

FROBENIUS ORBIT MAPS AND RANK-FOUR GROUP SCHEMES

ABSTRACT. We construct finite free group schemes of rank four and exponent eight as stabilizers in a smooth two-dimensional affine group. We first compute the complete local deformation ring of the invariant space of constants and squares in characteristic two, using six Grassmannian coordinates. For every Artinian specialization, evaluation at the identity has a finite free rank-four stabilizer. Its fourth-power morphism is $(x, y) \mapsto (2bxy, 2axy)$, and its eighth-power morphism is trivial. The fourth power is nonzero on the third-order neighbourhood of the universal deformation ring and on a specialization over $\mathbf{Z}[a, b]/(a^2b - 2, a^3, b^3)$. We give complete formulas for the latter group scheme. Right translation preserves the chosen quadratic space exactly when it contains the square of the scalar character. This identifies $(2a, 2b)$ as the common obstruction to fourth-power vanishing, normality of the stabilizer, and two-sidedness in an equivalent cyclic-module construction.

1. INTRODUCTION

An affine group scheme over a commutative ring T is an affine T -scheme $G = \operatorname{Spec} B$ whose multiplication, identity, and inverse are morphisms of schemes satisfying the group axioms. Equivalently, they give a group structure on $G(T') = \operatorname{Hom}_{T\text{-alg}}(B, T')$, naturally in every commutative T -algebra T' . It is *finite locally free of rank n* if B is locally on $\operatorname{Spec} T$ a free T -module of rank n ; *finite free* means that B is free over T . This rank does not count the points of G .

Grothendieck's power question asks whether $[n] : G \rightarrow G$, the morphism $g \mapsto g^n$ on every test algebra, must be constant at the identity when G has rank n . We use the formulation in the Gille–Polo re-edition of SGA 3, Exposé VIII, Remark 7.3.1 and N.D.E. (38) [2]. The answer is affirmative over a field [1, Exp. VII_A, Prop. 8.5], and hence over a reduced base by passage to the generic points [2, Exp. VIII, Rem. 7.3.1]. Deligne proved the assertion for commutative finite locally free group schemes over arbitrary bases [3, pp. 4–5]; see also [4, §§1 and 3]. A counterexample must therefore involve both a noncommutative group law and a nonreduced base. An explicit rank-four counterexample is recorded in [9] and formalized in Lean in the Mathlib module `Counterexamples.GrothendieckPower` [10]. That formalization uses a different Hopf-algebra presentation from the scalar-affine construction below; no identification of the two group schemes is made here. A comparison is given at the end of Section 5. For a general account of finite flat group schemes, including the rank-annihilation question, see Tate [6] and Schoof [4].

Schoof's earlier paper gives a positive result over certain nonreduced bases: if T is a local Artinian ring with maximal ideal $\mathfrak{m}_T$ and residue characteristic $p > 0$, and if $\mathfrak{m}_T^p = p\mathfrak{m}_T = 0$, then every finite flat group scheme over T is killed by its rank [5, Theorem 1.1]. In particular, this holds whenever $\mathfrak{m}_T^2 = 0$ [5, Corollary 1.2]. That paper also studies deformations of noncommutative special fibers; its Proposition 3.3 gives triviality after a finite faithfully flat extension, rather than necessarily over the original ring. Triviality after a faithfully flat extension should therefore be distinguished from triviality over the original base. The counterexample bases constructed below have a nonzero doubled parameter and do not satisfy Schoof's small-thickening hypotheses.

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This paper constructs such group schemes from an invariant space of quadratic functions. Write $k = \mathbf{F}_2$ for the field with two elements, and $T\{u_1, \dots, u_r\}$ for a free T -module with the indicated basis. Over k , translations of the affine plane preserve

$$\mathcal{U}_0 = k\{1, X^2, Y^2\}.$$

The orbit of evaluation at the origin is the Frobenius map $(x, y) \mapsto (x^2, y^2)$. Here *Frobenius* means the map induced by squaring functions in characteristic two. Its kernel is $\alpha_2 \times \alpha_2$, where $\alpha_2 = \operatorname{Spec} k[u]/(u^2)$ has the additive group law; over a k -algebra k' , its points are the elements $t \in k'$ with $t^2 = 0$. We deform this invariant space in the polynomial representation of a smooth group of scalar affine transformations. Invariance supplies a subgroup; a deformation of the Frobenius basis proves its finite flatness; affine iteration computes its powers. These three steps are proved separately.

