

COUNTABLE ORDINAL SURJECTIONS, CONSERVATIVITY, AND EXTERNAL HOD

ABSTRACT. We prove structural second-order conservativity of $\text{ZF} + \text{DC} + \text{PCMeas} + \text{SVC} + \text{KW}_1^{*,\omega}$ over $(\text{ZF} - \text{Foundation}) + \text{DC} + \text{PCMeas}$. A fixed-real core-model preparation removes SVC from the starting assumptions while preserving a prescribed rank and its full power set. A high Prikry sequence and a countably closed labeling forcing then supply ordinal-sequence surjections. Literal paired comparison removes the labeling coordinate from low-rank names, and external countable domination yields dependent choice. Finally, atomization and fiber purification remove Foundation from the base theory while preserving full second-order truth. For starting models with Foundation we also obtain rank-sentence equiconsistency without an SVC assumption. We then add the external HOD realization: sparse evaluated Cohen caps over a fresh choice envelope realize the sequence-generated model as $\text{HOD}_{\text{Ord}^\omega}$ over a securely coded ground. A rank-preserving class preparation gives the general realization and the three-way rank-theoretic equiconsistency, with a proper class of measurables in both the ambient universe and the resulting external HOD.

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1. CONSERVATIVITY WITHOUT SVC OR FOUNDATION

Write PCMeas for the assertion that measurable cardinals are unbounded in the ordinals. A measurable cardinal is uncountable and carries a nonprincipal, cardinal-complete ultrafilter; normality is required only in choice models. Write $\text{SVC}(S)$ when every nonempty set is a surjective

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image of $\eta \times S$ for some ordinal η . Finally, $\text{KW}_1^{*,\omega}$ says that every nonempty set is a surjective image of μ^ω for some ordinal μ , with this power computed in the model under discussion.

Put

$$T_- = (\text{ZF} - \text{Foundation}) + \text{DC} + \text{PCMeas}, \quad T_0 = \text{ZF} + \text{DC} + \text{PCMeas}, \quad T_s = T_0 + \text{SVC}.$$

The starting theory of the main theorem assumes neither SVC nor Foundation.

Theorem 1.1 (Structural conservativity). *The theory $T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$ is conservative over T_- for structural Π_2 assertions: for every fixed full second-order sentence φ , the assertion that φ holds in every set-sized structure has the same provability in the two theories.*

The core-model and atomization reductions originate in the research archive [1]. Sections 8–9 give their proofs and their application, including the fixed-base classical-core and recoding arguments. SVC is an intermediate property supplied by that preparation, not an assumption on the original countermodel.

Theorem 1.2 (Rank transfer without SVC). *Suppose $V \models T_0$ and $\alpha \in \text{Ord}$. In a set-generic extension of V there is a model M such that*

$$M \models T_0 + \text{SVC} + \text{KW}_1^{*,\omega}, \quad V_\alpha^M = V_\alpha^V, \quad \mathcal{P}(V_\alpha^V)^M = \mathcal{P}(V_\alpha^V)^V.$$

Moreover, $M = D(A)$ for a choice symmetric ground $D \models \text{ZFC} + \text{PCMeas}$ and a set $A \subseteq \mu^\omega$.

The model M contains a rank-preserving SVC inner model of V ; it need not contain all of V . Foundation removal concerns structural truth, so Theorem 1.1 does not assert that the non-well-founded membership structure of an arbitrary original universe becomes a literal rank of a well-founded model.

The forcing component has the following stronger containment statement once the preparatory SVC model has been obtained.

Theorem 1.3 (Local rank-to-sequence transfer). *Suppose $N \models T_s$ and $\alpha \in \text{Ord}$. In a set-generic extension of N there is an inner model M containing N such that*

$$M \models T_s + \text{KW}_1^{*,\omega}, \quad V_\alpha^M = V_\alpha^N.$$

More precisely, one can choose $\lambda > \alpha$ and $X = V_\lambda^N$ so that

$$\mathcal{P}(X)^M = \mathcal{P}(X)^N, \quad V_\lambda^M = X.$$

There are a choice symmetric ground $D \models \text{ZFC} + \text{PCMeas}$, an ordinal μ , and $A \subseteq \mu^\omega$ in M such that $M = D(A)$.

Here $D(A)$ is the transitive-base generated model: it contains D and the transitive closure of $\{A\}$. The local theorem is proved in Sections 2–7; Section 8 supplies its starting model from T_0 .

Corollary 1.4 (Rank-sentence equiconsistency). *For every first-order sentence σ , the theories*

$$T_0 + \exists \alpha (V_\alpha \models \sigma) \quad \text{and} \quad T_0 + \text{SVC} + \text{KW}_1^{*,\omega} + \exists \alpha (V_\alpha \models \sigma)$$

are equiconsistent.

Proof. Apply Theorem 1.2 to a witnessing rank. The converse is immediate. $\square$

Remark 1.5 (Role of the earlier preparation). The earlier core-model argument supplies a pointwise rank-preserving passage to SVC. We use that result directly, rather than attempting to derive literal rank equality from an abstract conservativity statement. Its corrected form uses a core model over the coding real inside relative HOD, finite-recoding invariance of its pure universe, and a set-forcing presentation over its own HOD. The unnecessarily stronger assertion $V = \text{HOD}(r)$ is not required. The earlier $\text{SVC} + \text{KW}_1^{*,1}$ conservativity result is compatible with this preparation, but is not an additional hypothesis of the main theorem.

The local construction uses a high Prikry sequence and a countably closed labeling forcing to obtain $Q \subseteq \kappa^\omega$ and $F : Q \rightarrow X$. Literal paired comparison removes the labeling coordinate from low-rank names. External countable domination gives DC in $N(Q, F)$. After the earlier atomization preparation, the same local construction gives an atom model; fibers of ordinal-sequence surjections then purify the protected structure and the SVC witness.

1.1. External HOD and rank theories. For $U \models \text{ZFC}$, put $\mathcal{H}^U = \text{HOD}_{\text{Ord}^\omega}^U$. An allowed parameter is one function from ω to the ordinals, together with finitely many ordinal parameters. The hereditary requirement is retained. For a fixed first-order sentence σ , write

$$\begin{aligned} E_1(\sigma) &= \text{ZFC} + \text{PCMeas} + [\mathcal{H} \models \exists \kappa (V_\kappa \models \sigma)], \\ E_2(\sigma) &= \text{ZFC} + \text{PCMeas} + [\mathcal{H} \models \text{PCMeas} \wedge \exists \kappa (V_\kappa \models \sigma)], \\ E_3(\sigma) &= T_0 + \exists \kappa (V_\kappa \models \sigma). \end{aligned}$$

The conjunction in E_2 is evaluated inside the external HOD.

Theorem 1.6 (External rank equiconsistency). *For every σ ,*

$$\text{Con}(E_1(\sigma)) \longleftrightarrow \text{Con}(E_2(\sigma)) \longleftrightarrow \text{Con}(E_3(\sigma)).$$

Theorem 1.7 (External realization). *Suppose $M \models T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$. For every ordinal α there is an outer model $U \models \text{ZFC} + \text{PCMeas}$ of M such that*

$$\mathcal{H}^U \models \text{PCMeas}, \quad V_\alpha^{\mathcal{H}^U} = V_\alpha^M.$$

Under the secure-ground hypotheses of Proposition 13.1, a further set-forcing extension of the fresh choice envelope gives $\mathcal{H}^U = M$.

These external results add a second construction to the KW theorem: sparse Cohen rows with evaluated terminal caps code a set of ordinal sequences over a recognizable choice envelope. The general preparation uses class forcing, so the outer model in Theorem 1.7 is not asserted to be a set-forcing extension of M . Theorem 1.6 concerns rank-sentence consistency, whereas Theorem 1.1 concerns structural second-order conservativity.

Formalization. Transitive-model notation presents the forcing and inner-model interpretations; consistency is not assumed to supply a transitive model. Set-forcing arguments use their forcing theorems, generated models use the definable transitive-base construction, and the Foundation-free start uses the interpreted cumulative hierarchy over a set of atoms. Equivalently these constructions are carried out over countable models with their internal well-foundedness and satisfaction notions. The KW and Foundation reductions in Sections 2–9 use only set forcing. The external realization later uses the progressively closed continuum-coding interpretation described in Section 12; its set arguments localize to set-sized initial segments.

2. SMALL FORCING AND GENERATED-MODEL PRESENTATIONS

Lemma 2.1 (Small forcing in ZF). *Let $V \models \text{ZF}$, let $\mathbb{B} \in V$ be a complete Boolean algebra, and let $\mathcal{U}$ be a nonprincipal κ -complete ultrafilter on κ , where $\kappa > h(\mathbb{B})$. For every V -generic $G \subseteq \mathbb{B}$, the filter*

$$\mathcal{U}^G = \{a \in \mathcal{P}(\kappa)^{V[G]} : \exists b \in \mathcal{U} (b \subseteq a)\}$$

is a nonprincipal κ -complete ultrafilter in $V[G]$. In particular, arbitrary set forcing over ZF preserves a proper-class tail of measurable cardinals.

Proof. For a name $\dot{a} \subseteq \check{\kappa}$, the range of

$$\zeta \mapsto \|\check{\zeta} \in \dot{a}\|_{\mathbb{B}}$$

is canonically well-orderable by least preimages. Its cardinality is below $h(\mathbb{B})$, hence below κ . The partition into Boolean-value fibers therefore has a unique $\mathcal{U}$ -positive fiber $b_{\dot{a}}$, with associated Boolean value $v_{\dot{a}}$. In the generic extension, this old fiber is contained in a if $v_{\dot{a}} \in G$, and is contained in $\kappa \setminus a$ otherwise. Thus the generated filter decides every subset of κ .

For completeness, take a name for a sequence $\langle a_i : i < \delta \rangle$, where $\delta < \kappa$. Localize and give the name a fixed default value outside the localization. For every i , form the unique positive Boolean-value fiber b_i in V . This is a uniform construction, so the entire sequence $\langle b_i : i < \delta \rangle$ belongs to V ; no choice of antichain witnesses is being made. If each actual $a_i \in \mathcal{U}^G$, its associated Boolean value belongs to G . Consequently

$$b = \bigcap_{i < \delta} b_i \in \mathcal{U}, \quad b \subseteq \bigcap_{i < \delta} a_i.$$

Nonprincipality is immediate. The resulting completeness also prevents κ from becoming a cardinal of size below κ : a short enumeration would express κ as a union of fewer than κ null singletons. $\square$

For a normal measure in a ZFC ground and forcing of size below its measurable cardinal, we also use the usual normal generated lift and the downward Lévy–Solovay theorem; see [6]. If $D \subseteq W \subseteq D[G]$ and the forcing is small relative to a normal D -measure $\mathcal{U}$ on κ , its restriction to W is the set

$$\mathcal{U}^W = \{a \in \mathcal{P}(\kappa)^W : \exists b \in \mathcal{U} (b \subseteq a)\}. \quad (2.1)$$

This definition belongs to W when $D \subseteq W$ and $W \models \mathbf{ZF}$. Normality and completeness follow by restricting the ambient lift. We never use unqualified downward absoluteness of measurability to an arbitrary inner model.

Lemma 2.2 (Bounded presentations). *Suppose $N \models \mathbf{ZF}$ and $M = N(Q, F)$, where $Q \subseteq \kappa^\omega$ and $F : Q \rightarrow X \in N$. Every particular set in M is the value of a bounded, set-sized evaluator from the whole sets Q, F , finitely many members of Q , parameters in N , and ordinal codes. The evaluator is absolute between transitive outer models containing the same inputs. For every nonempty $Y \in M$, there are an ordinal η , a nonempty set $Z \in N$, and a surjection in M*

$$\pi : \eta \times Z \times Q^{<\omega} \longrightarrow Y.$$

The whole sets Q, F remain fixed parameters of this surjection; they are not replaced by finitely many of their members.

Proof. We use transitive-base constructibility. For a sufficiently large ordinal θ , put

$$B_\theta = \text{TC}(\{V_\theta^N, Q, F\}), \quad J_0(B_\theta) = B_\theta, \quad J_{\xi+1}(B_\theta) = \text{Def}(J_\xi(B_\theta)),$$

and take unions at limits. Since the base is transitive, these levels are transitive and increasing. The standard construction of $N(Q, F)$ is the union, over the ground ranks and construction stages, of these hierarchies. In particular, a given $Y \in M$ is contained in some set-sized level $J_\beta(B_\theta)$. This is the transitive-base intermediate-model construction, not predicate constructibility omitting the members of the adjoined sets; see [3, 7].

Here is the finite-parameter coding at a fixed such level. A base element is a leaf. An element first produced at a successor stage is specified by a formula over the preceding set-sized level and finitely many parameters from that level. Substitute descriptions of those parameters. Transfinite induction on the construction stage shows that every element has a finite expression of this kind, with leaves in B_θ and with ordinal stage labels. At a limit stage use an earlier stage. At a successor stage only finitely many earlier descriptions are combined, so the expression remains finite. All

stages mentioned in a description of a member of $J_\beta(B_\theta)$ can be bounded by β (enlarging β by a finite amount does no harm).

An evaluator for such an expression uses only satisfaction for the indicated set structures $J_\xi(B_\theta)$. Satisfaction for a set structure, its definable-subset operation, and the transfinite recursion defining a bounded list of these levels are absolute between transitive outer models with the same base and ordinals. Thus the evaluator has the asserted absoluteness. No satisfaction predicate for the proper class M or N is used.

We next reduce its leaves. Apart from the whole sets Q, F , an element of $\text{TC}(\{Q, F\})$ is obtained by finitely many pairing and projection operations from individual members of Q and ground elements. Indeed, values of F and their hereditary members belong to N , and the members occurring in a sequence $q \in Q$ are built from natural numbers and ordinals below κ . The same statement holds for the ordered pairs coding the graph of F . Leaves from $\text{TC}(\{V_\theta^N\})$ are already ground elements. Consequently every expression can be converted to one using the fixed whole parameters Q, F , a finite tuple from Q , and a finite tuple from one set V_ρ^N , where ρ may be chosen uniformly from θ, X, κ .

Let $Z = (V_\rho^N)^{<\omega}$, increasing ρ so that Z is nonempty and contains all the required fixed ground data. The set of finite syntactic expressions, finite ordinal labels below the chosen bound, and finite type information is wellorderable, and so can be indexed by an ordinal η . Evaluate all these codes with all parameters in Z and all finite tuples from Q . Keep an output precisely when the code is valid and its value belongs to Y . This is a set-sized partial function in M , and its range is all of Y . Fix one $y_0 \in Y$ and send every unused argument to y_0 . The resulting total map is π . In particular, we have neither chosen a definition for every $y \in Y$ nor wellordered the ground parameter bank Z . $\square$

3. THE CHOICE SYMMETRIC GROUND FROM SVC

Proposition 3.1 (A fixed small presentation with a measurable tail). *If $N \models \text{ZF} + \text{SVC} + \text{PCMeas}$, there is a choice inner model D with*

$$D \models \text{ZFC} + \text{PCMeas}, \quad N = D(z)$$

for some $z \in N$. There are fixed set forcings $P \in N$ and $T \in D$ and a common extension

$$E = N[H] = D[G]$$

such that N is a symmetric submodel of $D[G]$ for a set-sized symmetric system in D .

Proof. By [7, Proposition 5.4], SVC gives a choice symmetric ground D and a set generator z with $N = D(z)$. Use [7, Lemma 4.2] to choose one fixed common-cover presentation in which N is actually a symmetric submodel. This qualification is useful: an arbitrary pair of forcings witnessing a common extension need not itself be the desired symmetric system; see [7, Note 4.6]. A sufficiently large pair of collapses supplies the required presentation.

It remains to show that this same D has a proper class of measurables. Put

$$\theta = h^N(\text{RO}^N(P)), \quad \tau = |T|^D.$$

These are fixed ordinals. For every N -measurable $\kappa > \max\{\theta, \tau, \omega\}$, Lemma 2.1 lifts its measure to $N[H]$. Thus κ is measurable in the choice model $D[G]$. Downward Lévy–Solovay for the D -forcing T , whose D -size is $\tau < \kappa$, puts a measure on κ in D . The bounds and the ground were fixed independently of κ , so $D \models \text{PCMeas}$. No absoluteness of the Hartogs calculation between N and D is used. $\square$

Now assume also $N \models \text{DC}$. We choose the ranks and cardinal used throughout the rest of the proof. Fix an SVC witness $S \in N$, the generator z , the protected rank, and the ground symmetric-system data. Choose an uncountable regular D -cardinal ρ above the size of T and the

rank bounds needed for this fixed presentation. Choose a sufficiently high beth fixed point λ of D with $\text{cf}^D(\lambda) = \rho$. For example, a strictly increasing continuous sequence of beth fixed points of length ρ , started high enough, has such a supremum. The forcing T preserves this cofinality. Since $D \subseteq N \subseteq D[G]$, it follows that $\text{cf}^N(\lambda) = \rho > \omega$.

Choose λ large enough that the transitive set $X = V_\lambda^N$ contains z , S , and the protected data. Then

$$N = D(X), \quad N \models \text{SVC}(X).$$

For the first equality, $D(X) \subseteq N$ and $z \in X \subseteq D(X)$; for the second, extend each given surjection on $\eta \times S$ by a default value on the rest of $\eta \times X$. Increase the bounds once more if necessary, and choose a measurable κ of D above them. The fixed forcing T is small below κ , so κ remains inaccessible in $D[G]$. All rank-bounded data below λ , including X and the old countable-condition posets introduced below, have size below κ there.

This construction changes neither N nor its protected ranks. The preliminary common choice cover is only an ambient model in which subsequent generics and size calculations can be chosen.

4. THE PRIKRY AMBIENT UNIVERSE

For the next four sections fix a symmetric presentation

$$D \subseteq N \subseteq D[G], \quad D \models \text{ZFC} + \text{PCMeas}, \quad N \models \text{ZF} + \text{DC} + \text{SVC}.$$

Write $\mathcal{S} = (\mathbb{A}, \mathcal{G}, \mathcal{F}) \in D$ for its symmetric system, and take G to be $\mathbb{A}$ -generic over D .

