

FAILURE OF SYMBOL GENERATION IN p -ADIC VANISHING CYCLES

ABSTRACT. For every odd prime p , we construct a regular strictly henselian local ring A whose generic fibre has étale cohomology classes in $H_{\text{ét}}^2(A[1/p], \mu_p^{\otimes 2})$ which are not sums of cup products of degree-one classes. The ring is the strict henselization of a local ring on a finite-type arithmetic scheme. A unit on a principal divisor gives an element of order p in $SK_1(A[1/p])$, detected by its localization boundary on the special fibre. Adams–Riemann–Roch and the connectivity of the motivic filtration show that this element has nonzero image in $H_M^3(A[1/p], \mathbf{Z}(2))$. The integral coefficient sequence then gives the cohomological obstruction. Only the eigenvalues 4 and 8 of ψ^2 are needed.

1. INTRODUCTION

The local groups of p -adic vanishing-cycle theory include $H_{\text{ét}}^q(A[1/p], \mu_{p^n}^{\otimes q})$, where A is a strictly henselian local ring of mixed characteristic $(0, p)$. They are the closed geometric stalks of the higher direct images for $j : \text{Spec } A[1/p] \hookrightarrow \text{Spec } A$ [2, §1.1]. A unit on the generic fibre defines a Kummer class in degree one; products of these classes are called *symbols*.

Bloch–Kato described these groups for smooth models using symbols and differential forms [2]. Fontaine–Messing and Kato developed the crystalline and syntomic approach [4, 6]. Hyodo treated semistable reduction [5], and Saito–Sato proved symbol generation for quasi-log-smooth pairs under their stated hypotheses [9, Theorem 1.1]. Lüdgers–Morrow obtained a comparison with Milnor K -theory for local, p -henselian, ind-smooth algebras over a complete mixed-characteristic discrete valuation ring, with infinite residue field [8, Theorem 1.3(i)].

We show that regularity of the total local ring alone does not imply symbol generation. The field norm-residue theorem [11] allows entries which need not be units on the generic fibre. Bhatt–Mathew’s syntomic image theorem further distinguishes symbols of total-space units from those of generic-fibre units [1, Theorem 1.8 and footnote 2]. Our example fails generation even by the latter, at a geometric stalk on a finite-type arithmetic scheme.

Fix an odd prime p and put $k = \overline{\mathbf{F}}_p$. Set

$$A_0 = (\mathbf{Z}[x, y, z]/(p - x^p - yz))_{(p, x, y, z)}, \quad A = A_0^{\text{sh}}, \quad U = \text{Spec } A[1/p]. \quad (1)$$

The strict henselization is taken with residue field k . Put $S = A/pA$, $\mathfrak{p} = (x, y) \subset S$, and let $P = V(\mathfrak{p})$ be its prime Weil divisor. Write $i : D = V(y) \hookrightarrow U$. We use K for the K -theory of perfect complexes and G for the G -theory of coherent sheaves, with their homotopy groups written additively. A unit class is denoted by $[u] \in K_1$, and a coherent-module class by $[M]_G \in G_0$. Set

$$SK_1(X) = \ker(\det : K_1(X) \rightarrow \Gamma(X, \mathcal{O}_X^\times)).$$

The map $\partial_p : K_1(U) \rightarrow G_0(S)$ is the Cartan map followed by the connecting homomorphism of the canonical localization sequence

$$G(S) \longrightarrow G(A) \longrightarrow G(U) \longrightarrow \Sigma G(S) \quad (2)$$

[12, V, Theorem 5.1 and Application 6.1].

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Theorem 1.1. *The ring A is regular strictly henselian of dimension three. The class $\tau = i_*[x] \in SK_1(U)$ has exact order p and satisfies*

$$\partial_p \tau = [S/\mathfrak{p}]_G \neq 0.$$

Its image under the integral motivic edge map

$$\epsilon_U : SK_1(U) \longrightarrow H_{\mathbf{M}}^3(U, \mathbf{Z}(2))$$

of Lemma 3.1 has exact order p . Consequently the cup-product map

$$H_{\text{ét}}^1(U, \mu_p) \otimes_{\mathbf{F}_p} H_{\text{ét}}^1(U, \mu_p) \longrightarrow H_{\text{ét}}^2(U, \mu_p^{\otimes 2}) \quad (3)$$

is not surjective.