1.1. Main results. Write $\mathbf{Z}$ for the integers and $\mathbf{Z}_2 = \varprojlim_j \mathbf{Z}/2^j\mathbf{Z}$ for the 2-adic integers; $\mathbf{Z}/4$ abbreviates $\mathbf{Z}/4\mathbf{Z}$. Put $S = \mathbf{Z}[[a, b]]$, the formal power-series ring. For a local ring T , write $\mathfrak{m}_T$ for its maximal ideal. The ambient group P over S consists of the transformations

$$g_v(Z) = v + \mu_v Z, \quad v = (x, y), \quad \mu_v = 1 + ax + by, \quad (1.1)$$

on the locus where μ_v is invertible, with $Z = (X, Y)^t$ a column of polynomial variables. All vector formulas use ordinary coordinate addition and scalar multiplication unless the group product $*$ is displayed. We write $T[X, Y]_{\leq 2}$ for polynomials of total degree at most two; this is a free module of rank six, not a subalgebra. An *invariant lift* over a local Artinian $\mathbf{Z}_2$ -algebra T with residue field k consists of $a, b \in \mathfrak{m}_T$ and a P_T -invariant rank-three direct summand

$$1 \in \mathcal{U} \subset T[X, Y]_{\leq 2}, \quad \mathcal{U} \otimes_T k = \mathcal{U}_0.$$

The polynomial coordinates are fixed. A basis $1, f_1, f_2$ is *normalized* when f_1, f_2 have zero constant terms and their coefficients of X^2, Y^2 form the identity matrix. Here a rank- r *subbundle* is a finite locally free submodule of rank r with locally free quotient; over the local rings used here it is a direct summand. We also call a rank-two or rank-three subbundle a two-plane or three-space. A *representation* is a group-scheme action by linear automorphisms; invariance means preservation after every base change, not just by the points over T . This deformation problem is defined precisely in Definition 2.2.

Evaluation is the functional $\operatorname{ev}_v(f) = f(v)$. For an action of a group scheme P and a section q_0 , the *stabilizer* $\operatorname{Stab}_P(q_0)$ has, over each test algebra, the elements g satisfying $gq_0 = q_0$. For a subbundle H_0 the condition is $gH_0 = H_0$. We write $\operatorname{Gr}_T^{\operatorname{sub}}(r, V)$ for the Grassmannian whose points over T' are the rank- r subbundles of $V \otimes_T T'$. The superscript specifies the subbundle convention rather than the quotient convention.

For a group scheme G , the notation $[m]_G = e$ means that g^m is the identity over every test algebra. Here e denotes the identity section, or the constant morphism with that value, according to its domain. The *exponent* is the least positive m with this property, when such an m exists. A power morphism need not be a group homomorphism. The *order of a point* $g \in G(T')$ is its order in that ordinary group: the least positive m for which $g^m = e$.

Theorem A (The universal family). *The functor of invariant lifts is prorepresented by the complete local $\mathbf{Z}_2$ -algebra D given in Theorem 2.6: its invariant lifts over each test ring T are naturally classified by continuous local maps $D \rightarrow T$. Let $D \rightarrow T$ be a continuous local $\mathbf{Z}_2$ -algebra homomorphism to a local Artinian ring with residue field k , and let $\mathcal{U}$ be the corresponding invariant lift. Put*

$$H_0 = \ker(\operatorname{ev}_0 : \mathcal{U} \rightarrow T), \quad G_T = \operatorname{Stab}_{P_T}(H_0) \subset P_T,$$

where the stabilizer is taken in $\operatorname{Gr}_T^{\operatorname{sub}}(2, \mathcal{U})$.

- (1) Rank. *The group scheme G_T is finite free of rank four. If $1, f_1, f_2$ is the normalized basis of $\mathcal{U}$, its coordinate algebra is*

$$B_T = T[x, y]/(f_1, f_2), \quad B_T = T\{1, x, y, xy\}.$$

Its special fiber over k is $\alpha_2 \times \alpha_2$.

- (2) Powers. *On the universal point $(x, y) \in G_T(B_T)$,*

$$[4](x, y) = (2bxy, 2axy), \quad [8] = e. \quad (1.2)$$

In particular, $[4] = e$ if and only if $2a = 2b = 0$ in T .