Choose a transitive rank $X = V_\lambda^N$ containing a generator for N over D , an SVC witness, and the structure to be protected. Choose λ sufficiently closed, with uncountable cofinality in the full choice cover; one may take a ground beth fixed point of cofinality greater than the size of the fixed small presentation. Then choose a measurable κ of D above $\lambda + \omega$ and all presentation bounds. In particular, arrange

$$N = D(X), \quad N \models \text{SVC}(X), \quad |X|^{D[G]} < \kappa, \quad \mathcal{S}, \text{RO}(\mathbb{A}) \in V_\kappa^D. \quad (4.1)$$

The forcing $\mathbb{A}$ and all subsequent choice-envelope forcing below κ have size less than κ . These bounds can be imposed before choosing κ : the presentation is fixed, and a small forcing has fewer than κ bounded-rank names below a sufficiently large ground inaccessible. The ground cofinality chosen for λ survives the small cover, hence $\text{cf}^N(\lambda) > \omega$ as well.

Let $\mathcal{U} \in D$ be a normal measure on κ , and let $\mathbb{P}_\mathcal{U}$ be its *old* Prikry forcing in D . Its ground wellordering belongs to N . The old measure base generates the normal lift in every small choice cover under consideration; therefore $\mathbb{P}_\mathcal{U}$ is dense in the Prikry forcing computed there from that lift.

Put $\nu = \omega_1^N$ and let

$$\mathbb{C} = \{k : k \text{ is an } N\text{-countable partial function } \nu \longrightarrow X\}^N,$$

ordered by reverse inclusion. This poset is frozen after being computed in N . Every condition belongs to X : its domain is bounded in $\nu < \lambda$, and its countable range has bounded rank below λ . We may use the dense initial-segment presentation $\bigcup_{\gamma < \nu} \gamma X$ computed in N . In the full choice cover $|\mathbb{C}| < \kappa$; increase the chosen bound on the small presentation if necessary.

Choose (c, K) generic for the actual product $\mathbb{P}_\mathcal{U} \times \mathbb{C}$ over the *full choice cover* $D[G]$, and set

$$W = N[c][K] = N[K][c].$$

The product uses the old posets just specified. This choice gives all the displayed genericities over N and, in addition, the full-cover genericity used in part (3) below. It makes no assertion that the old collapse is countably closed after adjoining c .

In N , DC makes $\mathbb{C}$ countably closed. Its generic union enumerates X with domain ν . By $\text{SVC}(X)$, $N[K]$ can wellorder every old set, and the same is then true of every set of names and

its evaluated range. Thus $N[K] \models \text{ZFC}$. The full cover $D[G][K]$ is a small forcing extension of D , so its normal lift of $\mathcal{U}$ restricts to the normal generated measure in $N[K]$. Product genericity makes c Prikry generic over this choice model. Consequently $W \models \text{ZFC}$.

Lemma 4.1. *In this setup:*

- (1) ${}^\omega X \cap W = {}^\omega X \cap N$;
- (2) every W -countable family of elements of N is contained in an N -wellorderable set;
- (3) $V_\kappa^{N[c]} = V_\kappa^N$.

Proof. For (1), first force with $\mathbb{C}$. The usual deciding-sequence argument uses only DC in N and countable closure of $\mathbb{C}$, and proves that $N[K]/N$ adds no countable sequence of elements of any fixed old set. In $N[K]$, X has size at most $\nu < \kappa$. The Prikry property for the lifted normal measure implies that the subsequent Prikry forcing adds no bounded subset of κ . Coding through the enumeration of X , a countable sequence from X in W therefore already lies in $N[K]$, and then in N . The same argument gives equality of the reals. We use the standard Prikry property and infinite-subsequence criterion in the form recorded in [2, Section 3].

For (2), bound the ranks of the countably many ground elements by an ordinal and place them in one set $Z \in N$. Let $\langle z_n : n < \omega \rangle \in W$ be the resulting sequence. View it as the value of a $\mathbb{C}$ -name $\dot{z} \in N[c]$. In W , for each n choose $k_n \in K$ which forces

$$\dot{z}(n) = \check{z}_n$$

over $N[c]$. Such a condition exists by the truth lemma; a fixed localization can be included in all the k_n . Since $\mathbb{C} \subseteq X$, (1) gives $\langle k_n : n < \omega \rangle \in N$. Inside $N[c]$, that array, Z , and $\dot{z}$ reconstruct $\langle z_n \rangle$ by the forcing relation: z_n is the unique $z \in Z$ for which $k_n \Vdash_{\mathbb{C}} \dot{z}(n) = \check{z}$. Hence the whole sequence belongs to $N[c]$.

Take a $\mathbb{P}_{\mathcal{U}}$ -name $\dot{v} \in N$ for it and let

$$B = \{z \in Z : \exists n < \omega \exists p \in \mathbb{P}_{\mathcal{U}} (p \Vdash \dot{v}(n) = \check{z})\}.$$

This is an N -set and covers every actual z_n . Fix the old wellordering of $\omega \times \mathbb{P}_{\mathcal{U}}$. Send each $z \in B$ to its least deciding pair. Distinct values have distinct pairs, so this is an injection into a wellordered set and induces an N -wellordering of B . The proof does not wellorder Z .

For (3), use the following generic-intersection fact, valid without Choice:

$$N_0 \subseteq E_0, \quad \mathbb{P} \in N_0, \quad h \text{ generic over } E_0 \implies N_0[h] \cap E_0 = N_0,$$

where the transitive models satisfy ZF and have the same ordinals, and the forcing is the same old poset in both models.

We give the proof. First, if h_0, h_1 are mutually $\mathbb{P}$ -generic over N_0 , then

$$N_0[h_0] \cap N_0[h_1] = N_0.$$

Induct on the rank of a common set x . Its members belong to N_0 by induction, so $x \subseteq V_\gamma^{N_0}$ for some ordinal γ . Choose names $\tau_0, \tau_1 \in N_0$ for x and a product condition (p_0, p_1) forcing their equality and that common value to be a subset of $V_\gamma^{N_0}$. For each $y \in V_\gamma^{N_0}$, p_0 must decide $y \in \tau_0$: otherwise two extensions giving opposite decisions, paired with one extension of p_1 deciding $y \in \tau_1$, contradict the forced equality. Hence

$$x = \{y \in V_\gamma^{N_0} : p_0 \Vdash_{\mathbb{P}} \check{y} \in \tau_0\} \in N_0.$$

This uses only Separation and the forcing theorem, not a choice of deciding conditions for all y .

Now let $x = \tau^h \in N_0[h] \cap E_0$, where $\tau \in N_0$ is a forcing name. The forcing theorem over E_0 gives $p \in h$ with

$$p \Vdash_{E_0, \mathbb{P}} \tau = \check{x}.$$

In a further auxiliary extension, choose h' generic over $E_0[h]$ with $p \in h'$. The filters h, h' are mutually generic over E_0 , hence over N_0 , and $\tau^{h'} = x$. The preceding intersection argument gives $x \in N_0$. The conclusion concerns membership in the original ground and is unchanged by this auxiliary extension. Equivalently, this argument can be carried out with the forcing theorem for the further product; no existence of transitive models is inferred from consistency.

Apply this fact with $E_0 = D[G]$, $N_0 = N$, and $\mathbb{P} = \mathbb{P}_{\mathcal{U}}$. The old forcing is dense in normal Prikry forcing for the lifted measure over the choice model E_0 . Small forcing preserves the inaccessibility of κ , and the Prikry property gives

$$V_{\kappa}^{E_0[c]} = V_{\kappa}^{E_0}.$$

For clarity, no-new-bounded-subsets suffices here because E_0 satisfies Choice and each old rank below the inaccessible κ has size below κ ; induction on rank then proves the displayed equality. If $x \in V_{\kappa}^{N[c]}$, the inclusion $N[c] \subseteq E_0[c]$ therefore puts x in E_0 . Generic intersection now puts x in N . The converse inclusion is immediate. This proves (3) directly, without a bounded symmetric-name reduction or a recomputation of the old collapse after Prikry forcing. $\square$

5. TAGGED CAPS AND THE LABELING FORCING

All sequences, carriers, records, and diagram witnesses in this section belong to N . The word *old* means “in N ,” including when a construction is inspected in an auxiliary extension. All asymptotic requirements are unchanged by modifying finitely many initial blocks. Fix $m_0 \geq 4$ and, for $m \geq m_0$, put

$$I_m = [2^m, 2^{m+1}), \quad w_m = 2^{\lfloor \sqrt{m} \rfloor}, \quad G_m = 2^{\lfloor m^{1/4} \rfloor}, \quad J_m = [2^m + w_m, 2^m + 2w_m].$$

For $m < m_0$ use $I_0 = \{0, 1\}$ and $I_m = [2^m, 2^{m+1})$ for $1 \leq m < m_0$, put $J_m = \{2^m\}$ and $G_m = w_m = 1$, and give every row the value 2^m there. Thus the blocks partition all of ω , including the index 0. Include $E_* = [0, m_0)$ in every cap and diagram exceptional set. The point 0 and all data in the finite initial interval $[0, 2^{m_0})$ are harmless finite carrier data. Thus every core is a subset of its block, while all separation assertions concern the eventual tail. An eventually positive sequence d_m is *super- G* if

$$\frac{d_m}{G_m^k} \longrightarrow \infty \quad \text{for every fixed } k < \omega.$$

Witness reserves below may be taken to be positive integer sequences by changing an initial segment. The core width w_m is super- G . A *row* is an old function f with $f(m) \in J_m$ and super- G distances from the two core endpoints. Write

$$A_f = \{f(m) : m < \omega\}, \quad f \prec g \iff g(m) - f(m) \text{ is super-}G.$$

A *row chain* is a set of rows linearly ordered by $\prec$. In particular its pairwise order and separation requirements hold on every sufficiently late block, not only outside an exceptional set.

The ideal of label errors is

$$\mathcal{J} = \{a \subseteq \omega : |a \cap n| = o(\log n)\}^N. \quad (5.1)$$

If $a \in \mathcal{J}$, its occupied block set $\{m : a \cap I_m \neq \emptyset\}$ has asymptotic density zero, and $|a \cap I_m| = o(m)$. Indeed the number of occupied blocks below m is bounded by $|a \cap 2^{m+1}| = o(m)$, which also bounds the load of the m th block. Since $m/G_m \rightarrow 0$, this load is $o(G_m)$.

Definition 5.1 (Cap records and diagrams). A *cap record* consists of an old carrier $a \subseteq \omega$, a countable row support L_a , an old density-zero set $Z_a \subseteq \omega$, and, for every $m \notin Z_a$, a labeling of the *entire* finite set $a \cap I_m$ by members of L_a . It satisfies:

- (1) $|a \cap I_m| = o(G_m)$ on all blocks, and $a \cap I_m \subseteq J_m$ whenever $m \notin Z_a$;

- (2) for each fixed $f \in L_a$, on all sufficiently late blocks outside Z_a , the canonical point $f(m)$ belongs to a and has tag f .

A *finite diagram* consists of finitely many *whole cap records* and finitely many specified canonical row points. A witness for the diagram is an old density-zero exceptional set, containing the exceptions of its cap records, and a super- G reserve d_m . Outside that exceptional set, equal points have equal tags; the convex envelopes of distinct represented tags occur in row order; the gaps between consecutive distinct-tag envelopes and the two core margins are at least d_m . A canonical point $f(m)$ has tag f .

A single tag may occupy several widely separated pieces; its diameter has no imposed bound. The finite-diagram requirement nevertheless controls *all* tags occurring in each selected cap block, including tags whose canonical-point requirements have not yet stabilized. Clause (2) stabilizes separately for each cap-row pair, not uniformly over all its tags. On a record's exceptional blocks there is no tag or canonical-point requirement; only its all-block load bound remains.

Definition 5.2 (The forcing $\mathbb{R}$ and designated tops). A condition consists of a countable row chain L , a coloring $\ell : L \rightarrow X$, and a countable nonempty family of immutable cap records supported on L . Fix once and for all the distinguished empty cap with empty support, exception set E_* , and empty labelings, together with the default witness $(E_*, \langle w_m : m < \omega \rangle)$ for its one-cap diagram and for the empty diagram. Require every condition to contain this record and these same default witnesses. This family is *finitely directed modulo finite inclusion*: for any finitely many of its carriers $a_1, \dots, a_k$, it contains a carrier b with $a_i \subseteq^* b$ for every i . For every finite diagram of its caps and canonical rows, the condition includes a witness as in Definition 5.1. Extension retains the rows, their colors, all cap records, and all previously assigned diagram witnesses.

A listed cap b is a *designated top* of this condition if its support is the entire current row chain and

$$a \subseteq^* b \quad \text{for every listed cap } a, \quad A_f \subseteq^* b \quad \text{for every } f \in L.$$

These containments hold modulo a finite set of *points*, including on exceptional blocks. They are additional properties of a designated top, not requirements on every cap record.

The condition with no rows and one empty cap is a largest condition, after fixing harmless default witnesses. All countable families and their assigned witness functions are actual old sets included in the condition. Conditions admit uniform codes in X : the countably many colors have ranks bounded below λ , because $\text{cf}^N(\lambda) > \omega$, and the other countable coding data and finite rank overhead remain below λ . A countable decreasing chain of conditions has its literal union as a lower bound. Every pair of rows and every finite diagram occurs at a previous stage, so its requirement and witness persist; finite directedness modulo finite inclusion persists for the same reason. The use of DC here is to establish countability of the union of the old countable families, not to select a class of new diagram witnesses.

Lemma 5.3 (Dense tops and insertions). *The following extensions are dense in $\mathbb{R}$.*

- (1) *A condition can be extended by a designated top containing every previous cap and every current canonical row modulo finite sets.*
- (2) *A new row of any prescribed color can be inserted at any old cut in the countable current row chain.*
- (3) *Every old $a \in \mathcal{J}$ can be included in a cap, and every fixed integer translate of a listed cap can be included in a cap.*

Proof. *Tops.* Enumerate the caps as a_i and the rows as f_i ; when a list is finite, omit nonexistent entries in the formulas below. Let $\mathcal{D}_n$ be the diagram of the first n whole caps and first n canonical

rows. Take increasing old density-zero sets U_n containing their assigned diagram exceptions and cap exceptions. Choose increasing thresholds M_n , $n \geq 1$, such that, for $m \geq M_n$,

$$\sum_{i < n} |a_i \cap I_m| + n \leq G_m/n,$$

and, off U_n , the diagram $\mathcal{D}_n$ has reserve at least $4G_m^{n+4}$. We may also require $w_m \geq 4G_m^{n+4}$ and

$$|U_n \cap t|/t < 1/n \quad (t \geq M_n).$$

All these are finitely many eventual requirements at each step. On $M_n \leq m < M_{n+1}$ define

$$b \cap I_m = \bigcup_{i < n} (a_i \cap I_m) \cup \{f_i(m) : i < n\}.$$

Take arbitrary finite initial data, for instance empty blocks. Let E contain the initial block indices and consist thereafter of the m which belong to U_n on the n th band. If t is in that band, the exceptional indices from all earlier bands lie in U_n ; consequently

$$|E \cap t|/t \leq 1/n + O(1/t),$$

so E is old and density zero.

On a block outside E , use the labels of $\mathcal{D}_n$. This copies every point of each selected cap, including every transient high tag, and equal points have consistent labels. The load is at most G_m/n . Every fixed old cap and row is included on *all* sufficiently late blocks. Give b support L and exception set E . It is a valid record, with a super- G reserve, for example G_m^{n+2} on the n th band.

For a finite diagram involving b and specified old caps and rows, take the band index large enough to include all those old items. Their points are then already in b , with the same tags outside E , and their union diagram is a subconfiguration of $\mathcal{D}_n$. Thus E together with a finite initial segment and the same bandwise reserve gives a *new* witness. No old witness is changed. The enlarged cap family is directed modulo finite inclusion because b dominates all its members. This also proves a useful version of the construction: one may start with any countable compatible family of records and diagram witnesses, temporarily lacking directedness; adjoining the constructed top makes it directed.

Insertion at a cut. First choose a designated top b . Enlarge the construction's exceptional set to include the exception of b and the witness exception for the diagram consisting of the whole cap b . On a good block, partition its finitely many active tag packets according to the given cut. Choose the new point in the gap between the last left packet and the first right packet, using a core endpoint if a side is empty. If both are empty use the core itself. The gap or margin has a super- G reserve, and its integer midpoint is separated from both bounding packets or endpoints by half that reserve minus one.

On exceptional blocks, use a separate slow interpolation of canonical rows. Enumerate the old rows and activate the first n only after their finite all-block order and margin bounds exceed, say, $4G_m^{n+4}$. Between the last active row on the left and first on the right, with core endpoints at empty sides, choose an integer midpoint. The activation thresholds tend to infinity, and every fixed row is eventually active. This defines an old row with the required cut and super- G margins on the exceptional blocks. On good blocks, every fixed old row eventually has its canonical point in b with the correct tag; the packet-gap construction gives the same required order and separation there. The two estimates together give the all-block eventual row inequalities.

Give the new row the requested color. For each new finite diagram, eventually its old cap points and canonical anchors lie in b and agree with b 's tags outside the corresponding old diagram exceptions. The just constructed gap supplies the new row's separation from every packet. Finite unions of these exceptions and the midpoint reserve supply new witnesses. This extends the row

chain without changing any old record, after which the top construction supplies a new designated top.

Errors and translates. For $a \in \mathcal{J}$, use a record with carrier a , empty support, and exception set its occupied block set. It has no good-block points requiring tags and has load $o(G_m)$. It is compatible with every old finite diagram after adjoining this density-zero exception. Apply the compatible-family version of the top construction.

