-coded information omitted from the intermediate model. Lemma 12.1 provides a *fresh* choice envelope with no new ordinal ω -sequences.

The high witnesses in Section 11 need not belong to the final $\mathcal{H}$. Their ambient existence makes A definable, and weak homogeneity controls hereditary descent. The proof requires neither tree-Prikrý completions of the Cohen caps nor a unique high witness for each code.

Proof dependencies and scope. The construction uses two nonstandard forcing results, Lemma 6.2 and Theorem 11.1. The first uses tagged finite interpolation and a countable diagonal argument. The second applies an existing approximation criterion to the displayed quotient order. Both are needed, for different parts of the proof. Lemma 8.8 supplies the separate fine-structural reduction. The following qualifications describe their scope.

- (1) The global rank-preservation proof uses literal equality of Q, F and of supported parameters, not merely isomorphism of their indexing structures.
- (2) The DC proof uses external countable domination and wellorderable search spaces; it does not assert that every ambient countable sequence belongs to the inner model.
- (3) The cap spectrum is computed over the actual recognizable envelope. It requires no classification of arbitrary incomparable grounds.

- (4) The secure preparation uses repeated $\diamond^*$ codes and retains a proper class of measurables. A subsequent countably closed set collapse gives global GCH in the choice envelope; the final evaluated cap forcing preserves it. The protected rank belongs to the target inner model and need not satisfy GCH.
- (5) The conclusion concerns first-order sentences true in a rank segment. It neither identifies the complete theories of the three ambient universes nor proves the internal assertion $V = \text{HOD}_{\text{Ord}^\omega}$ in the original model.

APPENDIX A. TWO BOOKKEEPING DETAILS

A.1. Names, parameters, and nonstandard models. All definability arguments use bounded-stage codes. To say that a set belongs to $D(A)$, specify a set-sized stage over a rank of D and the transitive closure of A . A relative-HOD definition can likewise be evaluated at a sufficiently large rank level, with that ordinal included in the code. No satisfaction predicate for the entire universe is required.

The forcing comparisons concern fixed names and formulas. Auxiliary collapses produce common generic realizations only to refute incompatible decisions. The Prikry-only names in Lemma 7.1 are constructed in the original model N . These facts allow the relative-consistency argument to be formalized over arbitrary countable models, without assuming transitivity.

For class preparation, use the conservative class-theoretic expansion for progressively closed coding. Each set needed in the proof appears in a set-sized initial segment. The corresponding forcing and reflection argument handles any specified finite list of set-theoretic axioms and formulas. The preparation is the high-start $\diamond^*$ version of [21, Appendix, Lemma 20]. Its closure and lifting checks are given in Lemma 12.3. The later GCH collapse and capped-history coding are both set forcing.

A.2. Uniformity of the high coding. The decoder uses three ordinal parameters: the presentation length τ , an initial protection bound, and the envelope rank η . The site recursion takes its size bounds from all of 2^τ , so the candidate forcing Q_e and cap cardinal ρ are defined before the spectrum is recovered. The final predicate asserts that e admits the two generic filters over the class $B = D(V_\eta)$. This is first-order because the posets and their dense subsets belong to the specified definable inner models.

Homogeneity requires the family of all presentation codes. Its common pointed collapse makes A OD without making any particular wellordering of $\text{TC}(\{A\})$ OD. The final HOD can therefore remain choiceless.

APPENDIX B. THE OBSTRUCTION WITHOUT MEASURABLES

The measurable hypothesis cannot simply be omitted, even from the comparison between the weakest external theory and the theory with dependent choice. Put

$$A_L \equiv \mathcal{P}(\omega) = \mathcal{P}^L(\omega) \wedge \omega_2 = \omega_2^L.$$

Lemma B.1. *In any model of $\text{ZF} + A_L$, every countable ordinal sequence belongs to L . Consequently*

$$\text{ZF} + A_L + \text{KW}_1^{*,\omega} \vdash \text{AC}.$$

If $U \models \text{ZFC}$ and $\mathcal{H}^U \models A_L$, then $\mathcal{H}^U = \text{HOD}^U$ and this inner model satisfies Choice.