- (3) Nonvanishing. *If $\mathfrak{n}$ is the maximal ideal of D , both $2a$ and $2b$ are nonzero in $A_4 = D/\mathfrak{n}^4$. Thus G_{A_4} has rank four and exponent exactly eight.*

Theorem 2.6 gives the ring D explicitly. Its six initial parameters $\lambda_1, \dots, \lambda_6$ are the coefficients of a general normalized lift of $\mathcal{U}_0$; they are not chosen in advance. Theorem 3.4 proves finite freeness, Proposition 4.3 computes the powers, and Theorem 4.7 proves their nonvanishing on A_4 .

The second result is a specialization of the same family.

Theorem B (The two-generator example). *Let*

$$R = \mathbf{Z}[a, b]/(a^2b - 2, a^3, b^3), \quad (1.3)$$

and let

$$B_R = R[x, y]/(x^2 + abx + b^2(1 + ax)y, y^2 + a^2y).$$

For every commutative R -algebra R' , the solutions $v = (x, y)$ of the two defining equations form a group under

$$v * v' = v + \mu_v v', \quad e = (0, 0), \quad v^{-1} = -\mu_v^{-1}v, \quad \mu_v = 1 + ax + by.$$

These operations define a group scheme $G_R = \text{Spec } B_R$. Its coordinate algebra is free over R on $1, x, y, xy$. The group is noncommutative and satisfies

$$[4](x, y) = (2bxy, 0) \neq e, \quad [8] = e.$$

Its special fiber over $R/(a, b) = \mathbf{F}_2$ is $\alpha_2 \times \alpha_2$.

Section 5 proves this theorem and specifies the specialization $D \rightarrow R$. Appendix B records the multiplication table, all Hopf maps, and the relevant representations. The ring R is obtained by prescribing an orbit derivative and imposing compatibility. It is not the universal truncation A_4 .

1.2. The structural interpretation. We use the *fppf topology*, whose coverings are jointly surjective families of flat morphisms locally of finite presentation. A statement holds fppf locally if it holds after such a covering. An fppf sheaf is a functor whose local sections glue uniquely for these coverings. The quotient P/G denotes the fppf sheaf associated to the functor of cosets for right multiplication, not an a priori quotient group. A *homogeneous space* here is a scheme with an action whose points are fppf locally in the orbit of its chosen point. For the topology, see [8, Tag 021L]; the torsor and quotient assertions used here are proved directly in Section 3.

The invariant space determines an affine plane of normalized functionals

$$\mathcal{X} = \{q \in \mathcal{U}^\vee : q(1) = 1\},$$

and $P_T/G_T \simeq \mathcal{X}$ as fppf sheaves. On its stabilizer, the residual action on this plane is multiplication by μ_v^2 . Proposition 4.3 identifies this action with the fourth-power displacement:

$$[4](v) = (\mu_v^2 - 1)v. \quad (1.4)$$

The bridge between the group and the quadratic space is the function $\sigma(X, Y) = (1 + aX + bY)^2$. The space is already invariant under left translation. Right translation requires just one additional condition: $\sigma \in \mathcal{U}$. Proposition 7.3 proves that this is equivalent to $2a = 2b = 0$. Thus the obstruction

is the failure to retain one particular quadratic function, not an unrelated coincidence of coefficient calculations.

Theorem 7.6 gives an algebraic counterpart. The invariant space is dual to a cyclic quotient $\mathcal{A}/K$ of a rank-six associative algebra: the class of 1 generates this quotient as a left module. A *left ideal* K satisfies $\mathcal{A}K \subseteq K$; it is *two-sided* if also $K\mathcal{A} \subseteq K$. The kernel K in this construction is always a left ideal. It is two-sided exactly when $2a = 2b = 0$, which is also the exact condition for G_T to be normal in P_T and for [4] to vanish. A subgroup is *normal* if conjugation by every point of the ambient group, after every base change, preserves it; we write $G \triangleleft P$. These equivalences are properties of this family, not general implications from nonnormality to excessive exponent.