For a fixed $j \in \mathbb{Z}$ and a listed cap a , take the new carrier

$$a \cup (a + j), \quad a + j = \{k + j : k \in a, k + j \geq 0\},$$

with the same support, retaining the original tags and giving translated points their source tags. On good sufficiently late blocks, core margins and distinct-tag gaps exceed $|j|$ by a super- G amount. Thus the translated points remain in the core and inconsistent distinct-tag overlaps cannot occur. The original copy keeps each fixed supported canonical point. Adjoin the old exceptional block set, its adjacent block indices, and a finite initial segment to the new exception set. This covers all boundary crossings. The new load at block m is bounded by a fixed sum of the old loads at $m - 1, m, m + 1$, and is $o(G_m)$, since the ratios of these neighboring G -values are bounded. For a new finite diagram, use the old diagram including a and its other constituents; subtracting $|j|$ from its eventual gaps and margins leaves a super- G reserve. This supplies consistent new witnesses while retaining all old ones. Finally apply the compatible-family top construction. $\square$

The dense initial-segment presentation $X^{<\omega_1^N}$ of $\mathbb{C}$ projects onto $\mathbb{R}$. Fix the uniform coding of conditions by elements of X and start at the largest condition. At a successor entry, adopt the decoded condition if it is a valid extension of the current one; otherwise keep the current one. At a countable limit take the literal union. This is a set-defined recursion in N and makes no global choice of lower bounds. Every extension of the current decoded condition is reached by appending its code, which proves the projection lifting property. Lemma 4.1(1) gives the same relevant countable X -strings and condition codes in $N[c]$; hence this same old projection works there. Write g for the projected generic.

5.1. The finite matching lemma. For an increasing bijection between finite frozen subsets of two integer intervals, including their endpoints, say it is L -balanced if every pair of corresponding consecutive frozen gaps d, d' satisfies

$$\min(d, L) = \min(d', L).$$

The endpoints will carry two distinct formal sentinel tags, less and greater than every genuine tag, used nowhere else.

Lemma 5.4 (Finite tagged interpolation). *Let A_i be finite prescribed subsets of integer intervals $[u_i, v_i]$, including the endpoints, for $i = 0, 1$. Let $h : F_0 \rightarrow F_1$ be an increasing frozen bijection, with $F_i \subseteq A_i$, matching the endpoints. Choose a common finite linear tag order T amalgamating the two tag orders over the frozen tags, and suppose:*

- (1) *the tag maps on A_i are weakly increasing, and frozen matched points have equal tags;*
- (2) *distinct-tag envelope gaps and the sentinel margins in each prescribed configuration are at least the positive integer D ;*
- (3) *h is L -balanced, where $L \geq D$.*

Take T to consist of the tags occurring on at least one side, including the sentinels. Let $n \geq |A_0| + |A_1| + |T|$ and suppose $D \geq 100(n + 1)^3$. Then there are finite extensions $A'_i \supseteq A_i$ and an

increasing tag-preserving bijection $h' : A'_0 \rightarrow A'_1$ extending h such that, with

$$\varepsilon = \left\lfloor \frac{D}{100(n+1)^3} \right\rfloor,$$

the map h' is ε -balanced and every distinct-tag envelope gap and sentinel margin in the final configurations is at least ε . Every tag occurs on both sides, and the total number of added final points is at most $|A_0| + |A_1|$, in particular $O(n)$. Tags initially present on only one side receive constructed partner packets, not independently prescribed partners.

There is also an untagged version: under the balance hypothesis, the same bound on D and the prescribed point count suffice for an increasing ε -balanced extension retaining all points, with the same linear addition bound and no tag-separation conclusion.

Proof. We first give a two-pass untyped procedure. Suppose matched endpoint intervals have lengths either equal and less than R , or both at least R , with at most n prescribed interior points in total and $R \geq (n+1)^2$. If the lengths are equal and short, translate the two patterns into one interval, take their union, and translate back. Otherwise put $d = \lfloor R/(n+1) \rfloor$ and insert the prescribed source points, with successive positive gap lengths $g_0, \dots, g_k$, where $k+1 \leq n+1$. Assign target gaps the preliminary lengths $\min(g_j, d)$. Their sum is at most $(n+1)d \leq R$, no greater than the target interval length. At least one source gap is at least d : otherwise their sum, the source interval length, would be less than R . Put all leftover target length in the first such gap. The resulting map copies short gaps exactly and leaves both members of every other gap pair at least d .

Now insert the originally prescribed target points, working backwards in each resulting gap. Translate in equal gaps shorter than d ; in long gaps repeat the allocation with threshold $\lfloor d/(n+1) \rfloor$. The resulting map is balanced at

$$q = \left\lfloor \frac{R}{(n+1)^2} \right\rfloor.$$

Every allocation is positive, and new interior points cannot collide with old matched endpoints. At most one partner is added for each originally prescribed point. This proves the untyped procedure.

Lower the frozen balance threshold from L to D and work separately in consecutive frozen gaps. If the two endpoint tags of such a gap are equal, tag monotonicity forces every prescribed point in the gap to have that tag. Use the untyped procedure with $R = D$ and give all added points that tag. This includes every short frozen gap: distinct endpoint tags or a genuine tag next to a sentinel would force the gap to have length at least D .

It remains to treat a frozen gap with distinct endpoint tags $s < t$. Both corresponding gaps have length at least D . The frozen tags locate the same list of required tags between s and t on both sides. On one side, call the closed envelope of the prescribed points of a represented tag its *packet*. These packets are ordered and consecutive packets have separating gaps of length at least D . Put

$$R = \left\lfloor \frac{D}{4(n+1)} \right\rfloor.$$

For consecutive nonempty packets with last and first points $u < v$, let $h_0 \leq n$ be the number of intervening tags missing on this side. Extend the left packet's zone to $u + R$, assign the j th missing tag the zone

$$[u + 2jR, u + (2j+1)R] \quad (1 \leq j \leq h_0),$$

and start the right packet's zone at $v - R$. Every empty guard between successive zones has length at least R , because

$$(2h_0 + 3)R \leq D \leq v - u.$$

At a frozen endpoint, begin or end the corresponding endpoint-tag zone at that endpoint. Thus every zone contains its entire prescribed packet and has length at least R ; an absent tag gets a zone of length R . No packet is compressed, whatever its diameter. Apply this on both sides with the same ordered tag list and match the two endpoints of corresponding zones and guards. The resulting skeleton extends the frozen endpoint map and is R -balanced. Sentinel zones are temporary: besides the original outer endpoint they contain only an artificial inner endpoint.

In each matched same-tag zone apply the untyped procedure to the original prescriptions and label each added point by that tag. Our bound on D gives $R \geq (n+1)^2$. Each active tag occurs on at least one side, so it acquires a partner point if it was missing on the other side. Leave the guards empty. The refined skeleton is balanced at

$$\left\lfloor \frac{R}{(n+1)^2} \right\rfloor = \left\lfloor \frac{D}{4(n+1)^3} \right\rfloor \geq \varepsilon.$$

Every artificial zone endpoint lies in a free interpacket gap; its partner is artificial too. Delete these endpoints in matched pairs, retaining the original frozen endpoints and all original prescriptions. A resulting consecutive gap is a sum of corresponding skeleton gaps. If a constituent pair is long at the displayed threshold, both sums are long; otherwise every constituent pair is equal, so the sums are equal. Balance therefore survives deletion. An entire empty guard remains between distinct final tags, including a sentinel and its neighboring genuine tag, so the output separation and margins survive as well.

Every added final point is the partner of an original prescription; artificial skeleton points have been removed. This gives the asserted linear bound. Adjacent frozen gaps agree at their shared frozen endpoint, so their maps glue. For the untagged version apply the untyped procedure with $R = D$ separately in each frozen gap and glue, omitting zones altogether. $\square$

Remark 5.5. Translation is used only in genuinely short equal same-tag gaps at the *current* threshold. It is unnecessary and can be incorrect to require translation in every gap shorter than the original L after the threshold has been lowered to D . The tag-preserving conclusion also does not identify two separately prescribed same-tag canonical points unless that pair was included in the frozen map.

For application to caps, adjoin the core endpoints as sentinels and take D_m no larger than the diagram reserve, the frozen map's balance reserve, and w_m . Each selected finite diagram uses entire prescribed cap blocks and only finitely activated canonical anchors. The number of its active tags is bounded by its point count, so these data have $n_m = o(G_m)$ after sufficiently slow activation. If D_m is super- G , then so is

$$\left\lfloor \frac{D_m}{100(n_m+1)^3} \right\rfloor : \quad \frac{D_m}{(n_m+1)^3 G_m^k} \geq \frac{D_m}{G_m^{k+3}} \longrightarrow \infty$$

for each fixed k , on all sufficiently late blocks. Thus the linear load increase remains $o(G_m)$ and the polynomial reserve loss remains super- G . A growing collection of diagram requirements must still be activated by an old slow diagonal; an arbitrary growing minimum of reserves is not asserted to be super- G . The finite construction is uniform, using the displayed leftmost allocations and the first long gap for leftover length, so old finite input arrays produce old output arrays. On exceptional blocks use the untagged version. Any finitely many initial blocks can be handled separately when map extensions are required only eventually, while retaining the prescribed root points.

6. PAIRED FORCING AND LITERAL COMPARISON

Throughout this section, all conditions, rows, cap records, and block maps are old, that is, belong to N . A designated top of a condition is a listed cap whose support is the entire row chain

and which contains each listed cap and each canonical row carrier modulo finite sets. The last containment holds on all sufficiently late blocks, including exceptional blocks. Finite directedness of cap families is under modulo-finite inclusion.

Fix a root condition r_* with a designated top b . To fix finitely many old carriers, first strengthen the root and pad a new top to contain them literally. No existing cap record or assigned witness is changed.

Definition 6.1. A paired condition over r_* consists of conditions $r_0, r_1 \leq r_*$, an order- and color-preserving bijection $\pi : L_0 \rightarrow L_1$ between their row chains fixing the root rows, designated tops B_0, B_1 containing b literally, and old increasing bijections

$$h_m : B_0 \cap I_m \rightarrow B_1 \cap I_m.$$

The maps fix b pointwise. Outside the core J_m the two tops agree and the maps are the identity. With core endpoints adjoined as sentinels, the maps are L_m -balanced on every block, for an old super- G sequence $\langle L_m : m < \omega \rangle$.

There is one old density-zero set E containing the exceptional sets of the two tops such that h_m transports all top tags by π whenever $m \notin E$. Moreover, for every fixed $f \in L_0$ there is $M_f < \omega$ such that

$$h_m(f(m)) = \pi(f)(m) \quad (m \geq M_f, m \notin E). \quad (6.1)$$

These canonical values belong to their respective tops on all sufficiently late blocks by the separate designated-top requirement. There is no canonical-match requirement on blocks in E . Each side's row chain still has super- G margins and row inequalities on all sufficiently late blocks.

An extension extends both side conditions and the row bijection, retaining all their immutable data. For some $M < \omega$, its block map extends the earlier map on its entire domain whenever $m \geq M$. Its comparison exception set and balance reserve may be newly supplied.

The exception in (6.1) is essential. Let $Z \subseteq [m_0, \omega)$ be an infinite old density-zero set, let $p(m)$ be the midpoint of J_m for $m \geq m_0$, and set $p(m) = 2^m$ on the initial blocks. Give $b = \{p(m) - 1, p(m) + 1 : m \in Z\}$ empty support and exception set $Z \cup E_*$. This is a valid root cap. Insert the row p on both sides with different colors. Matching its canonical point on every sufficiently late block, while fixing b , would force its partner to equal $p(m)$ on all sufficiently late $m \in Z$. The differently colored partner must be distinct from the native row p on that side, contradicting their super- G separation. Matching off E avoids exactly this obstruction.

Lemma 6.2 (Paired-cap lifting). *The paired forcing is countably closed in N . Both coordinate projections have the lifting property for arbitrary one-sided extensions. Any two extensions of r_* can be strengthened to the two coordinates of a paired condition.*

Proof. We give the scheduling and record verification as well as the three constructions.

Old diagonal scheduling. Consider countably many old density-zero sets, super- G reserves, eventual equalities and containments, and $o(G_m)$ load functions. At stage $n \geq 1$ select finitely many requirements, increasingly so that each fixed one is eventually selected. Requirements concerning a particular old top or map selected at stage n may also be included. Let U_n be the union of all exceptions selected up to stage n . Choose strictly increasing M_n such that, for $m \geq M_n$, all selected eventual assertions hold, the selected total point load and number of active anchors are at most G_m/n , and every selected reserve and w_m are at least CG_m^{n+10} . Here C is a fixed sufficiently large constant for Lemma 5.4. Also require

$$|U_n \cap t|/t < 1/n \quad (t \geq M_n).$$

Each choice is possible because it involves only finitely many asymptotic requirements. Set $s(m) = n$ on $[M_n, M_{n+1})$ and

$$E = \{m \geq M_1 : m \in U_{s(m)}\},$$

with finitely many initial blocks added if necessary. If t is in the n th band, all earlier exceptional blocks after M_1 belong to U_n , so $|E \cap t|/t \leq 1/n + M_1/t$. Thus E is old and density zero; each fixed selected exception is contained in E modulo finite.

The bandwise load is $o(G_m)$. The point-and-tag count is bounded by a fixed multiple of the total whole-cap load, the active anchors on both sides, and the sentinel count. The G_m/n schedule therefore gives $\nu_m \leq G_m$ eventually. For this count and input reserve $D_m \geq CG_m^{s(m)+10}$, enlarge the thresholds to accommodate rounding and obtain

$$\left\lfloor \frac{D_m}{100(\nu_m + 1)^3} \right\rfloor \geq G_m^{s(m)+5}.$$

This is super- G , and absorbs the polynomial loss of Lemma 5.4. Only finite minima are used at each stage. The exception set is a diagonal pseudounion, not a countable union.

The row order and canonical anchors. For one-sided lifting use the stronger source row order as an abstract order T , identifying old target rows by the existing row bijection. For initial two-sided amalgamation, amalgamate the native orders over the root: compare their cuts in the root order, and within a single cut put the foreign left rows before the foreign right rows, preserving each native order. Opposite non-root rows remain distinct abstract labels even when their native old functions coincide. Keep the inherited colors.

Every label is native on at least one side; a label native on both is already identified. Choose increasing finite H_n exhausting T . Include their native canonical-anchor diagrams, and the eventual containments and matches of identified anchors, in the schedule. On a good block in the n th band prescribe the *entire* input top blocks and the native anchors from H_n . Previously identified anchors are frozen matched pairs after their stabilization bounds and off their selected exceptions. Apply the tagged finite lemma. Define a missing canonical value to be the image or inverse image of its native anchor, with the intended tag.

On an exceptional block, retain each side's native anchors from H_n and independently interpolate the missing labels in the order T , using core endpoints as sentinels. Finite native chains have all-block super- G gaps and margins. Divide each gap among its finitely many missing labels; the schedule makes the resulting integer gaps and margins at least G_m^{n+5} . Then include *all these canonical values on both sides* in the prescribed point sets for the untyped finite interpolation. This puts the canonical points in their tops on exceptional blocks too, although their images need not be their canonical partners there. The additional cost is linear in $|H_n|$.

Before a missing row becomes active, assign it any fixed interior value on blocks $m \geq m_0$, and the required value 2^m on the initial blocks. Each fixed row, and each fixed pair, is permanently active eventually. The two block constructions therefore give old row functions with all-block super- G margins and pairwise inequalities in precisely the order T . A high-index tag in an input block receives its partner *packet* immediately, since the whole block is prescribed; its canonical anchor is activated later.

One-sided lifting. Given a paired condition with tops B_0, B_1 and map h , strengthen its first side arbitrarily and normalize the request to a designated top A_0 . On each sufficiently late block prescribe the entire A_0 block, the entire B_0 block if not already included, the entire B_1 block, and the active native anchors. Freeze h and the core sentinels. Outside the core add the union of the prescribed points on both sides and use the identity.

The source diagrams involving A_0, B_0 and active native anchors, and the target diagrams involving B_1 and its native anchors, supply old exceptions and reserves. Include the old paired exception and the selected canonical stabilization bounds. The old map's balance reserve, these diagram reserves, the native-row gaps, and w_m give the finite reserve minimum for the schedule. The prescribed loads are $o(G_m)$. The good- and exceptional-block constructions above consequently apply. The new maps eventually extend the entire old maps, fix b , and have all-block

super- G balance. They supply the extended target chain and color-preserving row bijection without changing a native row.

Initial two-sided amalgamation. Normalize the two given conditions to tops A_0, A_1 containing b literally. A previous top contains b modulo finite, so padding a new top by the finite remainder suffices. Amalgamate the abstract row orders as above, freeze the identity on b and the sentinels, and apply the same construction to the two entire top blocks. Include the finite diagrams comparing b and the native input tops. On exceptional blocks interpolate the active anchors separately, insert all their values into the untyped point sets, and retain the identity at b .

In both constructions use the identity on the union of all prescribed points on finitely many initial blocks if necessary. Non-root map extension is only eventual; the root remains fixed throughout. Sentinels are auxiliary: for each matched pair of core endpoints, retain both copies if either was a prescribed carrier point, and otherwise remove both copies from the new tops. This preserves the bijection. Prescribed good-block points are away from the endpoints by the margin requirement. Balance is measured with the sentinels adjoined. Numerical reserve requirements are asymptotic.

Records and immutable witnesses. Let C_i be the new top carriers, with full new row supports, the good-block tags just constructed, and exception set E . Their loads are $o(G_m)$ and their good-block distinct-tag and margin reserves are super- G . Every fixed canonical row belongs to its top on all sufficiently late blocks, including exceptional ones, and has its own tag on good late blocks. These are valid designated-top records.

Keep every native old record, labeling, exception set, and assigned finite-diagram witness literally. Each native cap a lies modulo finite in its input top and hence in C_i . A transported record on the other side, if included, has the exact carrier

$$a' = \bigcup_m h_m[a \cap C_i \cap I_m]$$

(or the corresponding inverse image), and transported support. Its exceptions include E , those of a , the old diagram exception comparing a to its input top, and a finite initial containment error. Off that finite union its labels are consistent with the top's labels, and each fixed supported row has its canonical point eventually. Its load is at most that of a . Do not add canonical corrections on exceptional blocks.