Proof. Fix an ordinal sequence s in the given universe W , and apply Jensen covering in the choice inner model $L[s]$; see [9, Theorem 1.1]. This use of covering does not assume Choice in W . The models L and $L[s]$ have the same reals, hence the same ω_1 , and ω_2^L remains a cardinal

in $L[s]$ because it remains a cardinal in W . Also $0^\#$ does not exist in $L[s]$, since it would be a nonconstructible real.

Covering gives a constructible set y containing $\text{ran}(s) \cup \omega_1$ whose cardinality in $L[s]$ is ω_1 . Its cardinality in L is at most ω_1 . Otherwise an L -injection of ω_2^L into y would collapse that cardinal in $L[s]$. Fix an L -injection $e : y \rightarrow \omega_1$. In $L[s]$, the sequence $e \circ s$ is bounded in ω_1 . An L -enumeration of a countable bound codes this sequence by a real. That real is constructible, so $s \in L$.

Thus each whole sequence space μ^ω is its constructible counterpart and is wellorderable. Under $\text{KW}_1^{*,\omega}$, least preimages wellorder every nonempty set. For the last assertion, $\mathcal{H}^U$ contains every ordinal ω -sequence of U . Applying the first assertion inside $\mathcal{H}^U$ makes every allowed sequence parameter constructible. Every constructible set is ordinal-definable in U , using its index in a canonical constructible wellordering. Replacing the sequence parameters by these ordinal definitions gives $\mathcal{H}^U = \text{HOD}^U$. $\square$

Proposition B.2. *There is a first-order sentence σ_0 such that*

$$\text{ZFC} \vdash \mathcal{H} \models \neg \exists \theta (V_\theta \models \sigma_0),$$

whereas, relative to $\text{Con}(\text{ZFC})$,

$$\text{Con}(\text{ZF} + \text{DC} + \exists \theta (V_\theta \models \sigma_0)).$$

The countermodel can also satisfy $A_L + \neg \text{DC}_{\omega_1}$.

Proof. Over L , force with the countable partial functions

$$P = \text{Fn}_{<\omega_1}(\omega_1 \times \omega_1, 2),$$

ordered by reverse inclusion. Permute the first coordinate, and use the normal filter generated by pointwise stabilizers of countable sets of coordinates. Let N be the hereditarily symmetric model. The forcing is countably closed and, by CH in L , has size ω_1 . It therefore adds no reals and preserves all cardinals. The same reals, ω_1 , and ω_2 occur in L , N , and the full extension. The countably complete symmetry filter and countable closure give $N \models \text{DC}$ by [28, Lemma 3.1].

Let g_i be the generic subset of ω_1 at coordinate i . The set $X = \{g_i : i < \omega_1\}$ belongs to N . Every ordinal-indexed map into X in N has range contained in its countable family of supported generics. Indeed, if a supported name for f satisfied $f(\xi) = g_i$ outside its support, transpose i with a coordinate absent from both the support and a condition deciding this equality. The condition and its image are compatible and force two distinct values at ξ . This is impossible. The generics on each countable support have a countable enumeration in N . Hence X is not wellorderable, and neither is $\mathcal{P}(\omega_1)$. The tree of injections from countable ordinals into X is closed under increasing countable sequences and has no maximal node. It has no ω_1 -branch, since such a branch would give an injection of ω_1 into X . Thus N also fails transfinite DC_{ω_1} , in the tree formulation of [28, Theorem 2.1].

Choose σ_0 to assert that there is no greatest ordinal; that $\lambda = \omega_1$ and $\mu = \omega_2$ exist, with a limit ordinal $\nu > \mu$; that every real is constructible; that $\lambda = \omega_1^L$ and $\mu = \omega_2^L$; and that $\mathcal{P}(\lambda)$ is not wellorderable. These are combined into one first-order sentence, with λ, μ, ν existentially quantified.

Suppose $V_\theta^W \models \sigma_0$, where $W \models \text{ZF}$. The first clause makes θ a limit, and the guard gives $\theta > \mu + \omega$. All functions between ordinals at most μ , and all subsets and binary relations on $\mathcal{P}(\lambda)$, belong to this rank whenever they belong to W . Thus its ω_1 and ω_2 are those of W , and its nonwellorderability assertion remains true in W . The local constructible hierarchy agrees with the actual one at every level below θ . To verify this, code finite assignments by finite function graphs: the satisfaction relation for a set structure and a hierarchy sequence through a fixed

level require only a fixed finite increase in rank. All such witnesses remain below the limit θ . In particular, every real of W is constructible.