For $X = \text{Spec}(\mathbf{Z}[x, y, z]/(p - x^p - yz))$ and $j : X[1/p] \hookrightarrow X$, this gives failure of symbol generation at the geometric point over (p, x, y, z) , by the description of the stalks of $R^q j_* \mu_p^{\otimes q}$ in [10, Tag 03Q9].

The K -theory class is the pushforward of the unit x on D , where $x^p = p$. Its special-fibre boundary is detected by a nontrivial line bundle. The remaining step is to detect its motivic weight. After inverting 2, Adams–Riemann–Roch gives $\psi^2 \tau = 4\tau$, whereas ψ^2 acts by 8 on the kernel of the weight-two edge. These eigenvalues differ by a unit in $\mathbf{Z}[1/2]$. Since p is odd, this proves nonvanishing of the integral motivic image. The argument uses motives only on the regular characteristic-zero scheme U .

2. THE TORSION CLASS AND ITS LOCALIZATION BOUNDARY

Lemma 2.1. *The ring A is regular strictly henselian of dimension three, p is prime in A , and U is a regular noetherian $\mathbf{Q}$ -scheme of dimension two with $\text{Pic}(U) = 0$. The ring S is a normal domain, regular away from its closed point, and*

$$\text{div}_S(y) = pP, \quad \text{cl}(P) \neq 0 \quad \text{in } \text{Cl}(S). \quad (4)$$

Proof. The equation of A_0 has nonzero linear term p in the regular local ring $\mathbf{Z}[x, y, z]_{(p, x, y, z)}$. Hence A_0 is regular of dimension three with parameters x, y, z . Strict henselization preserves these properties, so A is a noetherian regular strictly henselian local ring with the same parameters [10, Tags 06LK and 06LN].

Put $S_0 = A_0/pA_0 = (\mathbf{F}_p[x, y, z]/(x^p + yz))_{(x, y, z)}$. The y -adic valuation shows that $-yz$ is not a p -th power in $\mathbf{F}_p(y, z)$, so $X^p + yz$ is irreducible over that field and S_0 is a domain. This two-dimensional hypersurface is Cohen–Macaulay; its partial derivatives in y, z are z, y , and a prime containing both also contains x . Thus S_0 is regular off its closed point, and Serre’s criterion gives normality. Since $S \simeq S_0^{\text{sh}}$ [10, Tag 05WS], it too is a two-dimensional normal local domain [10, Tags 06DI and 06LK]. In particular, S is regular away from its closed point, and p is a nonzero prime element of A .

Factoriality of regular local rings and their localizations gives

$$A[1/p]^{\times} = A^{\times} p^{\mathbf{Z}}, \quad \text{Pic}(U) = 0 \quad (5)$$

[10, Tags 0AG0 and 0BCH]. The closed point of $\text{Spec } A$ is removed in U , while the height-two prime (y, z) survives: $A/(y, z)$ is a discrete valuation ring with uniformizer x , and $p = x^p \neq 0$ in this quotient. Therefore $\dim U = 2$.

Since $A/(x, y)$ is a discrete valuation ring with uniformizer z , $\mathfrak{p} = (x, y)$ is a height-one prime of S . It is the unique prime minimal over y , since $x^p = -yz$. At $\mathfrak{p}$, the element z is a unit and $y = -x^p/z$, so x is a uniformizer. This proves $\text{div}_S(y) = pP$. Since $p = x^p + yz \in \mathfrak{m}_A^2$, the images of x, y in $\mathfrak{m}_S/\mathfrak{m}_S^2$ are independent. They are therefore independent in $\mathfrak{p}/\mathfrak{m}_S \mathfrak{p}$, so $\mathfrak{p}$ is not principal. A height-one prime of trivial divisor class in a normal domain is principal. Hence $\text{cl}(P) \neq 0$, and it has exact order p . $\square$

Let $Y = \operatorname{Spec} S \setminus \{\mathfrak{m}_S\}$ and $\mathcal{L} = \mathcal{O}_Y(P \cap Y)$. Normality and regularity off the closed point give $\operatorname{Pic}(Y) = \operatorname{Cl}(S)$; thus $[\mathcal{L}]$ has order p and $\mathfrak{p}|_Y = \mathcal{L}^{-1}$. We write Picard groups additively.