For a new finite diagram, unite the relevant finitely many exceptions and enlarge its finite initial segment. Beyond it, all prescribed points and correctly tagged canonical anchors lie in C_i , and their tag envelopes are sub-envelopes of the new top's packets. The top reserve witnesses the diagram. Assign these new witnesses without changing any old one. Each new top dominates all native and transported records modulo finite, so the cap family is finitely directed. This verifies the lifting constructions as actual side extensions.

Countable closure. Let $t_0 \geq t_1 \geq \dots$ be an old decreasing sequence of paired conditions. Unite the side conditions, row bijections, and assigned witnesses. Each finite diagram occurred at a stage, so the side unions are conditions. Enumerate their caps, rows, finite diagrams, earlier maps, and exceptions.

At stage n choose a later $t_{k(n)}$ containing the first n enumerated items, with $k(n)$ increasing, and use its *whole* designated tops and map. Include in the schedule their eventual containment of the first n caps and canonical rows on all blocks; their agreement with the first n earlier maps; the first n canonical matches off their comparison exception; the diagrams of the *selected tops* together with the first n old caps and canonical anchors; their top and diagram exceptions; their map and diagram reserves; and their two top loads. Agreement comes from the extension order. Choose M_n by the schedule above.

On the n th band copy the selected whole top blocks and map. Every earlier map is eventually extended; every old cap and canonical row is eventually contained; and each fixed canonical match holds eventually off the diagonal E . The root and the identity off the core persist. Loads are $o(G_m)$, and all-block balance reserves are super- G . Copy the selected top labels. For every finite diagram involving a new top and specified old records, the corresponding diagrams with the selected stage top have been scheduled from some stage onwards. Their copied blocks and reserves witness the new-top diagram off E and a finite initial segment. Add only the two new tops and their new witnesses. They restore directedness and give a paired lower bound. No block contains an infinite pointwise union of cap loads. $\square$

6.1. The objects retained in the inner model. Let $g \subseteq \mathbb{R}$ be generic and let ℓ_g be its union coloring. Define

$$\mathcal{I}_g = \{a \in \mathcal{P}(\omega)^N : a \subseteq^* b \text{ for some cap } b \text{ appearing in } g\}.$$

Finite directedness and Lemma 5.3 make this ideal hereditary under old subsets, closed under finite unions and integer translates, and containing $\mathcal{J}$. Let $Q(c, g)$ consist of the increasing enumerations of finite modifications, using arbitrary ordinals below κ , of $c[a]$ for infinite old $a \in \mathcal{I}_g$.

Fix $x_0 \in X$. For $q \in Q(c, g)$ set

$$\iota_c(q) = \{n : c(n) \in \text{ran}(q)\}.$$

If a generic row f and an integer j satisfy $\iota_c(q) \triangle (A_f + j) \in \mathcal{J}$, put $F(c, g)(q) = \ell_g(f)$. If there is no such row and translate, put $F(c, g)(q) = x_0$. The index carrier differs finitely from an old carrier, so is old. Distinct rows have disjoint translated $\mathcal{J}$ -clouds: their super- G separation gives distinct canonical points on every sufficiently late block, and one point on each such block is not a $\mathcal{J}$ -error. Thus the definition is unambiguous. Lemma 5.3(2) makes F onto X .

A carrier already supported by a cap cannot acquire the cloud of a later row outside its recorded support. The finite diagram of the whole cap and that row separates the canonical point from all cap points on a density-one set of blocks. The reserve eventually exceeds any fixed translating integer. The difference from the translated row therefore has a point on a density-one set of blocks and is not in $\mathcal{J}$. This stabilizes localized label decisions.

Every member of Q is Prikry generic over N . For the ambient generic, verify this in the collapse-first choice cover $N[K]$, which contains every old carrier: c is generic there for the lifted measure, and Mathias's criterion, its infinite-subsequence consequence, and finite-change invariance apply. The old measure base gives genericity for $\mathbb{P}_{\mathcal{U}}$ and hence over N ; see [2, Section 3]. Only the evaluations of the fixed N -names Q, F are retained. The row order, tags, cap records, and master sequence are not separately adjoined.

Lemma 6.3 (Literal common-model comparison). *Let $r_0, r_1 \leq r_*$ and let $p_0, p_1 \in \mathbb{P}_{\mathcal{U}}$ be compatible. Fix finitely many supported Q -parameters whose old carriers lie literally in the root top b , with their finite ordinal decorations. In an auxiliary extension there are genuine $\mathbb{P}_{\mathcal{U}} \times \mathbb{R}$ -generics $(c_0, g_0), (c_1, g_1)$ over N , meeting the respective conditions, such that*

$$Q(c_0, g_0) = Q(c_1, g_1), \quad F(c_0, g_0) = F(c_1, g_1),$$

and the supported parameters are literally equal. In particular the two models $N(Q, F)$ are literally the same inner model there.

Proof. Work in the full choice cover of Section 4. Lemma 6.2 gives a paired condition whose coordinates strengthen r_0, r_1 . Force below it with the *old* paired forcing. Its lifting assertions involve fixed old condition sets and remain valid in this cover. Consequently each coordinate is a projection below that condition: for any dense open subset of the old $\mathbb{R}$ in the cover, strengthen

the selected coordinate into it and lift this strengthening. The resulting marginal filters g_i are genuinely $\mathbb{R}$ -generic over the cover and over N .

The old paired forcing and its data have size below κ in the cover by (4.1). A further forcing of size below κ makes its generic filter countable. Enumerate that filter and use its directedness to take a cofinal decreasing sequence. It need not belong to N . Its maps eventually extend each fixed earlier map. By copying one whole map per block beyond selected agreement thresholds, obtain an increasing block-preserving partial bijection h of ω , fixing b , which eventually agrees with each old top map in the filter.

Old localization of the carrier action. For $a \in \mathcal{I}_{g_0}$ choose an old paired condition t in the filter whose designated top contains a modulo finite. Directedness of the filter and the top property give such a t . Then

$$a' = h^t[a \cap B_0^t] \in N, \quad a' \subseteq B_1^t, \quad h[a \cap \text{dom}(h)] =^* a',$$

and $a \setminus \text{dom}(h)$ is finite. Thus a' is an old allowed carrier on the other side. The inverse argument works there. The possibly new real h has been used through one old map on each specified carrier.

For an old $d \in \mathcal{J}$ its localized image is again in $\mathcal{J}$: if $n \in I_m$, then

$$|h^t[d \cap B_0^t] \cap n| \leq |d \cap 2^{m+1}| = o(m) = o(\log n).$$

For a fixed generic row f and integer j , choose an old paired condition t containing f whose first top contains both A_f and $A_f + j$ modulo finite. Translate absorption, dense tops, and lifting make these requirements dense. Write $q = \pi^t(f)$. Once the balance reserve exceeds $|j|$,

$$h_m^t(f(m) + j) = h_m^t(f(m)) + j.$$

Indeed, each consecutive domain-top gap between these points is shorter than the reserve and has its length copied exactly. The core margins keep both endpoints in the same block eventually. The fixed row's canonical match holds eventually off the old E_t . Its image and $A_q + j$ therefore differ, modulo finite, by at most two points in each block in E_t . Up to block m there are at most $2|E_t \cap (m+1)| + O(1) = o(m)$ such points. This is an old $\mathcal{J}$ -error. Together with the error-image estimate it transports the entire translated cloud, preserving the color. Here, for a particular carrier in that cloud, choose one paired condition whose top covers that carrier, the canonical translate, and the specified error set; the finite collection of requirements is met by directedness and dense tops. Thus the same old map handles the whole comparison. Apply the inverse argument too. This uses a fixed row's error, not a countable union of errors. Whole exceptional cap blocks are transported literally, not discarded.

Genuine Prikry and product genericity. Choose a common strengthening $p_* = (s, A)$ of p_0, p_1 , including a common reservoir as well as a common stem. Normalize finitely many initial blocks of h to the identity on the whole block, including all integers below $|s|$. This preserves eventual agreement with each old map and still fixes b .

Amalgamate two ordered copies of ω , identifying n on the first with $h(n)$ on the second. Between identified points, interleave the finite remainders in any fixed order preserving the order of each side; do likewise in an empty block. Since h preserves finite blocks, the resulting order has type ω . Its increasing cofinal embeddings $e_i : \omega \rightarrow \omega$ satisfy

$$e_0(n) = e_1(h(n)) \quad (n \in \text{dom}(h)), \quad e_0(k) = e_1(k) = k \quad (k < |s|).$$

No cross-side points are identified except the specified h -pairs: in particular, $e_0(n) = e_1(m)$ if and only if $n \in \text{dom}(h)$ and $h(n) = m$.

All preparation precedes the Prikry forcing and has total forcing size below κ over the original choice ground. The old normal measure generates its normal lift to this prepared choice universe, with the old measure-one sets as a base. Choose a master sequence z Prikry generic over that entire universe through (s, A) , and put $c_i = z \circ e_i$. The maps e_i are in the prepared universe, so

Mathias's criterion makes each c_i Prikry generic there. The dense old presentation gives genericity for the old $\mathbb{P}_U$ as well. Each c_i contains s and has all subsequent entries in A , so meets p_i .

Thus c_i is $\mathbb{P}_U$ -generic over $N[g_i]$. Since g_i is $\mathbb{R}$ -generic over N and these are fixed old posets, the product lemma makes (c_i, g_i) genuinely $\mathbb{P}_U \times \mathbb{R}$ -generic over N . No Prikry generic was chosen before the comparison preparation.

Literal retained equality. For corresponding carriers, $c_0[a] =^* c_1[a']$. Closure under arbitrary finite ordinal modifications in the fixed definitions gives equality of the sets Q . For any q in this common set, the exact cross-side identification just proved gives

$$\iota_{c_1}(q) =^* h[\iota_{c_0}(q) \cap \text{dom}(h)].$$

Indeed, all but finitely many values of q lie on its allowed carrier in the first master, and that carrier is almost contained in $\text{dom}(h)$; the only common master values are the identified pairs. Both index carriers are old. The cloud correspondence and its inverse now give equality of the two functions F on every such q , including their default-label regions.

Because h fixes b , the two subsequences on every protected carrier $a \subseteq b$ agree literally. The same finite ordinal decorations therefore give the same actual parameters. These are equal evaluations of the fixed N -names Q, F in one ambient universe, so their generated models $N(Q, F)$ are literally equal. The carrier and error ideals and the definition of Q are never recomputed using new auxiliary reals. $\square$

7. RANK PRESERVATION AND DEPENDENT CHOICE

Put

$$M = N(Q, F), \quad Q = Q(c, g), \quad F = F(c, g),$$

using the literal evaluations of the fixed names from the preceding section. Thus $N \subseteq M \subseteq W$, and $M \models \text{ZF}$ by transitive-base constructibility.

Lemma 7.1 (Eliminating the labeling coordinate). *If $Y \in N$, $a \subseteq Y$, and $a \in M$, then $a \in N[c]$.*

Proof. By Lemma 2.2, fix a bounded set-sized evaluator for a . Its inputs are the whole sets Q, F , finitely many actual members $q_0, \dots, q_{d-1}$ of Q , fixed ground parameters, and fixed ordinal codes. Each q_i is the increasing enumeration of a finite ordinal modification of $c[a_i]$, for an old carrier a_i supported by a generic cap. Choose a product condition (p_*, r_*) in the actual generic which verifies these descriptions and the required presentation. Dense tops and finite padding give a further condition *in the actual generic* whose designated top b contains every a_i literally: first cover the finitely many carriers modulo finite and then add their remaining finite points to the new top. Rename this strengthened condition (p_*, r_*) . The old records are retained; only the newly adjoined top is padded. All the finite ordinal decorations are fixed parameters.

Let $\dot{a}$ be the name obtained from this evaluator and the fixed names for its inputs, using a harmless default outside the chosen localization. For $y \in Y$, define in N

$$\begin{aligned} E_y^+ &= \{p \leq p_* : \exists r \leq r_* (p, r) \Vdash \check{y} \in \dot{a}\}, \\ E_y^- &= \{p \leq p_* : \exists r \leq r_* (p, r) \Vdash \check{y} \notin \dot{a}\}. \end{aligned}$$

Their union is dense below p_* . Indeed, start with (p, r_*) and strengthen it to decide the indicated statement. No member of E_y^+ is compatible in the old Prikry order with a member of E_y^- . Otherwise Lemma 6.3, applied to their labeling witnesses, gives two genuine product-generic realizations with the same literal N, Q, F and the same supported q_i , as well as the same ground and ordinal parameters. The evaluator in Lemma 2.2 uses satisfaction only for the same set-sized constructibility levels, so its value is identical in both realizations. Opposite membership decisions are impossible.

Form the old Prikry name

$$\dot{b} = \{(\check{y}, p) : y \in Y, p \in E_y^+\}.$$

Its actual value is a . For every y , the truth lemma supplies a deciding product condition in the actual generic below the localization, and hence the actual sign in the appropriate $E_y^\pm$. The incompatibility just proved excludes the opposite sign in the Prikry filter. Thus $a = \dot{b}^c \in N[c]$. No family of maximal antichains has been selected. $\square$

Corollary 7.2. $\mathcal{P}(X)^M = \mathcal{P}(X)^N$ and $V_\lambda^M = X$.

Proof. Every M -subset of X belongs to $N[c]$ by Lemma 7.1. Its rank is at most $\lambda + 1 < \kappa$, so Lemma 4.1(3) puts it in N . This proves the power-set equality. If there were a new member of V_λ^M , choose one of least rank. All its members would belong to $V_\lambda^N = X$, making it a new subset of X , a contradiction. $\square$

Lemma 7.3 (External countable domination). *For every $\langle q_n : n < \omega \rangle \in W$ with $q_n \in Q$, there is an increasing $q^* \in Q$ such that*

$$\text{ran}(q_n) \subseteq^* \text{ran}(q^*) \quad (n < \omega).$$

Proof. In W , choose old carriers witnessing the q_n and conditions of g containing their supporting caps. Since g is a filter and $W \models \text{ZFC}$, choose a decreasing sequence $\langle r_n : n < \omega \rangle$ in g , each term below the witnesses chosen so far. The labeling conditions have codes in X , as verified in Definition 5.2. Lemma 4.1(1) returns the whole code array to N . Decoding there shows that this is an old decreasing sequence of actual conditions of $\mathbb{R}$.

For any old decreasing sequence $\langle r_n \rangle$, the set

$$D_{\vec{r}} = \{r \in \mathbb{R} : (\exists n \ r \perp r_n) \text{ or } (\forall n \ r \leq r_n)\}$$

is old and dense. Given a condition, recursively seek a common refinement with the successive r_n . If a step is impossible, the current condition is incompatible with that r_n . If every step succeeds, DC in N gives a descending sequence and countable closure gives a lower bound below them all. Since all r_n belong to g , a member of $g \cap D_{\vec{r}}$ must be below every r_n .

The dense-top lemma now gives a further member of g carrying a top b which contains, modulo finite, all the carrier witnesses. Let q^* be the increasing enumeration of $c[b]$. It belongs to Q , and its range contains each $\text{ran}(q_n)$ modulo the finite carrier errors and the finite ordinal decorations. This proves the assertion. The argument uses genericity only after the descending sequence has been shown to be old. $\square$

Lemma 7.4. $M \models \text{DC}$.

Proof. We first prove that every W -countable family from Q has an M -wellorderable cover contained in Q . For $\langle q_n : n < \omega \rangle$ choose q^* as in Lemma 7.3, and put

$$u_n = \{k < \omega : q^*(k) \in \text{ran}(q_n)\}.$$

The array $\langle u_n : n < \omega \rangle$ is coded by a single real of W , hence belongs to N by Lemma 4.1(1). Each u_n is infinite. Let v_n be the increasing enumeration of the set $q^* \restriction u_n$, equivalently the restriction of q^* to u_n reindexed by the increasing enumeration of u_n . The sequence $\langle v_n \rangle$ is definable in M from q^* and the old mask array, so belongs to M .

Take all increasing enumerations of finite modifications of the ranges of the v_n , allowing arbitrary finite ordinal additions and deletions below κ , and intersect this family with Q . Call the resulting set C . It contains every q_n : only finitely many values of q_n lie outside $\text{ran}(q^*)$. The set C is an image of a wellorderable set of parameters consisting of $n < \omega$ and finitely many ordinals below κ , with finite operation codes. Least preimages give a wellordering of C in M . This conclusion does not say that the original external sequence $\langle q_n \rangle$ belongs to M .

Let $R \in M$ be a serial relation on a nonempty set $Y \in M$, and prescribe $y_0 \in Y$. Choose a presentation in M

$$\pi : \eta \times Z \times Q^{<\omega} \rightarrow Y, \quad Z \in N,$$

as in Lemma 2.2. Seriality is absolute to W , since it quantifies only over the fixed sets Y, R . In $W \models \text{ZFC}$, choose an R -chain starting at y_0 and an argument of π for each entry. Lemma 4.1(2) gives an N -wellorderable set $B \subseteq Z$ covering the resulting countably many ground coordinates; intersect the cover with Z if necessary. Flatten the finite Q -tuples into one countable family and use the preceding paragraph to obtain its M -wellorderable cover $C \subseteq Q$. Then

$$Y_0 = \pi[\eta \times B \times C^{<\omega}]$$

belongs to M , is wellorderable there by least preimages, and contains the chosen external chain.