It remains to check the second constructible cardinal. Equality of reals gives $\lambda = \omega_1^L$ globally, and inner-model cardinal comparison gives $\gamma = \omega_2^L \leq \mu = \omega_2^W$. If $\gamma < \mu$, then γ is also a cardinal of the rank's local L : a local constructible bijection collapsing it would be a global constructible bijection. This puts an L -cardinal strictly between the asserted first and second uncountable L -cardinals, a contradiction. Therefore $W \models A_L$.

If $W = \mathcal{H}^U$ for a choice universe U , Lemma B.1 gives Choice in W , contradicting the retained nonwellorderability assertion. On the other hand, the symmetric model N satisfies σ_0 globally, so reflection gives a rank satisfying it. This is a formal relative-consistency construction; no transitive-model assumption is required. $\square$

The same sentence gives a rank-conservativity obstruction: $\text{ZF} + \text{KW}_1^{*,\omega}$ proves that no rank satisfies σ_0 , while $\text{ZF} + \text{DC}$ does not, assuming consistency. This proves that CM cannot be deleted from the stated theorem. It does not identify the weakest sufficient large-cardinal hypothesis.

APPENDIX C. A RAMSEY OBSTRUCTION

Let CR assert that the cardinals ρ satisfying $\rho \rightarrow (\rho)_2^{<\omega}$ are unbounded in the ordinals, and put $T_R = \text{ZF} + \text{DC} + \text{CR}$. In this section $\mathbb{R} = \mathcal{P}(\omega)$. The three sorts of arithmetic are ω , $\mathbb{R}$, and $\mathcal{P}(\mathbb{R})$. A Π_3^2 sentence has three alternating blocks of third-sort quantifiers, beginning universally, followed by a matrix with only number and real quantifiers. Finite tuples of third-sort variables may be combined by fixed recursive tags.

Theorem C.1 (Ramsey obstruction). *There is a parameter-free Π_3^2 sentence Φ such that*

$$T_R + \text{KW}_1^{*,\omega} \vdash \Phi, \quad \text{Con}(\text{ZFC} + \text{CR}) \implies \text{Con}(T_R + \neg\Phi).$$

Consequently, relative to $\text{Con}(\text{ZFC} + \text{CR})$, adding $\text{KW}_1^{,\omega}$ is not conservative over T_R for third-order arithmetic. The same example gives a structural Π_2 assertion separating these theories.*

We use the classical Dodd–Jensen core model. In a choiceless universe W , set $K = (K^{\text{DJ}})^{\text{HOD}^W}$. The semantic form of the separating implication is

$$[\neg\exists 0^\dagger \wedge \mathbb{R} = \mathbb{R}^K \wedge \omega_2 = \omega_2^K] \longrightarrow \text{Fodor}(\omega_1). \quad (\text{C.1})$$

We first prove this implication with $\text{KW}_1^{*,\omega}$, then give its third-order formulation and a counter-model.

Proof. The covering implication. Under CR, absence of $0^\dagger$ excludes an inner model with a measurable. Indeed, from such an inner model obtain its canonical $L[U]$, with measurable κ . Its wellordering gives an ordinal code $A \subseteq \alpha$ for the transitive closure of U , where $\alpha > \kappa$, $A \in L[U]$, and $L[A] = L[U]$. A Ramsey above $\alpha + \omega$ supplies indiscernibles for $L[A]$. Shifting above α gives a nontrivial self-embedding of $L[U]$ with critical point above κ , which implies $0^\dagger$. Here the relativized indiscernibles argument is applied to canonical structures with definable Skolem functions; it is available in ZF [30, Propositions 7–8] [29, p. 262]. The last implication is the standard nonrigidity characterization of $0^\dagger$ [31]. The possibly uncountable parameter sharp is not itself being identified with a real.