Proposition 2.2. *The pushforward $\tau = i_*[x]$ belongs to $SK_1(U)$, is killed by p , and satisfies $\partial_p \tau = [S/\mathfrak{p}]_G$. This boundary is nonzero; in particular, τ has exact order p .*

Proof. Since y is part of a regular system of parameters of A , the ring $B = A/yA$ is regular. Hence $D = \operatorname{Spec} B[1/p]$ is regular, and $x^p = p$ makes x a unit on D . Restriction of scalars along the regular closed immersion i preserves perfect complexes and defines $i_* : K_1(D) \rightarrow K_1(U)$. The class τ vanishes at the generic point of U , so its determinant is one. The projection formula gives

$$p\tau = i_*[x^p] = [p] \cdot [i_*\mathcal{O}_D] = 0, \quad (6)$$

since $0 \rightarrow \mathcal{O}_U \xrightarrow{y} \mathcal{O}_U \rightarrow i_*\mathcal{O}_D \rightarrow 0$ is exact [12, V, Projection Formula 3.12].

On $\operatorname{Spec} B$ one has $p = x^p$ and $D = \operatorname{Spec} B[1/x]$. The boundary of multiplication by x is the class of its cokernel B/xB . Naturality of (2) for $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$ and dévissage therefore give

$$\partial_p \tau = [A/(x, y)]_G = [S/\mathfrak{p}]_G \quad (7)$$

[12, V, Exercise 5.1]. The multiplicity is one: this is the boundary of x , not of x^p .

To detect this class, restrict to Y and take the determinant:

$$G_0(S) \longrightarrow G_0(Y) \simeq K_0(Y) \xrightarrow{\det} \operatorname{Pic}(Y).$$

The image of $[S/\mathfrak{p}]_G$ is $[\mathcal{O}_{P \cap Y}]$. The exact sequence $0 \rightarrow \mathcal{L}^{-1} \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{P \cap Y} \rightarrow 0$ shows that its determinant is $[\mathcal{L}] \neq 0$. $\square$

Equivalently, τ is multiplication by x on the perfect complex $[\mathcal{O}_U \xrightarrow{y} \mathcal{O}_U]$, in degrees $-1, 0$. This is a quasi-isomorphism, but not an automorphism of either free term; it cannot be cancelled as a difference of termwise automorphisms.

3. ADAMS OPERATIONS AND MOTIVIC DETECTION

For a regular noetherian $\mathbf{Q}$ -scheme X , let $F^r K(X) = \operatorname{Fil}_M^r K(X)$ be the integral motivic filtration, with $F^0 K(X) = K(X)$ and

$$\operatorname{gr}_M^r K(X) := \operatorname{cofib}(F^{r+1} K(X) \rightarrow F^r K(X)) \simeq R\Gamma_M(X, \mathbf{Z}(r))[2r]. \quad (8)$$

We use the filtration and regular-scheme comparison in [3, Theorem 3.6, Theorem 6.1(2), Remark 6.3]. For $\dim X \leq 2$, its connectivity bound is

$$\pi_n F^r K(X) = 0 \quad (n < r - 2). \quad (9)$$

This is [3, Theorem 3.6(1)] transported along the regular-scheme comparison (for noetherian schemes, valuative dimension equals Krull dimension). No finite-type hypothesis is required. The Adams operation ψ^2 preserves this filtration after inverting 2 and acts by 2^r on $\operatorname{gr}_M^r K(X)[1/2]$ [3, Theorem 4.14]. On regular affine schemes it agrees with the classical higher Adams operation [3, Appendix B.1]. Here $[1/2]$ localizes spectra or abelian groups, not the scheme X .

Lemma 3.1 (The two relevant weights). *For an integral regular noetherian $\mathbf{Q}$ -scheme X of dimension at most two, there is a natural surjection*

$$\epsilon_X : SK_1(X) \longrightarrow H_M^3(X, \mathbf{Z}(2))$$

induced by $F^2 K(X) \rightarrow \operatorname{gr}_M^2 K(X)$. On $(\ker \epsilon_X)[1/2]$, the operation ψ^2 is multiplication by 8. Consequently $\epsilon_X[1/2]$ is injective on the subgroup on which ψ^2 is multiplication by 4.