Inside M , form the tree T of finite R -chains in Y_0 starting at y_0 . Write $t \triangleleft s$ precisely when t properly extends s , so longer chains must receive smaller values of a rank function for this relation. This is a wellorderable set. If the extension relation were not well founded, there would be a nonempty subset of T with no terminal member in that subset. Recursively choose the least proper extension in that subset, using the fixed wellordering. This constructs an infinite branch in M without any choice principle. Consequently, absence of an internal branch would make the extension relation well founded and give it an ordinal rank function by well-founded recursion in ZF. That same function remains a rank function in W and rules out the chosen external branch. Thus an R -chain starting at y_0 exists in M , proving DC. $\square$

Proposition 7.5 (The sequence seed). *The model M satisfies $\text{ZF} + \text{DC} + \text{PCMeas} + \text{SVC} + \text{KW}_1^{*,\omega}$. There are an ordinal μ and a set $A \subseteq \mu^\omega$ with $M = D(A)$. Moreover, D is a symmetric ground of M .*

Proof. Because $N \models \text{SVC}(X)$ and $F : Q \rightarrow X$, the presentation in Lemma 2.2 gives, for every nonempty $Y \in M$, a surjection

$$\eta \times Q^{<\omega} \rightarrow Y$$

after increasing the ordinal factor. More explicitly, cover its ground bank Z by $\xi \times X$, replace that X -coordinate by $F(q)$, and combine the two ordinal factors and the finite Q -coordinates. Thus the single set $Q^{<\omega}$ is an SVC witness in M .

For each such Y , choose an ordinal μ large enough to contain η, κ and the finite coding symbols. An ordinal coordinate and a finite tuple of members of $Q \subseteq \kappa^\omega$ inject into μ^ω : code the tuple length and the ordinal in initial positions and interleave its finitely many sequences thereafter. Transfer the displayed surjection to the image of this injection, and extend it to all of μ^ω using one fixed element of Y as a default. This proves $\text{KW}_1^{*,\omega}$ with the power μ^ω computed internally in M .

To obtain one sequence seed for the whole model, define

$$R(p, q) \iff F(p) \in F(q) \quad (p, q \in Q).$$

Transitivity of X and surjectivity of F give

$$F(p) = F(q) \iff \forall t \in Q [R(t, p) \leftrightarrow R(t, q)].$$

Indeed, distinct sets in the transitive set X have a distinguishing member in X , and that member has an F -preimage. Hence equality of predecessor sets is an equivalence relation whose quotient relation is well founded and extensional, and is isomorphic under $[q] \mapsto F(q)$ to membership on X .

Use different finite tags for domain codes and relation codes, and interleave pairs of sequences. This produces a set $A \subseteq \mu^\omega$ which codes the whole domain Q and the whole binary relation R , for one sufficiently large ordinal μ . In $D(A)$ these two sets can be decoded. Their quotient relation is still well founded and extensional, so its unique Mostowski collapse recovers the same literal X

and the map F . Consequently $D(A)$ contains $N = D(X)$ and then $M = N(Q, F)$. Conversely $A \in M$ and $D \subseteq M$, so minimality of transitive-base constructibility gives $D(A) \subseteq M$. Therefore $M = D(A)$.

The full choice cover $D[G][c][K]$ is a set-forcing extension of D . Above the size of its forcing, every normal measure in D has its normal generated lift in this cover. For each such ground measure $\mathcal{U}$ on δ , its restriction to M is the set

$$\{a \in \mathcal{P}(\delta)^M : \exists b \in \mathcal{U} (b \subseteq a)\},$$

which belongs to M by Separation. It is a nonprincipal δ -complete ultrafilter: every M -sequence of fewer than δ members is a sequence in the full cover, where the generated lift is complete. The intersection belongs to M and satisfies the same ground-witness definition. Hence $M \models \text{PCMeas}$. Only a proper-class tail is needed; the selected κ was singularized.

Finally, $D \subseteq M \subseteq D[G][c][K]$ and $M \models \text{SVC}$. The intermediate symmetric-model theorem, in the form of [7, Corollary 5.10], makes D a symmetric ground of M . This conclusion is not being inferred just from the notation $D(A)$. $\square$

Proof of Theorem 1.3. Start with the $\text{ZF} + \text{DC} + \text{SVC} + \text{PCMeas}$ model N in the local statement. Proposition 3.1 supplies a symmetric-ground presentation of this same N over a choice ground $D \models \text{ZFC} + \text{PCMeas}$. Choose the protected rank $X = V_\lambda^N$ and the measurable κ as in (4.1), with λ above the required rank. The construction gives $N \subseteq M$, and Corollary 7.2 proves $\mathcal{P}(X)^M = \mathcal{P}(X)^N$ and $V_\lambda^M = X$. Lemma 7.4 and Proposition 7.5 give the required theory and the presentation $M = D(A)$ with D a symmetric ground. This proves the local theorem. Sections 8–9 now remove SVC and Foundation from the starting assumptions of the main structural-conservativity theorem. $\square$

8. THE CORE-MODEL PREPARATION REMOVES SVC

This section proves the preparation used in the earlier research [1]. We specify the relative construction and the applications of classical mouse theory; in particular, neither SVC nor a stronger unrelativized anti-large-cardinal hypothesis is assumed.

8.1. Classical mice over a fixed real. We use the fine coherent-measure mice of [8, Definition 3.24], over a real r . The bottom predicate is fixed by all iteration maps, and measure critical points are above its rank. Relative soundness, projecta, and hulls are computed in this expanded language. The fixed-base version of initial-segment comparison is used throughout: sound iterable mice projecting to their common base are initial-segment comparable. This is the relative form of [8, Lemma 3.39]; see also [10, Section 1.1]. A *selected r -sword* is the minimal sound, base-projecting iterable r -mouse with a coherent upper measure of order one. Its distinguished lower and upper measures are both total over the passive sword universe; the lower is an element and the upper is the active predicate. No proper initial segment has the order-one obstruction. We use the usual least-segment and sound-core normalization of this obstruction. Selection is of the transitive mouse with its predicates, not of a least real coding a mouse.

Lemma 8.1 (Noncreation of the selected sword). *Set forcing over a choice universe cannot create a selected r -sword for a ground real r .*

Proof. First, two selected witnesses over r are initial-segment comparable; minimality then makes them literally equal. Suppose a condition forces that a witness exists, and take mutually generic filters G_0, G_1 through that condition. Each extension supplies a countable code for its witness. Iterability of a supplied classical mouse code is Π_2^1 [9, Proposition 2], so both witnesses remain iterable in the product extension. The structural requirements are absolute for these set structures.

Uniqueness identifies their transitive universes and predicates there. The generic-intersection theorem returns the common structure Q to the ground.

Ground countability needs an argument. In the product extension, sound projection supplies a finite degree n and a finite ordinal tuple p with

$$Q = \text{Hull}_n^Q(\omega \cup \{r, p\}).$$

Both n and p are ground objects. The ground computes the same satisfaction relation for the fixed set structure Q and hence the same Skolem map on the countable set of formula codes and integer parameters. Its being onto Q is absolute. The ground therefore has a countable code for Q , whose iterability returns by the same Π_2^1 absoluteness. This contradicts nonexistence in the ground. The proof never applies absoluteness directly to the existential assertion that some mouse code exists. $\square$

In the no-sword case, $K^c(r)$ and $K(r)$ denote the classical coherent-measure construction with the bottom real retained. Concretely, replace the mice in [8, Definition 4.1] by mice over r . Small-mouse adjunction uses the longest good extension of the current sequence; initial-segment comparison makes it unique. The full-measure step uses the coherent-measure test, with countable completeness and the external cofinality condition for $K^c(r)$, and the universal-ultrapower test for the true core. The bicephalus comparison gives at most one such full measure. At limits take unions. In the thick-hull description of the true core use

$$\bigcap_{\Gamma \text{ thick}} \text{Hull}^{K^c(r)}(\Gamma \cup \{r\})$$

and take its transitive collapse. This is a relative construction from the beginning; adjoining r to an already constructed core is a different operation [11].

Lemma 8.2 (Fixed-base classical core). *Let $B \models \text{ZFC}$ and $r \in B$. If there is no selected r -sword, the construction just specified gives a choice inner model $K(r)^B$ containing r . It has the classical weak-covering and universal-core comparison properties in the category of r -premise, and*

$$K(r)^B = K(r)^{B[g]}$$

for set forcing over B .

Proof. Here are the fixed-base checks in the classical proof. All hulls contain r and the integers. Their collapses and all copying and iteration maps fix the bottom real. Relative condensation and initial-segment comparison therefore give the same candidate uniqueness and copying arguments. In the countably complete construction the evaluation of countably many representatives at one point in their measure-one intersection also preserves this predicate. These are exactly the operations used to establish iterability and extract the true core in [8, Section 4.1].

The weak-covering proof adds this fixed countable base to its hulls; none of its cardinal-tail bounds changes. A failure configuration is now an iterable premouse over that same base. The high Mitchell-order obstruction excluded in the classical theorem would contain an initial segment with a coherent order-one upper measure. Its minimal sound core over r is a selected r -sword. Thus the required obstruction is absent in this relative category. This reasoning makes no assertion about an arbitrary unrelativized inner model which might omit r . More explicitly, the covering diagram and its derived measures use collapses of hulls containing the base; the associated comparison maps are maps of r -premise. The order bound used in [8, Lemma 4.5 and Claims 4.23–4.24] is a bound on the measures of these constructed premise. The relative cutoff supplies that bound. The singular-strong-limit hull-size calculation and the hull construction of cofinality ω_1 are unchanged by including the countable base.

Thick classes and their hull intersections are computed with r included. The classical universality and true-core comparison proofs then apply with the same fixed parameter on both sides. Finally

use the classical set-generic-invariance proof, including its covering and thick-hull comparison, with that same base in the ground and extension. Forcing can add codes for old mice; those proofs do not assume otherwise. A fixed set forcing preserves the relevant successor-cardinal tail, so the old core remains universal; the forcing and thick-class comparison identifies the true-core hull. All these hulls contain r , and the forcing interpretation uses its ground check name. This is not clausewise absoluteness of K^c or of its countable-completeness test. Lemma 8.1 ensures that the relative obstruction remains absent in the extension. This gives literal equality of the two cores. The unrelativized application of this classical generic-invariance theorem is recorded explicitly in [4, Section 4, Lemma 4.2]. $\square$

8.2. The measure-restriction step. The convention for a core ultrafilter in classical maximality includes weak amenability. We verify it for the restriction used here before applying maximality.

Lemma 8.3 (Amenability of the normalized restriction). *Work in a choice universe C with the no-sword core $K = K(r)^C$. Suppose $F \in C$ is a normal measure on $\mathcal{P}(\kappa)^K$ (external to K), countably complete for C -countable families, and complete for K -sequences of length less than κ . Suppose also that κ is inaccessible in K . Then F is weakly amenable to K , and F belongs to the coherent sequence of K .*

Proof. Form the full class ultrapower $j : K \rightarrow Q = \text{Ult}(K, F)$. Countable completeness makes it well founded. Normality identifies the seed [id] with κ . Use pure full-class Łoś and image set-segment codes to obtain the fine presentation: each sufficiently long coded structure $(K \restriction \theta, \in, \vec{U} \restriction \theta, r)$ is a set of K by amenability of its existing sequence. Its image has the same local premouse axioms, these image structures cohere, and every ultrapower element lies in some image structure, by Replacement in K applied to its representative. Their union therefore supplies the fine class presentation of Q . This step uses all functions in K , not a degree ultrapower theorem presupposing weak amenability of F .

To verify iterability, take a countable hull of a sufficiently long set initial segment of Q . Choose representatives for its elements and take the countably many measure-one truth sets for the relevant set-structure formulas. Simultaneous evaluation at one point in their intersection embeds the collapsed hull into an initial segment of K . Copying supplies iterability of the hull, and the usual countable-hull reduction for classical iterations gives iterability of Q . Only set satisfaction is used here.

Let λ be a sufficiently large singular strong-limit cardinal of C with $\text{cf}^C(\lambda) > \kappa$. Weak covering gives $(\lambda^+)^K = \lambda^+$. For $\xi < \lambda$ the bank of K -functions $\kappa \rightarrow \xi$ has C -size less than λ ; hence $j(\xi) < \lambda$. All κ -functions into λ are bounded, so $j(\lambda) = \lambda$. Also $\lambda^\kappa = \lambda$ in C . The same counting below λ^+ and continuity at that regular cardinal give $j(\lambda^+) = \lambda^+$. Therefore

$$(\lambda^+)^Q = j((\lambda^+)^K) = \lambda^+.$$

Such λ form a stationary class: in any class club take a sufficiently high closure point of fixed regular cofinality above κ , also in the club of strong-limit cardinals. The classical successor-correctness criterion makes Q universal [8, Proposition 4.4].

The universal-core comparison theorem, with its proof, says that the Q side of the comparison with K does not move: Q is the nondropping comparison iterate of K [8, Theorem 4.12]. These premisses agree below κ , since j fixes the K -sets of rank below κ . For clarity about partial measures, the classical indexing rule is $\gamma = (\delta^{++})^{L[r, \vec{U} \restriction \gamma]}$ at critical point δ [8, Definition 3.24]. Its truncated-sequence model is an inner model of K , so this index is bounded by the corresponding double Hartogs bound in K . If $\delta < \kappa$, inaccessibility puts it below κ even when δ later becomes countable in the full core. Comparison therefore uses only measures with critical point at least κ . These internal amenable measures preserve the power set of κ , as do their nondropping direct

limits. Consequently

$$\mathcal{P}(\kappa)^Q = \mathcal{P}(\kappa)^K.$$

For a K -sequence $a = \langle a_i : i < \kappa \rangle$ of subsets of κ , Separation in Q forms

$$\{i < \kappa : \kappa \in j(a)(i)\} = \{i < \kappa : a_i \in F\}.$$

Power-set equality puts this trace in K . A K -enumeration of any K -set of K -size κ now gives weak amenability. The full hypotheses of classical ultrafilter maximality are satisfied, so F is a sequence measure [8, Corollary 4.13]. All maps in this argument fix r , as required by Lemma 8.2. $\square$

Lemma 8.4 (Measurables in one fixed relative core). *In a ZF universe with unbounded measurables, put $B = \text{HOD}(r)$. If B has no selected r -sword, then $K(r)^B \models \text{PCMeas}$.*

Proof. Fix an ambient measurable κ above the bottom real and an ambient κ -complete non-principal ultrafilter U . No normality in the choiceless universe is assumed. The restriction $u = U \cap \mathcal{P}(\kappa)^B$ is coded by a set of ordinals using a B -wellordering of $\mathcal{P}(\kappa)^B$. Relative Vopěnka gives a choice set-generic extension C of B , inside the ambient universe, containing that code. The core is unchanged by Lemma 8.2.

In C take the well-founded ultrapower of B by u and derive the normal B -measure D at the ordinal seed κ . Choose a representing function $f \in B$ for this seed. Then, for $a \in \mathcal{P}(\kappa)^B$,

$$a \in D \iff f^{-1}[a] \in u.$$

Thus D is a set of C . Its restriction F to the unchanged core is normal, countably complete for C -families, and complete for core sequences of length below κ . These facts follow from U and the displayed pushforward description; C is an inner model of the original universe.

The ambient measurability also makes κ inaccessible in the choice inner core. Regularity is downward absolute. If a smaller ordinal ξ admitted κ distinct subsets in the core, intersecting the ξ many sides selected by U would force two of them to have the same incidence pattern. This contradicts their distinctness. Core Choice therefore gives the strong-limit assertion. Lemma 8.3 puts F in the one fixed core $K(r)^B$.

Apply this argument separately for unboundedly many ambient measurables. It uses no class choice of their measures and no single cover containing all the restrictions. Normalization in this form also occurs in [14, Theorem 3.11]. $\square$

8.3. Recursive changes of the bottom real.

Lemma 8.5 (Recoding covariance). *Let r, s be mutually recursive reals in a common choice universe. The classical recipe above, and the selected sword followed by its upper-measure iteration when that sword exists, give the same pure membership universe after changing r to s .*

Proof. First $\text{HYP}(r) = \text{HYP}(s)$: every transitive admissible set containing either real contains the other by recursive comprehension. This common passive level lies below the first critical point κ . Indeed, in the passive premeasure structure P , κ is inaccessible; at its uncountable regular $\eta = \omega_1^P < \kappa$, the usual regular-cardinal Collection argument makes $L_\eta[r]$ admissible. The least r -admissible height is therefore below κ . This uses the premeasure structure: a partial measure's critical point can be countable in the eventual full mouse. Above the common level, rudimentary successor closures with the same measure sequence and limit unions give the same pure levels.

The two predicate languages translate into one another using finite oracle computations. With ω and the hereditarily finite sets named as bounds the translations are bounded. Their parameter-free Δ_1 definitions on the developed levels eliminate these bounds, preserving Σ_n and Π_n for $n \geq 1$. Hence

$$\text{Hull}_n(\Gamma \cup \{r\}) = \text{Hull}_n(\Gamma \cup \{s\}) \quad (n \geq 1).$$

For each ordinal tuple, theory and master-code predicates translate on formula codes without changing ordinal coordinates. Thus projecta, the least ordinal standard parameters, and pure sound-core collapses agree. This does not claim equality of the literal syntax of master codes. Degree-zero ultrapowers use the same whole function sets, and higher-degree ultrapowers use the corresponding definable functions. Their equivalence relations, membership relations, maps, and limit systems agree. Iterability and comparison are consequently preserved. Concrete uses of this real-premouse rearrangement appear in [12, p. 45] and [10, Lemma 3.5.2].

Rearrangement takes minimal sound swords to minimal sound swords. Initial-segment uniqueness identifies the selected structures, and the identical upper-measure iterations give identical stabilized pure universes.