Assume the antecedent of (C.1), and fix $s \in (\text{Ord}^\omega)^W$. Code its graph by a set of ordinals $\bar{s}$. Apter–Koeppke’s adjoining construction gives

$$C = (\text{HOD}^W)[\bar{s}] \models \text{ZFC}, \quad K \subseteq C \subseteq W, \quad K^{\text{DJ},C} = K$$

with $s \in C$ [30, Proposition 9]. The endpoint agreements give common ω_1 and ω_2 in these three models. Apply strong Dodd–Jensen covering inside C to $\text{ran}(s) \cup \omega_1$ [9, Theorem 1.2]. It has

a cover $y \in K$ with $|y|^C = \omega_1$. Correct ω_2 implies $|y|^K \leq \omega_1$. For a K -surjection $e : \omega_1 \rightarrow y$, the sequence of least e -indices of the values of s is bounded in ω_1 in C . A K -enumeration of a countable bound codes that sequence by a real. The real belongs to K , so $s \in K$.

Thus $(\xi^\omega)^W = (\xi^\omega)^K$ for every ordinal ξ . Each of these spaces has a K -wellordering. Under $\text{KW}_1^{*,\omega}$, least preimages under a sequence-space surjection wellorder every nonempty set. Hence AC, and therefore Fodor's lemma, follows.

The third-order formula. Use the classical one-measure lower-part presentation of K [34, Section 3]. Countable mousehood has a projective definition, and the corresponding countable-hull tests use only real codes [9, Lemma 3.32 and p. 92]. For $A \subseteq \mathbb{R}$, a certificate for $\text{KSet}(A)$ consists of a wellordered real domain, a well-founded extensional premouse relation on it, its measure predicate and satisfaction relation, actual-real labels, and a distinguished lower-part node whose elements are exactly A . Require every real-coded countable elementary hull to collapse to an iterable classical premouse. Also allow a constructible level certificate.

The domain wellorder makes the entire labelled certificate ordinal-codable. In the choice model obtained by adjoining that code to HOD , the countable-hull copying criterion gives iterability, and Proposition 9 of [30] identifies the core with K . This proves soundness. For completeness under $\mathbb{R} = \mathbb{R}^K$, choose the witness inside K , with both $\mathbb{R}^K$ and A in its lower part. A canonical hull containing every real and these two nodes has size at most ω_1 and its collapse fixes A . The small-mouse argument in the proof of [34, Lemma 3.1], applied inside K at ω_2^K , also gives this witness directly. CH in K supplies a real-domain code. Equality of reals preserves its countable-hull tests. Consequently

$$\mathbb{R} = \mathbb{R}^K \implies (\text{KSet}(A) \longleftrightarrow A \in K).$$

Under DC, a relation on a real domain is well-founded exactly when it has no real-coded descending omega-sequence. All the certificate conditions therefore have a second-order matrix after one existential third-sort quantifier: $\text{KSet}(A) \equiv \exists P k(A, P)$. Let CopyK say that every wellorder on a subset of the reals has an isomorphic real-domain wellorder with such a core certificate. Bundling the copy, certificate and isomorphism gives

$$\text{CopyK} \equiv \forall E \exists P c(E, P),$$

where c is second-order; malformed inputs permit a dummy certificate. On $\mathbb{R} = \mathbb{R}^K$, this is equivalent to $\omega_2 = \omega_2^K$. For if the cardinals agree, every real-domain order type is below ω_2^K and has a core copy. If $\omega_2^K < \omega_2^W$, an ambient real-domain wellorder of type ω_2^K has no such copy. The old CH bijection excludes the reverse cardinal inequality.

Write B for the second-order guard $\neg \exists 0^\dagger \wedge \forall x \in \mathbb{R} (x \in K)$. Here $0^\dagger$ uses its usual real-coded two-block indiscernible theory and countable well-foundedness tests. Let U code a wellorder w of all reals with countable proper initial segments, a set S of its points, and a regressive function on S . Such a w has type ω_1 . Fodor's assertion says either that a club avoids S , or that one fiber meets every club. An existential third-sort witness V stores a tag and either that first club or the single real naming the fiber. Thus its real-domain version is $\forall U \exists V \forall D q(U, V, D)$, where D tests clubs and q is second-order. Invalid U is accepted automatically. No family of clubs is selected.