1.3. Conventions and organization. All rings have an identity and ring homomorphisms preserve it. Coefficient rings are commutative; auxiliary algebras need not be, but their scalars are central. A module is *flat* if tensoring with it preserves exact sequences, and *faithfully flat* if, in addition, tensoring detects the zero module. Over affine schemes, a morphism is finite free of degree r if its coordinate algebra is free of rank r over the target algebra. A morphism is *locally of finite presentation* if locally its algebra is given by finitely many generators and relations. It is *smooth* if, in addition, it has the infinitesimal lifting property: for every square-zero closed immersion $Z_0 \hookrightarrow Z_1$ of schemes over the target, a morphism from Z_0 to the source extends locally on Z_1 to a morphism from Z_1 [8, Tag 02H6]. Here square-zero means that the defining ideal has square zero. Relative dimension is the dimension of the fibers; our smooth ambient examples are open subschemes of affine spaces.

We abbreviate $\mathrm{Spec} T$ to T in base subscripts and fiber products. Thus $X \times_T T'$ means $X \times_{\mathrm{Spec} T} \mathrm{Spec} T'$, and $V_{T'} = V \otimes_T T'$ denotes scalar extension of a module. For $X = \mathrm{Spec} B$, its coordinate algebra is $\mathcal{O}(X) = B$ and $X(T') = \mathrm{Hom}_{T\text{-alg}}(B, T')$. A *section* over T is a morphism $\mathrm{Spec} T \rightarrow X$ over T . For a local base with residue field k , the *special fiber* is $X_k = X \times_T k$; reduction of a module is $V \otimes_T k$. If $\psi : X \rightarrow Y$ is a morphism of affine schemes, $\psi^* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is pullback of functions. The *universal point* of $\mathrm{Spec} B$ is the B -point corresponding to id_B ; checking that point checks a morphism on every test algebra.

We write $\mathbf{A}_T^r = \mathrm{Spec} T[t_1, \dots, t_r]$. The additive and multiplicative group schemes are $\mathbf{G}_{aT} = \mathrm{Spec} T[t]$ and $\mathbf{G}_{mT} = \mathrm{Spec} T[t, t^{-1}]$, with addition and multiplication respectively. Thus $\mathbf{G}_{aT}(T') = (T', +)$ and $\mathbf{G}_{mT}(T') = (T')^\times$, the group of units. The group scheme $\mathrm{GL}_{r,T}$ has as its T' -points the invertible $r \times r$ matrices. A representation is *faithful* here if its homomorphism to a general linear group is a closed immersion. For a finite locally free module V , its dual is $V^\vee = \mathrm{Hom}_T(V, T)$. A frame is a chosen ordered basis; matrices act on columns, $(-)^t$ denotes transpose, and I_r is the identity matrix.

We use T for a commutative coefficient ring, imposing the Artinian hypothesis explicitly when required. We reserve A_4 for the universal truncation and R for (1.3). An *Artinian point* of D is a continuous local $\mathbf{Z}_2$ -algebra map $D \rightarrow T$, with T as in Definition 2.2. For algebraic statements about the universal family, a *D-algebra* means a commutative ring with a specified unital map from D ; no topology or Artinian hypothesis is imposed. Parameters and their images under scalar extension have the same names. Upper-case X, Y are polynomial variables; lower-case x, y are the coordinate functions of a group element. The notation μ denotes the function $1 + ax + by$, and μ_v its value at v . A *group character* is a group-scheme homomorphism to $\mathbf{G}_m$; in particular μ is the scalar character. An *algebra character*, used in Section 6, is instead a unital homomorphism from a T -algebra to T .

The proof follows the order of the construction. Section 2 computes invariant spaces. Section 3 constructs their finite flat stabilizers, and Section 4 proves the universal counterexample. Section 5 treats the explicit specialization. Sections 6–7 establish the structural interpretation. The appendices

contain additional information on the bases and complete formula tables. A guide to the principal notation begins on page 33.

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2. THE INVARIANT-SPACE DEFORMATION PROBLEM

Throughout this section, $S = \mathbf{Z}[[a, b]]$ and $k = S/(2, a, b) = \mathbf{F}_2$. We first specify the group and the deformation functor, and then calculate its representing ring.