For the core, use the actual recursion specified above. Candidate small mice correspond; their initial-segment chains and longest good extensions agree. Candidate full measures and the coherence tests agree. Countable completeness and the cofinality test are computed in the same universe; the universal-ultrapower test agrees by the correspondence of iterable extensions. Intrinsic uniqueness therefore gives the same new measure. At limits take the same unions. Small-mouse adjunction may skip passive stages, but all measure entries below the common passive level are empty. A candidate with a nonempty sequence lies above that level, and a shorter passive candidate contributes no different measure entry. Thus no first discrepancy in the sequence can arise. Thick hulls containing the bottom real agree by the displayed hull equality, so their pure collapses agree as well. $\square$

8.4. The choice model and its own HOD.

Lemma 8.6 (Stabilized sword model). *If a choice universe B has a selected r -sword, its upper-measure iteration gives a choice inner model C_r containing r , with unbounded measurables and an externally iterable fine-structural coherent-measure presentation.*

Proof. Use the active presentation of the sword P : its upper measure is total over the passive universe, weakly amenable, and of order one; the lower measure belongs to the mouse and is also total over that passive universe. Iterate the upper measure and its images, taking well-founded direct limits. These total top-measure iterations have no model or degree drops. Their critical points κ_α increase continuously and are unbounded in Ord^B . For each fixed η , the fine segment through η is unchanged once $\eta < \kappa_\alpha$. Let E_∞ be the stabilized sequence and put

$$C_r = J_{\text{Ord}}[r, E_\infty].$$

Induction on the hierarchy index shows that every set-sized segment is the identical segment of a sufficiently late iterate of P . Thus E_∞ is coherent and amenable, and r is retained. This is the standard good Ord-iteration of the order-one mouse [15, Theorem 14].

At κ_α , coherence leaves the lower measure in the next iterate, while weak amenability of the upper measure preserves its entire power-set bank. All later critical points exceed κ_α , and a sufficiently late stage also fixes the lower measure as a set. Both bank and measure stabilize. The surviving measure is therefore total in C_r , and the unbounded critical sequence gives PCMeas^{C_r} .

For completeness, full Replacement of the original sound mouse is not needed to obtain ZFC in the final class. The set-stage iteration, the stabilized sequence, and its entire hierarchy are definable in B from P . For any fixed finite list of formulas relativized to this class, B -Replacement bounds witness stages for tuples from each level. Closing under these bounds produces arbitrarily high reflecting levels. Separation follows by defining the desired subset over such a level. For Power Set, the collection of all C_r -subsets of a given set is first a B -set; its entry stages are bounded by B -Replacement. A later level contains and defines the whole collection. The same bounding and finite reflection argument supplies Replacement for C_r -definable functions. The

other pure axioms follow from the full hierarchy. Canonical wellorders of set levels, using their truncated amenable sequence data, belong to later levels and give Choice.

Finally, any set-sized segment is a segment of an iterable set iterate of P . A proposed failure of iterability reduces to a countable hull and copies to such an iterate. This supplies the external iterability of the stabilized fine presentation. $\square$

Lemma 8.7 (Internal HOD ground). *Each of the two models C_r just constructed is a set-forcing extension of HOD^{C_r} .*

Proof. Both final coherent sequences have order zero. In the core branch, a positive-order entry would yield a selected sword by the least-segment and sound-core normalization. In the sword branch, the same minimality makes its top upper measure the only positive-order entry; its critical point escapes to Ord. The stabilized entries therefore have order zero.

Use the usual normal-measure coding of an already fine coherent-measure premouse by short type-1 extenders. For a measure U over its designated segment, derive E from its ultrapower j_U . The component at the seed κ recovers U , and the other components are its pullbacks by representing functions. The identification

$$[b, f]_E \mapsto j_U(f)(b)$$

is onto the measure ultrapower because the seed κ generates it, and it respects membership and embeddings. Applying this coding to the local fine-structural codes transports permissible definable functions, sound cores, and degree ultrapowers; initial segments, drops, and direct limits correspond. This is the standard short-measure presentation equivalence, with one fixed indexing and trivial-completion convention throughout. We began with fine mice; no upgrade of a coarse structure is asserted.

There is no additional tree-strategy hypothesis here. At distinct critical points these short measures do not overlap: the supports of lower measures are below the next measurable critical point in the relevant premouse segment. An order-zero measure leaves no measure at its own critical point after it is used. A first genuine return of a normal tree to an earlier model would require a new critical point below an earlier short support. Distinct critical points are excluded by nonoverlap and prefix agreement; the same critical point is excluded by order-zero coherence. After removing padding the tree is therefore a classical normal linear iteration, with its usual possible drops. Classical iterability supplies the well-founded unique limit branch. This is the nonoverlap regime described in [13, Lecture 5]. Thus the fine presentation M_r is externally $(0, \omega_1^B + 1)$ -iterable in its choice background B , and its pure universe satisfies Power Set.

Apply extender definability [5, Theorem 1.1] in its real-relative form. The relative condensation-stack argument is used explicitly in [16, proof of Lemma 5.4]. The resulting parameter may be taken as

$$a = (r, M_r \restriction \omega_1^{C_r}) \in C_r.$$

The ordinal here is the model's own ω_1 , and the theorem requires external iterability, not a strategy belonging to C_r . The definable hierarchy and its wellorders give $C_r \models V = \text{HOD}(a)$. By Choice in C_r , code $\text{TC}(\{a\})$, its membership relation, and its marked root a by a set of ordinals $t \in C_r$. Then $C_r \models V = \text{HOD}(t)$ as well.

Vopěnka's theorem inside C_r supplies a set forcing in $D_r = \text{HOD}^{C_r}$ and a D_r -generic g such that

$$C_r = D_r[g].$$

Here the relative HOD recovered by the theorem is all of C_r , by the preceding set-parameter definability. See [17, Theorem 6.2.7]. There is no identification of D_r with the original background $\text{HOD}(r)$. $\square$

Lemma 8.8 (Core-model preparation). *In a ZF universe with a proper class of measurables there is a uniform assignment, for every real r , of a choice inner model C_r such that:*

- (1) $r \in C_r$ and $C_r \models \text{ZFC} + \text{PCMeas}$;
- (2) the pure universe of C_r is unchanged by the finite recursive recodings induced by permuting a collapse enumeration;
- (3) C_r is a set-forcing extension of $D_r = \text{HOD}^{C_r}$.

Proof. Work in $B_r = \text{HOD}(r)$. If there is a selected r -sword, use Lemma 8.6; otherwise use Lemmas 8.2 and 8.4. Finite recursive recodings s satisfy $B_s = B_r$. Lemma 8.5 identifies the branch, the construction, and the resulting pure universe in this common background. Lemma 8.7 gives the final clause. All recipes have been specified without an additional choice of a real code or a background wellorder. $\square$

8.5. Rank and dependent choice.

Proposition 8.9 (Rank-preserving passage to SVC). *Suppose $V \models T_0$. For every ordinal β there is an inner model $N \subseteq V$ satisfying $T_0 + \text{SVC}$ such that*

$$V_\beta^N = V_\beta^V, \quad \mathcal{P}(V_\beta^V)^N = \mathcal{P}(V_\beta^V)^V.$$

One can arrange a choice ground $D \models \text{ZFC} + \text{PCMeas}$ and a choice set-forcing cover $C = D[g]$ with $D \subseteq N \subseteq C$.

Proof. Choose $\lambda > \beta + \omega$ with $\text{cf}^V(\lambda) > \omega$, and let $B = V_\lambda^V$. Such a limit ordinal is available under DC; for example, start sufficiently high and add ω_1^V . Collapse B to be countable and let r code membership and equality of entries in its generic enumeration. Lemma 2.1 preserves a proper-class tail of measurables through this fixed collapse. Apply Lemma 8.8 in the collapse extension and put $C = C_r$.

Inside C , first quotient the coded enumeration by its equality relation and then take its well-founded extensional collapse. This recovers the same transitive set B . Define

$$N = \text{HOD}_{B \cup \{B\}}^C,$$

allowing individual elements of B as parameters as well as the whole set B . This relative hereditary-definability construction satisfies ZF and contains B and all its elements.

Finite permutations of the ω -domain witness weak homogeneity of the collapse: move the finite domain of one condition away from that of another. They replace r by exactly the recodings covered by Lemma 8.8(2), while fixing the actual ground set B and its members. Thus every fixed ordinal-and- B definition over C is invariant under the weakly homogeneous group. The usual hereditary rank induction gives $N \subseteq V$. Consequently

$$V_\lambda^N = B = V_\lambda^V.$$

In particular, both the rank V_β and all its subsets have been retained.

For every nonempty $Y \in N$, the relative-HOD definition gives a surjection in N from $\eta \times B^{<\omega}$ onto Y , for some ordinal η . This enumerates all bounded definition codes and all finite parameter lists; it does not choose a definition for each element of Y . Finite lists from the limit rank B can be coded into B , so $N \models \text{SVC}(B)$.

For DC, let $R \in N$ be serial on Y , fix $y_0 \in Y$, and take $f : \eta \times B \rightarrow Y$ in N . At a current y put

$$\xi(y) = \min\{\xi < \eta : \exists b \in B (y R f(\xi, b))\}.$$

Use DC in V to choose successive b_n at these least ordinal coordinates. The array $s = \langle b_n : n < \omega \rangle$ belongs to B , since $\text{cf}^V(\lambda) > \omega$ and λ is a limit. It therefore belongs to N . Inside N , recursion using s , y_0 , and the least-coordinate rule reconstructs the entire chain. Thus $N \models \text{DC}$.

Finally $D = \text{HOD}^C \subseteq N \subseteq C = D[g]$. Downward Lévy–Solovay gives a measurable tail in D , since $C \models \text{PCMeas}$ and the forcing is fixed. For every sufficiently high ground measure U , its generated lift in C restricts to the set

$$\{a \in \mathcal{P}(\delta)^N : \exists b \in U (b \subseteq a)\}$$

in N . This is the complete nonprincipal ultrafilter required for PCMeas^N . The ground D and cover C are fixed before taking this tail. If desired, the intermediate symmetric-model theorem and SVC^N make D a symmetric ground of N . $\square$

Proof of Theorem 1.2. Apply Proposition 8.9 above $\alpha + 1$ to obtain $N \subseteq V$. Apply the local Theorem 1.3 to N , protecting that same rank and its power set. The resulting M has the displayed rank and power-set equalities with V , and has the stated sequence-seed presentation. All required generics may be chosen over the original collapse extension, so the construction takes place in a set-generic extension of V . It asserts $N \subseteq M$, not $V \subseteq M$. $\square$

9. REMOVING FOUNDATION BY ATOMIZATION AND FIBERS

We give the final reduction from the Foundation-free starting theory. The atomization setup and fiber purification are the earlier argument of [1, 23 September, 06:07 UTC]. We spell out the version of the local forcing theorem that its application needs. In this section ZFA has Foundation above a fixed set $\mathcal{A}$ of atoms; Ker denotes the pure kernel.

9.1. The local construction with atoms.

Lemma 9.1 (The choice-envelope version). *Let $N \subseteq E$ be ZFA models with the same ordinals and the same set $\mathcal{A}$ of atoms. Suppose:*

- (1) $N \models \text{DC} + \text{SVC}(X)$, where $X = V_\lambda^N(\mathcal{A})$ is a sufficiently high limit rank, $\text{cf}^N(\lambda) > \omega$, and $\mathcal{A} \subseteq X$;
- (2) E satisfies Choice, $\mathcal{A}$ is countable in E , and $\text{Ker}(E) = D[g]$ for a pure $D \subseteq N$ satisfying $\text{ZFC} + \text{PCMeas}$ and fixed set forcing;
- (3) a measurable $\kappa > \lambda + \omega$ of D is chosen above the size of this fixed forcing and the sizes in E of X , the old collapse, and the cap and comparison posets.

Then the local Prikry-labeling construction produces $N \subseteq U$ with

$$U \models \text{ZFA} + \text{DC} + \text{PCMeas} + \text{SVC} + \text{KW}_1^{*,\omega}, \quad \mathcal{P}(X)^U = \mathcal{P}(X)^N, \quad V_\lambda^U(\mathcal{A}) = X.$$

Proof. Only the pure D and the choice envelope E are needed for this version. Use the old pure Prikry poset from D , including its old wellordering. The old normal measure base generates the normal lift in E and its small choice extensions. Compute the countably closed collapse in N and choose the product generic over E , exactly as in Section 4. Internal countable closure and condition coding use $\text{cf}^N(\lambda) > \omega$; no requirement on $\text{cf}^E(\lambda)$ is imposed.

For completeness, the small-forcing assertions with atoms reduce to the pure ones. Fix in E an enumeration of its atoms. Recursively code an atom by its tagged index and a set by the tagged set of its members' codes. Every resulting code is pure and belongs to $\text{Ker}(E)$. Choice in E codes any set forcing by an ordinal condition set and a pure order. Names can be translated using the same atom indices. For a pure value, decoding its name uses no atoms; hence the pure kernel after forcing is the corresponding set-forcing extension of the old pure kernel. This coding is used only in the envelope; its atom enumeration is not adjoined to N or U .

The collapse-first proof of Lemma 4.1(1)–(2) therefore applies to atomful X and arbitrary ground values. SVC makes $N[K]$ a choice model with atoms, and Prikry no-new-bounded-subsets gives the same countable- X freshness and wellorderable ground-family covers.

Here is the required rank version of part (3). Since $|\mathcal{A}|^E < \kappa$ and κ remains inaccessible in the choice envelope, every old atom-rank below κ has size below κ . Prikry no-new-bounded-subsets then preserves all these ranks: a least-rank new set would be a subset of a small old rank bank, and its pullback by an old wellordering would be a new bounded subset of κ . Atoms themselves are unchanged. The generic-intersection proof also works by rank over the common atom set; atoms are its initial case and already belong to N . Thus

$$V_\kappa^{N[c]}(\mathcal{A}) = V_\kappa^N(\mathcal{A}).$$

Use transitive-base constructibility with atoms retained at every successor stage, for example

$$J_{\xi+1} = \mathcal{A} \cup \text{Def}(J_\xi; \in, \text{Atom}),$$

rather than discarding the atoms when taking definable subsets. Lemma 2.2 still gives finite expressions: atoms are ground leaves, and the atom predicate has the same interpretation in the compared universes. No equality $N = D(X)$ is used in this presentation lemma.

All cap and paired constructions concern integer carriers and their literal colors in X . A color may be an atom. They never recover a color from its members, so the proofs in Sections 5–6 are unchanged. The common atom predicate makes bounded-evaluator absoluteness apply as well. Labeling elimination and the atom-rank equality just proved yield $\mathcal{P}(X)$ preservation.

The old condition-array argument, the real-coded masks, and the wellorderable-tree argument prove DC as before. Bounded presentations and $F : Q \rightarrow X$ give SVC and KW for every nonempty set of U , including sets containing atoms. Finally the full choice envelope after these set forcings has pure kernel a set-forcing extension of D , by the coding paragraph above. A sufficiently high tail of ground measures has generated lifts there, and the old-base definition used in Proposition 7.5 restricts those lifts to measures belonging to U . This proves PCMeas^U .

The pure sequence-seed and symmetric-ground conclusions of Proposition 7.5 are not part of this atom lemma. In particular, the predecessor-set quotient does not distinguish distinct atoms, and a pure $D(A')$ cannot contain them. Neither conclusion is needed below. $\square$

9.2. Preparation of the protected atom structure. Start with $V \models T_-$ and a full second-order counterexample $\mathcal{M}$ to a fixed sentence. Use its relational multisorted presentation: a first-order sort S_0 , and for every arity k occurring in the sentence, a sort representing all V -subsets of S_0^k , with the incidence relation. The familiar two-sorted presentation is the unary or tuple-coded case. Replace the combined domain by a set $\mathcal{A}$ of formal atoms, using distinct tags for the sorts.

Form the full interpreted cumulative hierarchy $\mathcal{H}(\mathcal{A})$ using the original universe's power sets. Its atoms are opaque labels; the original members of a label do not become members of the atom. Thus non-well-founded membership in V introduces no cycles in this interpreted hierarchy. It satisfies ZFA. It satisfies DC because an ambient chain through a set of its coded objects has bounded atom-rank and belongs to the hierarchy. Its pure kernel has the original ordinals and their full power sets, so the original measures and the ordinal-indexed families testing their completeness remain available. Hence

$$\mathcal{H}(\mathcal{A}) \models \text{ZFA} + \text{DC} + \text{PCMeas}.$$

The atomized structure remains full: inverse images under the atom-label bijection identify its subsets and finite-arity relations with all the original ones.

Choose a limit λ of uncountable cofinality in this hierarchy, high enough to contain the structure and the full power sorts, and set

$$B = V_\lambda^{\mathcal{H}(\mathcal{A})}(\mathcal{A}).$$

Collapse B to be countable. Its real code r records membership, equality of repeated enumeration entries, and which entries are atoms. The small-forcing proof of Lemma 2.1 works with fixed

atoms, so a measurable tail remains. Apply the earlier core-model Lemma 8.8 in the pure kernel of this collapse extension, obtaining $C_r = D[g]$.

Construct the choice companion E explicitly. Let I be the least enumeration indices of the distinct atoms, as determined by r . Inside C_r form the hierarchy over the formal atom set indexed by I ; then relabel those formal atoms by the actual atoms using the collapse enumeration. The resulting E satisfies Choice, has pure kernel C_r , and contains the literal B . For the last assertion, quotient the code by its equality predicate and perform its well-founded collapse with the specified atom labels. Those labels distinguish different atoms and distinguish them from the empty set. Uniqueness of this collapse gives the same B .

The companion is invariant under the finite permutation group used in Proposition 8.9. The pure C_r is unchanged, and the induced bijection between its two atom-index sets belongs to that same pure model. Transporting the hierarchy along it gives the identical actual companion. Therefore

$$N = \text{HOD}_{B \cup \{B\}}^E,$$

with the atom predicate retained, descends to $\mathcal{H}(\mathcal{A})$ by the same hereditary homogeneity argument. It contains B and has $V_\lambda^N(\mathcal{A}) = B$. Relative-HOD presentations give $\text{SVC}(B)$, and the least-ordinal-coordinate DC proof in Proposition 8.9 applies using the countable closure of B in $\mathcal{H}(\mathcal{A})$.

The pure $D = \text{HOD}^{C_r}$ is contained in N : definitions over the definable pure kernel of E are available over E . Ground measure lifts restrict to N , giving PCMeas^N . This supplies a pure ground inside N and a choice envelope above it. It does not assert a choice ground with atoms inside N or apply the pure SVC symmetric-ground theorem to an atom model.