The required Φ is the following normal form:

$$\forall U \exists (E, V) \forall (P, D) [\neg B \vee \neg c(E, P) \vee q(U, V, D)]. \quad (\text{C.2})$$

All prefix variables have the third sort. If CopyK fails, a single E with no certificate makes the matrix true. If it holds, a certificate for the selected E reduces the matrix to the Fodor clause. Thus (C.2) is equivalent, over $\text{ZF} + \text{DC}$, to (C.1). This proves the advertised Π_3^2 upper bound, without claiming minimality.

The countermodel. Start with $\text{ZFC} + \text{CR}$ and pass to its classical core D . Mitchell's Ramsey preservation theorem gives $D \models \text{ZFC} + \text{GCH} + \text{CR}$ [29, Theorem 1]. Also $K^D = D$ and D has no $0^\dagger$.

Apply Karagila's local symmetric construction at $\kappa = \mu = \omega_1^D$ [32, Section 4]. First add $g : \kappa \rightarrow \kappa$ by bounded conditions, and set $S_\beta = \{\xi < \kappa : g(\xi) \geq \beta\}$. Then use the countable-support product of the club-shooting orders on the S_β , with the normal filter fixing old countable sets of coordinates. Let N be the symmetric model.

The combined forcing has a countably closed dense presentation of size ω_1 . A dense condition synchronizes the club maxima and chooses the value of g there above every active index. Countable unions have lower bounds of the same form, and CH bounds the number of codes. The reals and cardinals are unchanged, and the countably complete support filter gives DC in N . Every ordinal-coded object of N belongs to a bounded head $D[G_\alpha]$, and every such head is contained in N .

Put $T_* = \kappa \setminus \Delta_{\beta < \kappa} S_\beta$. Then $f = g \upharpoonright T_*$ is regressive and each fiber of value η misses the retained club through $S_{\eta+1}$. The set T_* is stationary in N . Indeed, a retained club belongs to a head $D[G_\alpha]$. Fusion in that head produces a limit point $\delta > \alpha + 1$ of the club at which one may set $g(\delta) = \alpha$. Appending δ to every active coordinate below α is legal, while $\delta \notin S_{\alpha+1}$. This is stationarity in N ; the full choice extension retains an additional diagonal club.

For any old Ramsey $\rho > \omega_1$, a coloring in N of $[\rho]^{<\omega}$ belongs to one bounded head. That head is a choice extension by a forcing of size $\omega_1 < \rho$. Small-forcing preservation supplies a homogeneous set there [33, Theorem 4.1], so that set also belongs to N . Hence $N \models \text{CR}$. Classical mouse absoluteness and set-forcing invariance identify $K^N = D$; unchanged reals preserve the $0^\dagger$ predicate. Thus the antecedent of (C.1) holds in N , while its conclusion fails. The old CH bijection transports this counterexample to the third-order formula. These forcing and inner-model constructions give the stated consistency implication. $\square$

For the structural conclusion, use a finite signature with number and real sorts partitioning its domain, the arithmetic operations on the number sort, and a membership relation between the two sorts. A full second-order sentence asserts the Peano axioms with second-order induction on the first sort and requires every subset of that sort to be coded by exactly one element of the second. Such a structure is isomorphic to the canonical structure on $\omega \sqcup \mathcal{P}(\omega)$. Translate third-sort variables in Φ into second-order subsets of its real sort. The universal assertion that every such frame satisfies this translation is equivalent to Φ in ZF. It has the structural Π_2 form of Definition 1.6. The Π_3^2 bound concerns third-order arithmetic, whereas this Π_2 classification concerns the Lévy hierarchy.

In particular, the Ramsey replacement of Theorem 1.7 fails even with Foundation retained in the starting theory, and hence also with Foundation omitted. This argument does not assert that $\text{KW}_1^{*,\omega}$ alone implies Fodor's lemma, nor does it supply a Ramsey replacement counterexample for the separate external rank-equiconsistency theorem without a further rank translation.

APPENDIX D. A STRONG-CARDINAL SEPARATION FOR THE CHANG MODEL

We give the separation used in the introduction to explain the choice of $\text{HOD}_{\text{Ord}^\omega}$ in Theorem 1.1. Throughout this appendix, $C^U = L(\text{Ord}^\omega)^U$ is computed in an ambient universe $U \models \text{ZFC}$.