2.1. The ambient group. For a point $v = (x, y)$, set $\ell(v) = ax + by$ and $\mu_v = 1 + \ell(v)$.

Proposition 2.1 (The scalar-affine group). *Over $S = \mathbf{Z}[[a, b]]$, the scheme*

$$P = \operatorname{Spec} S[x, y, (1 + ax + by)^{-1}] \quad (2.1)$$

is a smooth group scheme of relative dimension two with identity 0 and operations

$$v * w = v + \mu_v w, \quad \mu_{v*w} = \mu_v \mu_w, \quad v^{-1} = -\mu_v^{-1} v. \quad (2.2)$$

Its action $g_v(Z) = v + \mu_v Z$ preserves $\mathcal{V} = S[X, Y]_{\leq 2}$. Its special fiber is the translation group $\mathbf{G}_{a_k}^2$.

Proof. Linearity of ℓ gives

$$1 + \ell(v + \mu_v w) = (1 + \ell(v))(1 + \ell(w)).$$

Thus the transformations are closed under composition and inversion, with the displayed operations. Associativity follows from composition. The underlying scheme is an open subscheme of the affine plane, hence is smooth. Affine substitution preserves polynomial degree, and the specialization $a = b = 0$ gives translations. $\square$

For any commutative ring T and $a, b \in T$, write $P_{a,b} = \operatorname{Spec} T[x, y, (1 + ax + by)^{-1}]$ with the same operations. When the coefficients are understood, we also write P_T .

We use the left action $g \cdot f = f \circ g^{-1}$ on functions. A submodule is preserved by all these operators after every base change if and only if it is preserved by forward substitutions $f \mapsto f \circ g$: inversion permutes the elements of $P_{a,b}(T')$ for every test algebra T' . We use forward substitution for the coefficient calculation. The representation has basis

$$\mathcal{V} = S\{1, X, Y, X^2, XY, Y^2\}.$$

On the special fiber it contains $\mathcal{U}_0 = k\{1, X^2, Y^2\}$, since $(X + x)^2 = X^2 + x^2$ and $(Y + y)^2 = Y^2 + y^2$.

2.2. The functor and its graph coordinates.

Definition 2.2 (Invariant lifts). Let $\text{Art}_{\mathbf{Z}_2, k}$ be the category of local Artinian $\mathbf{Z}_2$ -algebras with a specified identification $T/\mathfrak{m}_T \simeq k$ of their residue field. Here *local* means that there is a unique maximal ideal, and *Artinian* means that descending chains of ideals stabilize. In these test rings the maximal ideal is nilpotent: some power is zero. Morphisms are local $\mathbf{Z}_2$ -algebra homomorphisms inducing the identity on k . For T in this category, let $\mathcal{D}(T)$ be the set of triples $(a, b, \mathcal{U})$ such that $a, b \in \mathfrak{m}_T$ and

$$1 \in \mathcal{U} \subset T[X, Y]_{\leq 2} \quad (2.3)$$

is a $P_{a,b}$ -invariant rank-three direct summand reducing to $\mathcal{U}_0$. The affine coordinates are fixed, and no change of coordinates is divided out. Thus *lift* or *deformation* means the specified embedded subbundle with this reduction, not an isomorphism class obtained by changing the ambient coordinates. A test-ring map sends $(a, b, \mathcal{U})$ to its scalar extension; this defines the functor $\mathcal{D}$.

Since $a, b \in \mathfrak{m}_T$, they generate a nilpotent ideal. Thus μ_v is a unit on all of $\mathbf{A}_T^2$, and $P_T = \mathbf{A}_T^2$ as a scheme.

Lemma 2.3 (The six graph coordinates). *Let $T \in \text{Art}_{\mathbf{Z}_2, k}$ and let $\mathcal{U} \subset T[X, Y]_{\leq 2}$ be a rank-three direct summand containing 1 and reducing to $\mathcal{U}_0 = k\{1, X^2, Y^2\}$. No invariance is assumed. Then $\mathcal{U}$ has a unique normalized basis $1, f_1, f_2$ with*

$$\begin{aligned} f_1 &= X^2 + \lambda_1 X + \lambda_2 Y + \lambda_3 XY, \\ f_2 &= Y^2 + \lambda_4 X + \lambda_5 Y + \lambda_6 XY, \end{aligned} \quad (2.4)$$

where $\lambda_i \in \mathfrak{m}_T$. Conversely every such six-tuple defines a direct summand with the stated reduction.