We now have precisely the inputs of Lemma 9.1, with $X = B$. In particular $\text{cf}^N(\lambda) > \omega$, since $N \subseteq \mathcal{H}(\mathcal{A})$. The envelope can see B and λ as countable; that causes no problem. The old condition sets have codes in B and are small in E . Choose a sufficiently high D -measurable κ after these set-sized bounds are fixed. The lemma yields

$$U \models \text{ZFA} + \text{DC} + \text{PCMeas} + \text{SVC} + \text{KW}_1^{*,\omega}$$

with the protected power sets unchanged. The original multisorted counterexample is consequently still full in U .

9.3. Fiber purification and the final theorem.

Lemma 9.2 (Fiber purification). *If $U \models \text{ZFA} + \text{DC} + \text{PCMeas} + \text{SVC} + \text{KW}_1^{*,\omega}$, then its pure kernel P satisfies $T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$. Every full second-order structure in U has a full isomorphic copy in P .*

Proof. For the nonempty combined domain S of a relational presentation, take a surjection $f : \kappa^\omega \twoheadrightarrow S$ in U and put

$$j(s) = f^{-1}\{s\}.$$

The nonempty disjoint fibers make j injective. Each fiber is pure, being a set of ordinal sequences. Thus the whole range $j[S]$, its sort predicates, and all transported finite-arity relations are pure sets and belong to P . The possibly atomful graph of j need not belong to P . Empty domains can be treated directly.

Fullness follows at each relevant arity. For $Y \in P$ with $Y \subseteq j[S_0]^k$, the pullback under j^k is a U -subset of S_0^k , so it is represented by an element of the full power sort in U . Its image represents exactly Y . Conversely each represented subset in the transported structure is pure and belongs to P . In the application, preservation of the protected power sets says that these U -relations are exactly the relations of the original counterexample. Thus the copied structure has the same full second-order truth.

The pure kernel satisfies ZF, including Foundation. A DC chain through a pure set is itself pure. Measures on ordinals and all families testing their completeness are pure as well, so PCMeas descends. The models have the same κ^ω ; a surjection from it onto a pure set has pure graph. Hence KW descends.

Finally fix one SVC witness C in U . Fiber-code it in the same way to a pure copy C^b . For any nonempty pure Y , compose an SVC surjection $\eta \times C \twoheadrightarrow Y$ with the inverse of this fiber bijection. The resulting map $\eta \times C^b \twoheadrightarrow Y$ has pure graph and belongs to P . The same fixed C^b works for every Y , proving SVC in P . $\square$

Proof of Theorem 1.1. Given a full second-order counterexample in a model of T_- , atomize it, perform the earlier core-model preparation, and apply the local atom theorem as above. Lemma 9.2 gives a pure model of $T_0 + \text{SVC} + \text{KW}_1^{*,\omega}$ containing a full counterexample to the same sentence. The countermodel contrapositive proves the asserted structural conservativity. The reverse implication of provability is immediate because the target theory extends T_- . All constructions are interpretations and set-forcing constructions over the starting model; no external transitive-model hypothesis is used. $\square$

10. EXTERNAL HOD AND HOMOGENEOUS DESCENT

Lemma 10.1. *If $U \models \text{ZFC}$, then $\mathcal{H}^U \models \text{ZF} + \text{DC}$. It contains every ordinal ω -sequence of U and is closed under U -countable sequences of its elements.*

Proof. The usual hereditary-definability proof of ZF applies to the definable class of allowed parameters. For example, the internal power set of $x \in \mathcal{H}^U$ is

$$\{y \subseteq x : y \in \mathcal{H}^U\},$$

which is definable from a definition of x , and whose elements are hereditary members of the class. Separation and Replacement are handled in the same way.

Let $\langle x_n : n < \omega \rangle \in U$, with each $x_n \in \mathcal{H}^U$. In U , choose a definition of x_n from an ordinal sequence and ordinal parameters. Use bounded-stage definitions: choose a rank level at which the defining formula has the required unique value. Interleave the sequence parameters, the rank levels, the ordinal parameters, and the formula codes. The result is one $q \in (\text{Ord}^\omega)^U$ from which the entire sequence $\langle x_n \rangle$ is definable in U . Its hereditary members already belong to $\mathcal{H}^U$, so the sequence belongs to $\mathcal{H}^U$. Every $q \in (\text{Ord}^\omega)^U$ is itself hereditarily definable from q . Finally, choose a dependent-choice chain in U through a serial relation belonging to $\mathcal{H}^U$ and apply the countable closure just proved. $\square$

Lemma 10.2 (Homogeneous descent). *Let $M \models \text{ZF}$, and suppose $U = M[G]$ is a weakly homogeneous set-forcing extension with*

$$(\text{Ord}^\omega)^U = (\text{Ord}^\omega)^M.$$

Then $\mathcal{H}^U \subseteq M$. No ordinal-definability assumption on the forcing is needed.

Proof. Induct on hereditary rank. Suppose $x \in \mathcal{H}^U$ and every member of x is already in M . Choose β with $x \subseteq V_\beta^M$. A definition of x in U uses an allowed parameter q belonging to M and ordinal parameters. Weak homogeneity makes the largest condition decide the corresponding uniqueness assertion and, for each $y \in V_\beta^M$, the assertion that y belongs to its unique value. The forcing relation therefore defines x as a subset of V_β^M inside M . This completes the induction. $\square$

Proposition 10.3. *For every σ , $\text{Con}(E_1(\sigma))$ implies $\text{Con}(E_3(\sigma))$.*

Proof. Let $U \models E_1(\sigma)$, put $H = \mathcal{H}^U$, and choose κ with $V_\kappa^H \models \sigma$. Choose $\lambda > \kappa + \omega$ of uncountable cofinality in U , and set $X = V_\lambda^H$. This X is ordinal-definable in U , belongs to H , and is closed under U -countable sequences of its elements by Lemma 10.1.

Collapse X to be countable over U , coding equality as well as membership and quotienting repetitions before collapse. Apply Lemma 8.8, and again set $W = \text{HOD}_{X \cup \{X\}}^{C_r}$. The same collapse-homogeneity argument now gives more than $W \subseteq U$: the collapse forcing is definable from the OD set X , so every descended definition uses X and finitely many elements of X , as well as ordinals. The set X is OD in U , and each of those finitely many elements is hereditarily definable from an ordinal ω -sequence. Combining their parameters shows $W \subseteq H$.

Since $X \in W$, we have $V_\lambda^W = X$. The presentation $\text{SVC}(X)$ and the ambient countable closure of X give DC in W . The same generated-measure argument gives PCMeas. Therefore $W \models E_3(\sigma)$. $\square$

11. SPARSE EVALUATED COHEN CAPS

We now work entirely over a choice model B . The purpose of this section is to code a *family of coherent rows*, not merely the union of its bit positions.

We use the convention $\text{Add}(\gamma, 1) = 2^{<\gamma}$, ordered by extension; its dense partial-function presentation gives the same forcing. In this section, all such posets are computed in the model shown in the superscript.

Theorem 11.1 (Fixed-base spectrum). *Let $B \models \text{ZFC}$ have a proper class of inaccessible cardinals, and let $\emptyset \neq F \subseteq 2^\tau$ belong to B , where τ is infinite. There are a schedule of inaccessible coding cardinals, uniformly defined forcings Q_e for $e \in 2^\tau$, and a cap cardinal ρ , all definable from B and ordinal parameters independently of F , such that the weakly homogeneous forcing P_F defined below has the following spectrum in $B[G]$:*

$$e \in F \iff \text{there is a } B\text{-generic } h_e \subseteq Q_e, \text{ and an } \text{Add}(\rho, 1)^{B[h_e]}\text{-generic over } B[h_e]. \quad (11.1)$$

The first coding cardinal can be placed above any prescribed bound, and the resulting forcing is correspondingly distributive.

11.1. The schedule and the actual forcing. Order the physical sites $s = (\xi, i)$, for $\xi < \tau$ and $i < 2$, lexicographically with ξ first. Distinct rows have independent copies at a shared physical site. Choose inaccessible cardinals γ_s recursively, with the first above $2^{|\tau|}$ and the prescribed protection bound, and with

$$|\text{RO}(S_{<s}^{\text{univ}})|^B < \gamma_s. \quad (11.2)$$

The universal prefix has copies for all possible rows in 2^τ , not only the members of F . Thus this schedule is independent of the family being coded. We may always choose the least inaccessible above the displayed bound.

For $e \in 2^\tau$, let

$$Q_e = \prod_{\xi < \tau}^{\text{full}} \text{Add}(\gamma_{(\xi, e(\xi))}, 1)^B.$$

Let ρ be an inaccessible above the Boolean completion of the entire universal lower product. Define

$$P_F = \prod_{d \in F}^{\text{full}} (Q_d * \text{Add}(\rho, 1)^{B[h_d]}). \quad (11.3)$$

The terminal forcing is computed in its *own row extension*. The actual extension retains evaluated terminal generics, not generics for the larger raw termspaces.

The full lower product

$$S_F = \prod_{d \in F}^{\text{full}} Q_d$$

can be flattened by physical sites. Denote its aggregate generic at s by G_s . Since $|F| < \gamma_0 \leq \gamma_s$, the aggregate site forcing is a product of fewer than γ_s copies of $\text{Add}(\gamma_s, 1)^B$. It is $< \gamma_s$ -closed in B and adds no shorter ordinal sequences over B .

For each d , let T_d be a full set of total names for cap conditions: the largest condition of Q_d forces each member to be a cap condition, and every total name has an equivalent member. Use all bounded Boolean-name codes for graphs on $\rho \times 2$ with this property, allowing equivalent codes. This is a canonical set from Q_d and ρ ; no auxiliary wellorder is a parameter of the forcing. Smallness of $\text{RO}(Q_d)$ and regularity of ρ bound the possible lengths of each name below ρ . Normalize forced unions and Boolean mixes to these Boolean-name codes. Lower automorphisms act on their Boolean values and preserve this full name system. The product

$$S_F \times \prod_{d \in F} T_d$$

with the ground termspace orders projects onto P_F , retaining the identical lower coordinate. For the lifting property, mix each desired stronger term on its strengthened lower cone with the old term elsewhere. Thus this projection commutes with the projection to S_F . Each ground termspace is $< \rho$ -closed. After the lower forcing, of size below ρ , its full product is $< \rho$ -distributive by Easton's lemma. Hence the actual cap quotient adds no ordinal sequences of length below ρ over the lower extension. This is a use of a closed *dominating* forcing; it does not identify the actual extension with the larger termspace extension. The underlying term-forcing facts are discussed in [19, Section 3].

These observations also give the protection assertion: with the first site above a regular θ , the dominating forcing is $< \theta$ -closed in B , so the actual forcing is $< \theta$ -distributive. Taking $\theta > |V_\eta^B|$ preserves V_η^B by rank induction. Canonical lower-row automorphisms lift through the name for the recomputed Cohen forcing; the latter is forced homogeneous. Coordinatewise automorphisms then witness weak homogeneity of the full product.

11.2. Localization to physical sites.

Lemma 11.2. *Suppose a B -generic $h_e \subseteq Q_e$ belongs to the P_F -extension. Then*

$$B[h_e] \subseteq B[G_s : s \text{ is selected by } e].$$

Proof. The whole lower generic h_e can be coded by a subset of a ground ordinal below ρ . The cap quotient adds no such subsets, so h_e is in the lower extension. Let $k_s \subseteq \gamma_s$ be its component at an e -selected site.

If there is a site after s , its first cardinal exceeds the Boolean size of the prefix through s . The ground full tail is closed up to that cardinal. Easton's lemma over the smaller prefix therefore makes it add no subsets of γ_s ; at a final site there is no tail. This verifies the tail distributivity needed here from (11.2). Thus

$$k_s \in B[G_s][G_{<s}].$$

The forcing producing $G_{<s}$ has size below γ_s . Consequently the pair

$$(B[G_s], B[G_s][G_{<s}])$$

has the $< \gamma_s$ -approximation property. The set k_s is approximated over $B[G_s]$: a $B[G_s]$ -set of ordinal positions of size below γ_s is already in B , since G_s adds no short ordinal sequences; and the restriction of the B -Cohen generic k_s to such a set belongs to B . Therefore

$$k_s \in B[G_s]. \tag{11.4}$$

For each site, fix in B a wellordering of a bounded set of names for subsets of γ_s . Choose the least name evaluating to the actual k_s . Their indices form an ordinal sequence of length $|\tau|$. The lower forcing adds no such sequence, so the entire vector of indices belongs to B . It follows that the vector of all k_s belongs to the extension by the selected sites alone.

This vector determines the full product generic h_e . Indeed, a ground product condition belongs to a generic filter exactly when it is compatible with the union of the generic bits at every coordinate: otherwise genericity for the dense set deciding compatibility with that condition gives a contradiction. Hence h_e itself belongs to the selected-site extension. $\square$

11.3. The quotient and its term order. We give the quotient calculation explicitly, since replacing this order by raw term forcing would destroy the spectrum.

Let $E = B[h_e]$. Use the old vector of names in Lemma 11.2 to name h_e over the selected sites. Restrict to an actual positive ground cone which forces that this name is Q_e -generic over B . This is an ordinary forcing statement: the ground dense subsets of Q_e form a set. The membership Boolean values of the named filter generate a complete subalgebra of that selected-site completion, on the cone, whose induced extension is exactly E . Its size is below ρ ; no injectivity of the map from the whole Q_e completion is required. The omitted sites remain independent product directions over this algebra. Quotient positivity depends only on the selected lower projection, since every terminal name has been totalized. Restricting to this cone leaves the actual extension under discussion unchanged. Let P denote a dense old-condition presentation of that quotient. Let

$$T = \left(\prod_{d \in F} T_d \right)^B$$

be the old set of vectors of row terms. The quotient of the *actual* forcing (11.3) is represented by $P \times T$ with order

$$(s', t') \leq (s, t) \iff s' \leq s \text{ and } \forall d \in F \ s'_d \Vdash_{Q_d}^B t'_d \leq t_d. \quad (11.5)$$

The comparisons on the right are the original ground forcing comparisons. There is no quantification over all new E -names for terminal conditions.

We use Jakob's criterion [18, Definition 3.5 and Theorem 3.8]: an iteration-like order with square- ρ -cc base and strongly $< \rho$ -distributive term order has the $< \rho$ -approximation property. Here strong $< \rho$ -distributivity means that any ρ -long sequence of dense open sets admits a descending sequence meeting them; it does not require a final lower bound. We verify the hypotheses for (11.5).

Lemma 11.3. *If $e \notin F$, the quotient order (11.5) has the $< \rho$ -approximation property over E .*

Proof. The quotient base and its square have size below ρ . For any fixed old lower condition s , the section order on T is $< \rho$ -closed in B : take names for unions of descending cap conditions below s_d , separately in each row, and use default values outside that cone. The extension E/B is by forcing of size below ρ . Strong $< \rho$ -distributivity of these fixed ground section orders is preserved by this chain-condition forcing, as in the preservation observation preceding Jakob's approximation criterion. The term order is the disjoint union of these section orders, and is therefore strongly $< \rho$ -distributive in E .

The order is based, and its refinement property follows immediately from the forcing relation. For the projection property, if $(s', t') \leq (s, t)$, mix in row d the name t'_d below s'_d with t_d off that Boolean cone. The resulting old vector t'' is a section extension below s , and below s' it is equivalent to t' . All inputs to this mixing are old sets, and the mixing is performed in B .

It remains to verify mixing for two section extensions t^0, t^1 below t at one fixed s . For every $d \in F$, choose the least ξ_d with $d(\xi_d) \neq e(\xi_d)$. The site

$$s_d = (\xi_d, d(\xi_d))$$

is omitted by e and active in row d . Keeping the selected sites unchanged, refine s in two ways by assigning opposite values at a fresh bit in this d -copy, simultaneously for every $d \in F$. This is a legal old condition: $|F|$ is below the first coding cardinal, and the product has full support. Both refinements remain positive in the quotient because only omitted sites were changed. Their projections are incompatible in every actual row.

In row d , Boolean-mix t_d^0 and t_d^1 below these incompatible row projections, and retain t_d elsewhere. The resulting vector t^* is old and is a section extension of t at s . Below the respective quotient refinements it specializes to t^0 and t^1 . This is exactly the mixing property. Jakob's theorem now gives the stated approximation property. $\square$

Proof of Theorem 11.1. If $e \in F$, its planted row and cap give the right side of (11.1). Conversely, suppose the right side holds and $e \notin F$. Lemma 11.2 and Lemma 11.3 give the $< \rho$ -approximation property for $(B[h_e], B[G])$. But a recomputed $\text{Add}(\rho, 1)^{B[h_e]}$ -generic is a new subset of ρ all of whose $B[h_e]$ -small restrictions belong to $B[h_e]$, contradicting approximation. The preservation and homogeneity assertions were verified after (11.3). $\square$

Corollary 11.4 (Coding an orbit of presentations). *Let $x \in B$. There is an arbitrarily highly distributive, weakly homogeneous forcing, uniformly definable in B from x and ordinal parameters, which makes x OD relative to the class B . If B is ordinal-definable in the final extension, then x is OD there.*

Proof. Choose an infinite cardinal τ large enough to code $(\text{TC}(\{x\}), \in, x)$. Let $F_x \subseteq 2^\tau$ be the family of all its pointed well-founded extensional presentation codes. Domain and root data are included, and unused positions are allowed. This family is nonempty and is defined from x, τ without selecting a presentation.

Apply Theorem 11.1. Relative to the class B , the first-order capped-history predicate recovers exactly F_x . Every member of this nonempty family has the same pointed Mostowski collapse, namely x . Hence x is OD relative to B . All schedules use the universal family 2^τ to obtain their bounds, so the decoder does not need F_x in order to define the candidate forcings. $\square$

Remark 11.5. No continuum-function comparison, large cap width, or singularization is needed in Theorem 11.1. Sparse site localization supplies the omitted quotient directions. Nor is a universal Cohen source retained: the ground termspace establish preservation, while the quotient argument is about the evaluated forcing (11.5).