Define

$$E_1^C(\sigma) = \text{ZFC} + \text{CM} + [C \models \exists \kappa (V_\kappa \models \sigma)].$$

Let CS assert that strong cardinals are unbounded, with strongness given by its usual first-order extender formulation, and put

$$B_{\text{str}} = \text{ZFC} + \text{CS} + \text{“there is an inaccessible limit of strong cardinals.”}$$

The following single first-order sentence will separate the theories:

$$\begin{aligned} \sigma_{\text{str}} \equiv & \exists \lambda \exists \nu [\lambda, \nu \in \text{Ord} \wedge \lambda < \nu \\ & \wedge \nu \text{ is a limit ordinal} \\ & \wedge V_\lambda \models \text{ZFC} + \text{CS}]. \end{aligned} \tag{D.1}$$

Here the rank and its satisfaction relation are expressed by set witnesses, including satisfaction of the recursive set of all ZFC axioms. The ordinal ν is a height guard: in a witnessing outer rank V_κ , it ensures $\lambda + \omega \leq \nu < \kappa$.

Theorem D.1 (Chang rank separation). *If $U \models \text{ZFC}$, no full rank $V_\lambda^{C^U}$ satisfies $\text{ZFC} + \text{CS}$. In particular, $E_1^C(\sigma_{\text{str}})$ is inconsistent, whereas*

$$\text{Con}(B_{\text{str}}) \implies \text{Con}(E_3(\sigma_{\text{str}})).$$

Thus, relative to B_{str} , substituting the Chang model for $\mathcal{H}$ destroys even the equiconsistency of theories (1) and (3). The prohibition on Chang ranks requires no ambient measurable cardinals. Strong cardinals enter the consistency bound for the separating instance of theory (3). For a tautological σ , the two Chang analogues remain consistent relative to $\text{ZFC} + \text{CM}$.

D.1. The imported mouse construction. We use Mitchell's Chang-sharp construction. The relevant results are [25], Theorem 1.5, Sections 1.3 and 3, Definition 4.1, and Proposition 1.4.

The construction uses a set mouse over all the reals with a sufficiently long last extender. Section 3 reduces the hypothesis to set-sized data, and Definition 4.1 specifies the additional approximation system. The relevant mice use the linear iteration theory below a strong cardinal. Iterating the final extender gives a continuous class $I = \{\kappa_\alpha : \alpha \in \text{Ord}\}$. Definition 3.9 gives uniformly interpreted term codes from the mouse. These terms represent every element of Chang. Proposition 1.4 extends appropriate order-preserving maps of I to elementary embeddings of Chang.

The large cardinal existence input is that $\text{ZFC} + \text{CS}$ supplies such a mouse over its entire real set, with the required iterability. Gappo and Sargsyan explicitly identify this upper bound in the introduction to [26]. We call the full set data specified by Mitchell a *Mitchell datum*. This abbreviation includes the approximation system; a bare premouse with a long extender is not sufficient.

All structural requirements on a fixed datum are first-order assertions about set structures, their satisfaction relations, functions, and ordinals. Their interpretation is absolute when the data and the relevant countable sequences are shared. The remaining nonlocal requirement is iterability. The next section addresses that requirement rather than assuming it is absolute.

D.2. Countable hull iterability transfers from a Chang rank.

Lemma D.2. *Suppose $W = V_\lambda^{C^U} \models \text{ZFC}$. Then W and U have the same reals and the same ω_1 . If $a \in W$, every U -countable sequence of elements of a belongs to W .*

Proof. The Chang model is closed under countable sequences formed in U . In particular it contains every real and every countable ordinal sequence of U .

If $a \in W$ and $s : \omega \rightarrow a$ belongs to U , then $s \in C^U$. Its rank is bounded by $\max\{\text{rank}(a), \omega\}$ plus a fixed finite ordinal. This is below λ , since $W \models \text{ZFC}$. Hence $s \in W$.

Every real has bounded finite rank above ω , so the real sets agree. If an ordinal in W is countable in U , its enumeration belongs to W by the preceding argument. Applying this to ω_1^W , if necessary, proves equality of ω_1 . $\square$

Lemma D.3. *Let W be as above. Suppose $M \in W$ is a set premouse in the linear iteration setting of [25]. If W regards every countable elementary hull of M as iterable through all countable*

linear iterations, then the same assertion holds in U . Moreover M has no ill-founded set-length linear iteration in U .