Proof. Projection to the coefficients of $1, X^2, Y^2$ is an isomorphism on the special fiber of the submodule, hence an isomorphism of free rank-three T -modules. Its inverse sends the first basis vector to 1. The other two inverse images have zero constant terms and the prescribed pure-square coefficients, leaving precisely the six displayed coefficients. Their reductions are zero because the two normalized inverse images reduce to X^2 and Y^2 ; hence $\lambda_i \in \mathfrak{m}_T$. Conversely that projection is a left inverse for every displayed graph basis. $\square$

The word *normalized* means that the constant terms of f_1, f_2 vanish and their X^2, Y^2 coefficient matrix is I_2 . Modulo the constant line, a nearby two-plane is the graph of a linear map from $T\{X^2, Y^2\}$ to $T\{X, Y, XY\}$. The six entries of that map are the *graph coordinates* in

$$\text{Gr}_S^{\text{sub}}(2, \mathcal{V}/S \cdot 1); \quad (2.5)$$

see [8, Tag 089R]. Each of the two square directions can acquire components in X, Y, XY . The values of the parameters are nilpotent on Artinian test rings; the formal parameters themselves need not be nilpotent in the representing ring.

2.3. The invariance calculation. For the normalized polynomials, put $F = (f_1, f_2)^t$. Its *Jacobian* is the matrix of formal polynomial derivatives

$$J_F(X, Y) = \begin{pmatrix} \partial_X f_1 & \partial_Y f_1 \\ \partial_X f_2 & \partial_Y f_2 \end{pmatrix}.$$

Write $L = J_F(0, 0)$, and use L also for the linear map $v \mapsto Lv$. Thus $F = Q_2 + L$ denotes the sum of its homogeneous degree-two and degree-one parts, and the differential at v is $dF_v(w) = J_F(v)w$. The *polarization* of Q_2 is

$$B_{Q_2}(v, w) = Q_2(v + w) - Q_2(v) - Q_2(w).$$

It is a symmetric bilinear map over any coefficient ring.

Let q_1, q_2 be the standard basis of the rank-two module of values. Later these vectors are identified with the functionals dual to f_1, f_2 and vanishing on 1. We use the coefficient vectors

$$A_1 = \begin{pmatrix} \lambda_1 \\ \lambda_4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \lambda_2 \\ \lambda_5 \end{pmatrix}, \quad C_{12} = \begin{pmatrix} \lambda_3 \\ \lambda_6 \end{pmatrix}.$$

Thus $L = J_F(0)$ has columns A_1, A_2 , and C_{12} is the coefficient of XY .

Proposition 2.4 (Equivalent invariance conditions). *Let T be any commutative ring, and choose $a, b, \lambda_1, \dots, \lambda_6 \in T$. Define $P_{a,b} = \text{Spec } T[x, y, (1 + ax + by)^{-1}]$ by the scalar-affine law (2.2), and let f_1, f_2 be the polynomials (2.4). Put $\mathcal{U} = T\{1, f_1, f_2\}$ and $F = (f_1, f_2)^t = Q_2 + L$. No nilpotence hypothesis is needed. The following conditions are equivalent:*

- (1) $\mathcal{U}$ is invariant under $P_{a,b}$;
- (2) the polynomial identity

$$F(v + \mu_v w) = F(v) + \mu_v^2 F(w) \tag{2.6}$$

holds in $T[x, y, s, t]^2$ for independent pairs $v = (x, y)$ and $w = (s, t)$;

- (3) the polarization satisfies

$$B_{Q_2}(v, w) = \ell(v)L(w); \tag{2.7}$$

- (4) the coefficients satisfy the eight scalar equations

$$\begin{aligned} a\lambda_1 &= 2, & b\lambda_2 &= 0, & \lambda_3 &= a\lambda_2 = b\lambda_1, \\ a\lambda_4 &= 0, & b\lambda_5 &= 2, & \lambda_6 &= a\lambda_5 = b\lambda_4. \end{aligned} \tag{2.8}$$

Proof. Invariance and the orbit identity. First note that multiplication by $1 + ax + by$ is injective on $M[x, y]$ for every T -module M : the least nonzero homogeneous component of a polynomial is unchanged. The map to its localization at μ_v is therefore injective. Polynomial identities obtained on $P_{a,b}$ consequently hold before localization.