Proof. In weight zero, $\pi_n \operatorname{gr}_M^0 K(X) = 0$ for $n > 0$ [3, Theorem 3.6(3), Theorem 6.1(2)]. In weight one, the low-degree comparison gives

$$\pi_2 \operatorname{gr}_M^1 K(X) = 0, \quad \pi_1 \operatorname{gr}_M^1 K(X) = \Gamma(X, \mathcal{O}_X^\times)$$

[3, Corollary 7.11]. Thus $\pi_1 F^1 K(X) \simeq K_1(X)$, and the map from this group to $\pi_1 \operatorname{gr}_M^1 K(X)$ is the determinant, as can be checked on local rings. The weight-one cofiber sequence therefore identifies $\pi_1 F^2 K(X) = SK_1(X)$.

By (9), $F^4 K(X)$ is 2-connective. Hence

$$\pi_1 F^3 K(X) \simeq \pi_1 \operatorname{gr}_M^3 K(X) = H_M^5(X, \mathbf{Z}(3)).$$

Since $F^3 K(X)$ is 1-connective, the weight-two cofiber sequence gives

$$H_M^5(X, \mathbf{Z}(3)) \longrightarrow SK_1(X) \xrightarrow{\epsilon_X} H_M^3(X, \mathbf{Z}(2)) \longrightarrow 0. \quad (10)$$

After inverting 2, ψ^2 acts by $2^3 = 8$ on the leftmost group, so it acts by 8 on $\ker \epsilon_X$. A class in this kernel with eigenvalue 4 is killed by $8 - 4 = 4$, and is therefore zero after inverting 2. $\square$

Lemma 3.2. *One has $\psi^2 \tau = 4\tau$ in $K_1(U)[1/2]$.*

Proof. For a regular closed immersion, Adams–Riemann–Roch gives

$$\psi^2 i_*(\xi) = i_*(\theta^2(N_i^\vee) \psi^2(\xi)) \quad \text{in } K_1(U)[1/2]$$

[7, Theorem 4.5 and Remark 4.6(a), p. 432]. Here N_i is the normal line bundle, and the Bott class of a line bundle L is $\theta^2(L) = 1 + [L]$. The conormal line of the principal divisor D is trivial, so $\theta^2(N_i^\vee) = 2$. Also $\psi^2[x] = [x^2] = 2[x]$. Applying the formula to $\xi = [x]$ gives the result. Apply the theorem with base scheme U and the trivial group scheme. Both U and D are affine of finite type over this base and have the resolution property, and i is a regular closed immersion. The closed-immersion formula requires no completion of K -theory. The operations in this formula agree with the classical higher Adams operations by [7, Remark 2.8(a), p. 424], and hence with the filtered operations used here by [3, Appendix B.1]. $\square$

Proof of Theorem 1.1. Lemma 2.1 and Proposition 2.2 establish the assertions about A and τ . Define $c = \epsilon_U(\tau) \in H_M^3(U, \mathbf{Z}(2))$. The class τ remains nonzero after inverting 2, since it has order p and p is odd. Lemmas 3.1 and 3.2 therefore give $c \neq 0$. Since $p\tau = 0$ and ϵ_U is an integral homomorphism, $pc = 0$. Thus c has exact order p .

The coefficient triangle and the multiplicative diagonal motivic-étale comparison yield

$$0 \longrightarrow H_M^2(U, \mathbf{Z}(2))/p \longrightarrow H_{\text{ét}}^2(U, \mu_p^{\otimes 2}) \xrightarrow{\beta} H_M^3(U, \mathbf{Z}(2))[p] \longrightarrow 0 \quad (11)$$

[3, Theorem 4.41(1), Remark 4.42]. The map β is the connecting homomorphism for multiplication by p on the integral motivic complex, transported by the comparison. Since $\operatorname{Pic}(U) = 0$, Kummer theory makes every degree-one class the class of a global unit [10, Tag 03PK]. It lifts to $H_M^1(U, \mathbf{Z}(1)) = \Gamma(U, \mathcal{O}_U^\times)$, so products of degree-one classes lift integrally and have zero Bockstein. A preimage of $c \neq 0$ under (11) cannot be a sum of such products. $\square$

Remark 3.3. The construction and nonvanishing of τ in Section 2 also hold for $p = 2$. The restriction to odd primes enters only when inverting 2 to separate Adams weights. It imposes no further restriction: in particular, the argument applies to $p = 3$. We do not assert an identification of all of $SK_1(U)$ with motivic cohomology, or of all finite-coefficient $K_2(U)$ with étale cohomology.

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