Proof. Include the extender predicates and the final extender in the structure being considered. Let $X \prec M$ be any elementary substructure countable in U . An enumeration of X is a countable sequence from the fixed set underlying M , so Lemma D.2 puts X and an enumeration in W . Set satisfaction is absolute, so W recognizes its elementarity. The two universes form the same transitive collapse $\bar{M}$.

A countable premouse, a countable putative linear iteration, and a descending sequence witnessing ill-foundedness can be coded by reals. The relevant code relations are statements of second-order arithmetic. Since W and U have exactly the same reals, they agree on these relations and on the assertion that every countable legal linear iteration of $\bar{M}$ is well-founded. Thus every countable hull remains countably iterable in U .

For completeness, suppose there were a bad set-length linear iteration of M in U . Retain the iteration through a first bad model, together with a descending-sequence witness for its formal ultrapower or direct-limit relation. In U , take a countable elementary hull of a sufficiently large set structure containing these data and the definitions of the iteration operations. Its transitive collapse gives a countable bad linear iteration of the collapse of a countable elementary substructure of M .

The collapse preserves the iteration rules. Formal ultrapowers use functions from the relevant source structure, and elementarity supplies representatives of their quotient elements in the hull. The same reasoning applies to the direct-limit construction. The retained descending sequence witnesses that the last relation is still ill-founded. This contradicts the preceding paragraph.

This argument concerns linear iteration. It does not assert preservation or existence of a strategy for arbitrary uncountable branching iteration trees. $\square$

D.3. Why the datum cannot belong to Chang.

Proposition D.4. *A Mitchell datum over all the reals of U , with the requisite external linear iterability, cannot belong to C^U .*

Proof. Suppose the datum belongs to $C = C^U$. Write (M, E) for its mouse and final extender. The canonical linear iteration by E can be computed inside C at every set-length stage. Its computation agrees with that in U : the source structures and their functions are the same, formal ultrapowers and direct limits are the same, and well-founded collapses are unique. Transfinite recursion in C supplies the set-length iteration systems. Thus the class

$$I = \{\kappa_\alpha : \alpha \in \text{Ord}\}$$

and the interpretation of Mitchell's term codes are definable over C from set parameters in the datum.

This last definability assertion can be read directly from Definition 3.9 of [25]. Its ingredients are the canonical iteration maps, countable sequences of term codes, composition, and satisfaction in a set-sized Chang construction stage. In particular, the apparently class-length term $i_{\text{Ord}}(f)(a)$ is evaluated at the first set stage α for which all entries of the finite ordinal tuple a are below κ_α . The value $i_\alpha(f)(a)$ is below κ_α and is fixed by every subsequent iteration map. Thus evaluation is uniformly first-order definable from the set data. No truth predicate for the whole Chang model is required. The auxiliary approximation system and generic objects used in the proof of indiscernibility do not enter the interpretation of these terms.

Define an increasing map on ordinal indices by

$$f(\alpha) = \begin{cases} \alpha, & \alpha < \omega_1, \\ \omega_1 + \alpha, & \alpha \geq \omega_1, \end{cases} \quad s(\kappa_\alpha) = \kappa_{f(\alpha)}.$$

The map preserves successors and limit-index status. It is continuous at limit indices of countable cofinality. Its only initial discontinuity is at ω_1 , whose cofinality is uncountable in both universes. Therefore s satisfies the hypotheses of Mitchell's Proposition 1.4.

Using the term representation of Chang, define

$$j(\tau(B)) = \tau(s[B]).$$

The equality and formula preservation furnished by the sharp make this definition independent of the chosen representation and make $j : C \rightarrow C$ elementary. The relation defining j quantifies over set term codes and countable suitable sets B ; all its ingredients are definable over C from the datum. In particular j is a parameter-definable class of C . It is nontrivial because

$$j(\kappa_{\omega_1}) = \kappa_{\omega_1 \cdot 2} > \kappa_{\omega_1}.$$

Suzuki's theorem [27], which requires ZF rather than Choice, excludes such a parameter-definable nontrivial elementary self-embedding of C . This is a contradiction. $\square$

D.4. The separation.