Assume (1). Work with the universal element over $T[x, y, \mu_v^{-1}]$. Invariance under forward substitution follows from the inversion argument above. Comparing the X^2, Y^2 coefficients forces the component of $f_i \circ g_v$ in the span of f_1, f_2 to be $\mu_v^2 f_i$. Its constant term is $f_i(v)$. This gives (2) after localization, and hence as a polynomial identity by injectivity. Conversely, (2) writes every $f_i \circ g_v$ as an element of $\mathcal{U}$ after arbitrary base change. Apply the same identity to g_v^{-1} to obtain equality of the translated submodules, not just inclusion. Thus $\mathcal{U}$ is invariant, proving (2) $\Rightarrow$ (1).

Polarization. Homogeneity and linearity give the exact identity

$$F(v + \mu_v w) - F(v) - \mu_v^2 F(w) = \mu_v (B_{Q_2}(v, w) - \ell(v)L(w)). \tag{2.9}$$

Injectivity of multiplication by μ_v , already proved above, gives (2) $\Leftrightarrow$ (3).

Coefficient comparison. Evaluating (3) on the four pairs of source basis vectors gives exactly

$$aA_1 = 2q_1, \quad bA_2 = 2q_2, \quad bA_1 = aA_2 = C_{12}. \tag{2.10}$$

Its scalar components are (2.8). Conversely these equations determine all four values of the bilinear error, so that error is zero. This proves (3) $\Leftrightarrow$ (4). $\square$

Each chain defining λ_3 or λ_6 contains two equations. The factors of two come from the diagonal polarization of a square; the equality of the mixed values comes from symmetry. Appendix C gives the two separate substitution errors.

Corollary 2.5 (The Jacobian criterion). *Let T be a commutative ring, let $a, b \in T$, and let $F \in T[X, Y]^2$ have total degree at most two and satisfy $F(0) = 0$. Put $\mu_v = 1 + ax + by$ for $v = (x, y)$. Then the polynomial identity*

$$F(v + \mu_v w) = F(v) + \mu_v^2 F(w)$$

is equivalent to

$$J_F(X, Y) = (1 + aX + bY)J_F(0). \quad (2.11)$$

No normalization of the quadratic coefficients is required.

Proof. Write $F = Q_2 + L$ as the sum of its homogeneous quadratic and linear parts. Then $dF_v(w) = L(w) + B_{Q_2}(v, w)$. The Jacobian identity is therefore (2.7), which is equivalent to the orbit identity by the exact defect formula (2.9). That formula does not use normalized coefficients. $\square$

This criterion uses the degree-two hypothesis. It is not a claim that representations in positive characteristic are determined by their differentials.

2.4. The representing ring. A local ring D with maximal ideal $\mathfrak{n}$ is *complete* here for its $\mathfrak{n}$ -adic topology, so $D \simeq \varprojlim_j D/\mathfrak{n}^j$. The completion of a local ring U at its maximal ideal $\mathfrak{m}_U$ is $\hat{U} = \varprojlim_j U/\mathfrak{m}_U^j$. A map of local rings is *local* if it carries the maximal ideal into the maximal ideal. The notation $\text{Hom}_{\text{cont, loc}, \mathbf{Z}_2}(D, T)$ means continuous local $\mathbf{Z}_2$ -algebra homomorphisms, using the maximal-ideal topologies; the topology of an Artinian test ring is discrete. To say that D *prorepresents* $\mathcal{D}$ means that these Hom sets are naturally in bijection with $\mathcal{D}(T)$ for all test rings T .

Theorem 2.6 (The universal invariant space). *Let I denote the ideal generated by the eight relations (2.8) in each indicated coefficient ring. The functor $\mathcal{D}$ of Definition 2.2 is prorepresented by*

$$D = \mathbf{Z}_2[[a, b, \lambda_1, \dots, \lambda_6]]/I. \quad (2.12)$$

This is the complete local ring at $\mathcal{U}_0$ of the invariant-subspace locus in (2.5). Eliminating the two mixed coefficients gives

$$D \simeq \frac{\mathbf{Z}_2[[a, b, \lambda_1, \lambda_2, \lambda_4, \lambda_5]]}{(a\lambda_1 - 2, b\lambda_5 - 2, a\lambda_4, b\lambda_2, a\lambda_2 - b\lambda_1, a\lambda_5 - b\lambda_4)}. \quad (2.13)$$