12. A FRESH ENVELOPE AND A SECURE GROUND

Lemma 12.1 (A fresh choice envelope). *Suppose $M \models \text{ZF} + \text{DC} + \text{SVC}$. There is a weakly homogeneous set-forcing extension $B = M[K] \models \text{ZFC}$ with*

$$(\text{Ord}^\omega)^B = (\text{Ord}^\omega)^M.$$

Proof. Choose an SVC witness S , and force with countable partial functions from ω_1^M to S . Countable closure and ground DC give no new countable sequences of ground elements. Permutations of ω_1 moving two countable domains apart witness weak homogeneity. The generic union maps ω_1 onto S , hence wellorders S by least preimages. The SVC surjections then wellorder all old sets. For a new set, use an old name: its collection of possible named elements is an old set and is now wellorderable, so evaluation wellorders the value. Thus the extension satisfies Choice. $\square$

Definition 12.2. The *Continuum Coding Axiom* CCA says that for every ordinal θ and every $a \subseteq \theta$ there is an ordinal ζ such that, for every $\beta < \theta$,

$$\beta \in a \iff 2^{\aleph_{\zeta+\beta+1}} = \aleph_{\zeta+\beta+2}.$$

A choice ground D will be called *secure* here if it satisfies CCA. The two features used are that $D \subseteq \text{HOD}^U$ in every further set-forcing extension U , and that D is definable in U as its mantle.

Lemma 12.3. *A model of ZFC + PCMeas has a progressively closed class-forcing extension satisfying ZFC + CCA + PCMeas, preserving any prescribed rank. Moreover, if $D \models \text{ZFC} + \text{CCA}$ and U is a set-forcing extension of D , then $D \subseteq \text{HOD}^U$ and D is the mantle of U .*

Proof. We use the progressively closed continuum-coding iteration of [20], with the following local preparation and gaps. All computations within a block are in its current choice universe. Suppose $a \subseteq \theta$ codes the history before this block. Choose a regular λ_0 above θ and the preservation bound. Perform a full-support *set* iteration through stages $\xi \leq \theta + 3$. At stage ξ , force with

$$\text{Coll}(\lambda_\xi^+, (2^{\lambda_\xi})^{\text{current}}),$$

using partial functions of size less than the current λ_ξ^+ . After the stage put $\lambda_{\xi+1} = (\lambda_\xi)^+$; at limit indices take suprema. The collapse is $< \lambda_\xi^+$ -closed, preserves that successor, and adds no subsets of λ_ξ . It makes the old power set have size λ_ξ^+ , so it forces GCH at λ_ξ . Subsequent tails are $< (\lambda_{\xi+1})^+$ -closed over that prefix and preserve the equation and the earlier cardinals. At a limit the supremum of the earlier preserved cardinals is again a cardinal. Consequently the final λ_ξ form an actual consecutive-cardinal interval with GCH throughout it. The whole cleanup is $< \lambda_0^+$ -closed; larger cardinals may have been collapsed and are recomputed, not presumed preserved.

Now apply the local Easton theorem from [20, Section 1.2] on the successor sites $\lambda_{\beta+1}$, $\beta < \theta$: give their continua the successor or double successor according as $\beta \in a$ or $\beta \notin a$. The requisite local hypotheses hold with $\delta = \lambda_1$ and $\gamma = \lambda_{\theta+3}$: $2^{<\delta} = \delta$, GCH holds on $[\delta, \gamma)$, and all target values lie below γ . The Easton function is nondecreasing even at adjacent sites. This bounded forcing preserves cardinals, is $< \lambda_1$ -closed, and realizes precisely the stated values. If ζ is the ordinal index of λ_0 , the code is read at $\aleph_{\zeta+\beta+1}$ using just ζ, θ . Thus no global GCH preparation is assumed.

Retain Reitz's history bookkeeping, placing each individual Cohen coordinate as a contiguous binary block in a code for the previous generics. Cleanup generics are included in later histories. Only the pre-cleanup history is coded at this block; there is no self-coding requirement. Every ordinal set in an initial extension occurs, by density, on a fresh interval in a sufficiently high later Cohen coordinate: a condition can be extended there to spell the set. The next history code records that whole coordinate, so the set receives a continuum code. The same argument above each bound gives arbitrarily high codes.

Before each new block choose a ground measurable κ above the ground cardinality of the entire preceding set forcing. Start the block above κ , with closure strictly above it, and put later blocks beyond the preceding coding intervals. The small prefix lifts a normal measure at κ ; the tail adds neither subsets of κ nor the short sequences needed to test its completeness and normality. Thus unboundedly many selected ground measurables survive. The first closure bound also exceeds the cardinality of the rank to be protected.

Use the ordinary progressively closed class iteration with Easton support at its physical forcing locations and trivial gaps up to the chosen bounds. Full support above referred only to a set-sized cleanup, not to full support at class limits. The standard tame, set-name framework of [20] supplies the forcing theorem and ZFC; every set appears in a set-sized initial extension. These are

genuine tail closure cuts, so the prescribed rank and the selected measurable class are preserved. We do not claim preservation of every original measurable.

CCA gives arbitrarily high continuum codes for every ground set of ordinals. Above the size of a subsequent set forcing, the relevant cardinals and GCH truth values are preserved. Reading a sufficiently high code, with its ordinal starting point and length, makes each ground set OD in U . Coding transitive closures gives $D \subseteq \text{HOD}^U$.

CCA implies the Ground Axiom [20]. If W is any other ground of U , downward directedness of grounds gives a common ground below D and W ; see [21, 7]. The intermediate-model theorem makes it a ground of D . The Ground Axiom inside D therefore makes that common ground equal to D . Every ground of U contains D , and D is itself a ground. Hence D is the mantle of U . $\square$

Lemma 12.4 (Recognizing an envelope). *Suppose D is secure and $B = D(A, K)$ is a set-forcing extension of D satisfying ZFC. Choose η with $A, K \in V_\eta^B$. If U is a further set-forcing extension preserving V_η^B , then the actual class B is definable in U from the ordinal η by*

$$B = D(V_\eta^U).$$

Proof. The right-hand model is constructed inside B when $V_\eta^U = V_\eta^B$, and contains D, A, K . Thus it equals B . Lemma 12.3 identifies D inside U as its mantle. $\square$

Remark 12.5. The definability of the class B does not imply $B \subseteq \text{HOD}^U$. Likewise, V_η^U is OD but need not belong to HOD. This is the distinction that permits the decoder to recognize the choice envelope without importing its new wellorderings into the target inner model.

13. EXACT EXTERNAL REALIZATION

Proposition 13.1 (Realization over a secure ground). *Suppose*

$$D \subseteq M = D(A) \subseteq B = M[K],$$

where $D \models \text{ZFC} + \text{CCA} + \text{PCMeas}$, $A \subseteq \mu^\omega$, B is a set-forcing extension of D satisfying ZFC, and B/M is weakly homogeneous with no new ordinal ω -sequences. Then there is a further set-forcing extension U of B such that

$$U \models \text{ZFC} + \text{PCMeas}, \quad \mathcal{H}^U = M, \quad M \models \text{PCMeas}.$$

Proof. Choose η with $A, K \in V_\eta^B$. Apply Corollary 11.4 over B , starting above $|V_\eta^B|$, to the set A . Let U be the resulting extension. Lemma 12.4 defines the actual envelope B inside U , so the exact capped-history spectrum makes A OD in U .

The combined forcing from M to U is weakly homogeneous. Indeed, the name for the coding forcing is given by a uniform definition from the old A and ordinal parameters; automorphisms of the first forcing carry that name to a forced equal name. They therefore lift to the iteration, and the second forcing is forced weakly homogeneous. If a local name for a second-stage automorphism is needed, give it the identity as default off a condition forcing it to be an automorphism. This uses localized Boolean mixing, not a choiceless maximum principle.

The first extension adds no ordinal ω -sequences by hypothesis, and the second is highly distributive over the choice model B . Lemma 10.2 gives $\mathcal{H}^U \subseteq M$. Conversely $D \subseteq \text{HOD}^U \subseteq \mathcal{H}^U$, and $A \in \mathcal{H}^U$: it is OD in U and every member of A is an allowed ordinal-sequence parameter. Minimality of $D(A)$ therefore gives $M \subseteq \mathcal{H}^U$.

The entire extension U/D is by set forcing. All sufficiently high normal D -measures lift to U , and their restrictions (2.1) belong to M . Thus both models have a proper class of measurables. $\square$

The following preparation makes the preceding proposition applicable without assuming that an arbitrary choiceless intermediate model preserves DC through class forcing.

Proposition 13.2 (Installing the secure ground). *Suppose $M = D(A) \models \text{ZF} + \text{DC} + \text{PCMeas} + \text{SVC}$, where $D \models \text{ZFC} + \text{PCMeas}$ is a symmetric ground and $A \subseteq \mu^\omega$. Given an ordinal α , there are*

$$D^* \subseteq M^* = D^*(A) \subseteq B^* = M^*[K]$$

satisfying the hypotheses of Proposition 13.1, with

$$V_\alpha^{M^*} = V_\alpha^M.$$

Proof. Take a fresh envelope $B = M[K]$ from Lemma 12.1. The symmetric-ground relation composes with set-forcing extensions. Since both D and B satisfy Choice, D is an actual set-forcing ground of B , by [7, Lemma 5.8]. Write $B = D[b]$ using a forcing presentation in D .

Let $\mathbb{P} \in D$ be the presentation with generic b . Choose in D a regular $\delta > \max(|\mathbb{P}|, \omega)$, above a uniform $\mathbb{P}$ -name bound for the cardinality of the protected B -rank. Such a bound is available before evaluating b , using a set of names for that rank. Choose the preparation $\mathbb{Q}$ of Lemma 12.3 in D , independently of b , with every set-sized initial segment $\mathbb{Q}_\beta < \delta$ -closed in D . Take a coherent class generic J over B for this frozen D -iteration in the set-name framework just specified. For every β , the ordinary product lemma gives

$$D[b][J_\beta] = D[J_\beta][b].$$

Since $\mathbb{P}$ is δ -cc, Easton's lemma makes the frozen $\mathbb{Q}_\beta < \delta$ -distributive in B . We do not claim that its ground closure persists in B . This is the small-forcing/class-preparation commutation used in [20, Appendix, Lemma 121].

Put $D^* = D[J]$. Every dense subset of the fixed $\mathbb{P}$ in D^* appears in some $D[J_\beta]$, where b is generic by the product lemma. Thus b is generic over D^* , and

$$B^* = D^*[b] = \bigcup_{\beta} B[J_\beta] \models \text{ZFC}, \quad M^* = D^*(A).$$

Every short ordinal sequence in B^* appears at a set stage, so set-stage distributivity proves that B^*/B adds none of length less than δ . It also preserves the protected rank: a first new set in that rank would be a new subset of an old set of cardinality less than δ , coded by its old wellorder as a short ordinal sequence. Thus, by freshness of B/M and the inclusions $M \subseteq M^* \subseteq B^*$,

$$(\text{Ord}^\omega)^{M^*} = (\text{Ord}^\omega)^{B^*} = (\text{Ord}^\omega)^M. \quad (13.1)$$

We claim

$$M^* \cap B = M. \quad (13.2)$$

Every individual set of $D^*(A)$ belongs to $M[J_\beta]$ for some set-sized initial segment of the class iteration. To see this, take a bounded transitive-base construction witnessing its membership and then a stage at which the required rank of D^* has stabilized. The forcings for K and J_β are old set forcings and are mutually generic over M , so

$$M[J_\beta] \cap M[K] = M.$$

The usual product-intersection proof works over ZF : compare two names for a first new set of a given rank and use product conditions deciding membership. This proves (13.2). Together with preservation of the protected B -rank, it gives $V_\alpha^{M^*} = V_\alpha^M$.

The same filter K is generic over M^* for the *frozen* old collapse forcing. Every dense set in M^* appears in some $M[J_\beta]$, and mutual genericity makes K meet it. Moreover $B^* = M^*[K]$. One inclusion is automatic inside B^* . For the other, $D^* \subseteq M^*$ and $b \in B = M[K] \subseteq M^*[K]$, so $D^*[b] \subseteq M^*[K]$. The old automorphisms of the collapse still witness weak homogeneity, while (13.1) supplies the required absence of new allowed parameters.

Finally, $D^* \models \text{ZFC} + \text{CCA} + \text{PCMeas}$ by construction. This proves all the hypotheses of Proposition 13.1. No assertion that M^* directly inherited DC through the class preparation was used. After realizing it as $\mathcal{H}^U$, Lemma 10.1 supplies DC. $\square$

Proof of Theorem 1.7. From SVC, take a choice symmetric ground D and a transitive generator T with $M = D(T)$; see [7, Proposition 5.4]. The choice ground may be taken to have a proper class of measurables: make a set-generic choice envelope, use Lemma 2.1, and apply downward Lévy–Solovay to the resulting choice-ground presentation. Alternatively, in the application from Theorem 1.2, the ground D already has this property.

By $\text{KW}_1^{*,\omega}$, choose a surjection $f : (\mu^\omega)^M \rightarrow T$. Code its domain and the relation $f(p) \in f(q)$ by a single set A of ordinal ω -sequences. Equality of predecessor sets and the well-founded extensional collapse, as in Proposition 7.5, recover T and f . Hence $M = D(A)$. Apply Proposition 13.2 and then Proposition 13.1. $\square$

14. THE THREE CONSISTENCY IMPLICATIONS

Proof of Theorem 1.6. The implication $\text{Con}(E_2(\sigma)) \Rightarrow \text{Con}(E_1(\sigma))$ is immediate. If $U \models E_2(\sigma)$, Lemma 10.1 shows that $\mathcal{H}^U$ is a model of $E_3(\sigma)$. Thus $\text{Con}(E_2(\sigma)) \Rightarrow \text{Con}(E_3(\sigma))$.

Proposition 10.3 gives $\text{Con}(E_1(\sigma)) \Rightarrow \text{Con}(E_3(\sigma))$ without assuming that the original external HOD already contains a proper class of measurables.

For the remaining direction, let $V \models E_3(\sigma)$ and choose κ with $V_\kappa^V \models \sigma$. Apply Theorem 1.2 above κ to obtain a sequence-generated model M with $V_\kappa^M = V_\kappa^V$. Apply Theorem 1.7 protecting that rank. Its resulting U satisfies ZFC + PCMeas, and

$$\mathcal{H}^U \models \text{PCMeas}, \quad V_\kappa^{\mathcal{H}^U} = V_\kappa^M = V_\kappa^V.$$

Hence $U \models E_2(\sigma)$. This proves the cycle of implications. $\square$

Why this is an external theorem. In the original Prikry ambient universe, the master sequence c is present. Any external $\mathcal{H}$ contains every ambient ordinal ω -sequence, including c . Consequently, the final external HOD must not simply be taken inside that original ambient universe: the master can expose real-coded information which the intermediate model deliberately omitted. Lemma 12.1 constructs a *fresh* choice envelope without new ordinal ω -sequences.

Likewise, the high witnesses in Section 11 need not belong to the final $\mathcal{H}$. Their existence in the ambient universe makes A definable, while weak homogeneity controls which hereditarily definable objects descend. This is why the proof neither completes the Cohen caps by tree-Prikry forcing nor requires a unique high witness for each code.

Proof dependencies and scope. The two nonstandard forcing results carrying the construction are Lemma 6.2 and Theorem 11.1. The former is a tagged finite-interpolation and countable-diagonal argument; the latter is an application of an existing approximation criterion to the explicitly displayed quotient order. They have different roles and neither substitutes for the other. Lemma 8.8 is the separate fine-structural reduction. In particular:

- (1) The global rank-preservation proof uses literal equality of Q, F and of supported parameters, not merely isomorphism of their indexing structures.
- (2) The DC proof uses external countable domination and wellorderable search spaces; it does not assert that every ambient countable sequence belongs to the inner model.
- (3) The cap spectrum is computed over the actual recognizable envelope. No classification of arbitrary incomparable grounds is invoked.
- (4) The secure preparation retains a proper class of measurables. No claim is made that the Prikry cardinal or every coding-site measurable survives.
- (5) The conclusion concerns first-order sentences true in a rank segment. It does not identify the complete theories of the three ambient universes, or establish the internal assertion $V = \text{HOD}_{\text{Ord}^\omega}$ in the original model.

APPENDIX A. TWO BOOKKEEPING DETAILS

A.1. Names, parameters, and nonstandard models. All uses of definability can be expressed through bounded-stage codes. A statement that a particular set is in $D(A)$ means that it occurs at a specified set-sized stage over a specified rank of D and the transitive closure of A . Similarly, a particular relative-HOD definition can be evaluated at a sufficiently large rank level, whose ordinal is included in its code. Thus the arguments never require a satisfaction predicate for the entire universe.

The forcing comparisons are assertions about fixed names and fixed formulas. Auxiliary collapses are used only to produce common generic realizations refuting incompatible decisions. The Prikry-only names obtained in Lemma 7.1 are constructed in the original model N . These observations are also what permits the relative-consistency argument to be formalized over arbitrary countable models rather than assuming transitive models.

For the class preparation, one may work in the usual conservative class-theoretic expansion used for progressively closed coding. Any set needed in the proof is captured at a set-sized initial segment, and any specified finite list of set-theoretic axioms and formulas is handled by the corresponding forcing and reflection argument. The CCA construction is the standard class-forcing interpretation from [20], with additional gaps to preserve the chosen measurable cardinals.

A.2. Uniformity of the high coding. There are three ordinal parameters in the decoder: the presentation length τ , an initial protection bound, and the envelope rank η . The site recursion uses all of 2^τ in its size bounds. Therefore the candidate forcing Q_e and cap cardinal ρ can be defined before the unknown spectrum is recovered. The final predicate says that e admits the two generic filters over the class $B = D(V_\eta)$; this is a first-order assertion, since the posets and their dense subsets belong to the indicated definable inner models.

Taking all presentation codes, rather than one chosen code, is essential for homogeneity. Recognizing their common pointed collapse makes A OD without making any particular wellordering of $\text{TC}(\{A\})$ OD. This is compatible with the final HOD being choiceless.

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