Proof of Theorem D.1. Suppose $W = V_\lambda^{C^U} \models \text{ZFC} + \text{CS}$. Inside W , the imported mouse construction produces a Mitchell datum over $\mathbb{R}^W$. Lemma D.2 gives $\mathbb{R}^W = \mathbb{R}^U$ and the same countable sequences from each fixed set in the datum. The bijection witnessing $|M| = |\mathbb{R}|$ remains a bijection externally. All elementarity and internal cardinal computations in the fixed structures remain correct. The height of M still has cofinality ω_1 : its cofinal ω_1 -sequence remains cofinal, while any new countable cofinal sequence would belong to W by bounded closure. Thus the structural requirements hold externally. Lemma D.3 transfers the required linear iterability. The datum belongs to $W \subseteq C^U$, contradicting Proposition D.4.

If a rank $V_\kappa^{C^U}$ satisfies σ_{str} , the height guard puts $\lambda + \omega$ below κ . Induction on the witnessed cumulative hierarchy identifies its nested rank with the actual $V_\lambda^{C^U}$. Every subset required at an earlier hierarchy stage is present in the outer rank. Finite assignments and the full set satisfaction relation also lie below $\lambda + \omega$, so satisfaction agrees. This gives precisely the impossible W above, proving inconsistency of $E_1^C(\sigma_{\text{str}})$.

For the consistency direction, work in a model of B_{str} , and let δ be an inaccessible limit of ambient strong cardinals. Then $V_\delta \models \text{ZFC}$. Every ambient strong $\rho < \delta$ is strong in V_δ : for each target rank V_η below δ , choose the usual bounded extender E derived from an ambient strongness embedding. Inaccessibility of δ puts its support, the extender, and the required power sets below δ .

To check the target-rank containment, represent each $x \in V_\eta$ in the ambient extender ultrapower as $[a, f]_E$. Replace f by the function that keeps its values in $V_{\eta+1}$ and uses $\emptyset$ elsewhere. It represents the same x , since the ultrapower image of $\eta + 1$ exceeds η . The new function has rank below δ . Hence the natural factor map from the ultrapower of V_δ has every element of V_η in its range. Its transitive collapse fixes these elements by induction on rank. Thus the ultrapower of V_δ contains V_η , as required. This proves $V_\delta \models \text{CS}$.

The ambient universe satisfies CM, since strong cardinals are measurable and are unbounded. It satisfies ZFC, hence ZF + DC. The rank $V_{\delta+\omega \cdot 2}$ satisfies σ_{str} , using $\lambda = \delta$ and $\nu = \delta + \omega$. This realizes $E_3(\sigma_{\text{str}})$. $\square$

The obstruction is a set-sized iterable datum that a full Chang rank satisfying ZFC + CS would contain. The height guard in σ_{str} ensures that the nested rank and its satisfaction relation have their intended meanings.

D.5. The anti-inner-model question. The separation above leaves open whether restricting the available inner models restores the comparison. Use the standard Mitchell-order convention, so a measurable cardinal has order at least one, and let

$$A_2 \equiv \text{“there is no inner model of ZFC with a measurable } \nu \text{ satisfying } o(\nu) \geq 2.”$$

The Mitchell order is computed in the indicated inner model. Equivalently, the excluded configuration consists there of two normal measures U_0, U_1 on the same ν with $U_0 \in \text{Ult}(M, U_1)$. Here M is the inner model, and the ultrapower uses its functions. We use the semantic inner-model formulation of A_2 ; any formal version of the following question must fix a corresponding mouse or scheme formulation on both sides.

Question D.5. Can an anti-inner-model hypothesis restore the equiconsistency of theories (1) and (3) after the Chang substitution? In particular, for every fixed first-order sentence σ , is

$$\text{Con}(E_1^C(\sigma) + A_2) \longleftrightarrow \text{Con}(E_3(\sigma) + A_2)?$$

In this question A_2 is appended to both ambient theories. The left-hand witnessing rank is still computed inside Chang, and the right-hand rank is computed in the $\text{ZF} + \text{DC} + \text{CM}$ universe. The restriction permits order-one measurable cardinals and does not exclude measurable limits of measurable cardinals. No equiconsistency under this restriction is asserted here. Whether the Chang separation itself can be obtained from ordinary measurable-cardinal strength also remains open.

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