Explicitly, continuous local $\mathbf{Z}_2$ -algebra maps $D \rightarrow T$ correspond naturally and bijectively to triples $(a, b, \mathcal{U}) \in \mathcal{D}(T)$, by taking the images of the parameters. The tautological (universal) invariant space is $D\{1, f_1, f_2\}$, whose base change along a map $D \rightarrow T$ gives the corresponding invariant lift. Here

$$f_1 = X^2 + \lambda_1 X + \lambda_2(1 + aX)Y, \quad f_2 = Y^2 + \lambda_4 X + \lambda_5(1 + aX)Y. \quad (2.14)$$

Proof. Lemma 2.3 identifies unequivariant lifts with six-tuples in $\mathfrak{m}_T$. Proposition 2.4 makes equivariance equivalent to the relations I . Hence

$$\mathcal{D}(T) \simeq \text{Hom}_{\text{cont, loc}, \mathbf{Z}_2}(D, T)$$

naturally in T . The algebraic graph chart has ring $S[\lambda_1, \dots, \lambda_6]$, and its invariance locus is cut out by these same equations. Completion at $(2, a, b, \lambda_1, \dots, \lambda_6)$ gives D . Eliminating $\lambda_3 = a\lambda_2$ and $\lambda_6 = a\lambda_5$ yields the remaining assertions. $\square$

The four remaining coefficients form the derivative matrix. Their relations are equivalently

$$L \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = 2I_2, \quad L \begin{pmatrix} b \\ -a \end{pmatrix} = 0, \quad L = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_4 & \lambda_5 \end{pmatrix}. \quad (2.15)$$

Corollary 2.7 (The characteristic constraint). *The ideal $I \subset \mathbf{Z}[a, b, \lambda_1, \dots, \lambda_6]$ contains 4. Hence $4 = 0$ in every commutative ring satisfying (2.8), in particular in D .*

Proof. Over $\mathbf{Z}[a, b, \lambda_1, \dots, \lambda_6]/I$, take exterior products in the free rank-two value module. The product is alternating over every ring, so (2.10) gives

$$4(q_1 \wedge q_2) = (aA_1) \wedge (bA_2) = (bA_1) \wedge (aA_2) = C_{12} \wedge C_{12} = 0. \quad (2.16)$$

Since $q_1 \wedge q_2$ is a basis, the coefficient 4 vanishes. $\square$

Remark 2.8 (Scope of universality). The functor parametrizes embedded invariant subspaces in the prescribed representation, not arbitrary deformations of α_2^2 . It is not just a closed locus in the two-parameter base: the subspace is part of the data, and

$$D/(a, b) = k[[\lambda_1, \lambda_2, \lambda_4, \lambda_5]].$$

On the other hand, setting all six λ_i equal to zero forces $2 = 0$ and leaves $k[[a, b]]$. Thus the literal space of constants and squares is invariant precisely in characteristic two. No claim about other notions of lifting Frobenius is intended.

3. EVALUATION AND FINITE FLAT STABILIZERS

In this section, T may be any commutative ring with coefficients $a, b, \lambda_1, \dots, \lambda_6$ satisfying (2.8). Put $\mathcal{U} = T\{1, f_1, f_2\}$, with the normalized polynomials (2.4), and write $F = (f_1, f_2)^t$. In particular this applies to every commutative D -algebra. The group $P_T = P_{a,b}$ is the principal open where μ is invertible in $\mathbf{A}_T^2$; it is the whole plane whenever a, b are nilpotent, in particular on Artinian deformation bases and after every scalar extension of such a base. Theorem 3.4 treats that Artinian case, and Proposition 3.6 treats the general case.

3.1. The value plane and its origin. For an action of P_T on a T -scheme with a chosen section q_0 , its *orbit map* is $g \mapsto g \cdot q_0$. The fiber at q_0 is the scheme-theoretic stabilizer. The *dual action* on linear functionals is defined by $(g \cdot q)(f) = q(g^{-1} \cdot f)$.

Definition 3.1 (The value plane and the evaluation kernel). For the invariant space $\mathcal{U} = T\{1, f_1, f_2\}$ over T as above, define the affine T -scheme of normalized functionals by

$$\mathcal{X} = \{q \in \mathcal{U}^\vee : q(1) = 1\}, \quad q_0 = \text{ev}_0|_{\mathcal{U}}. \quad (3.1)$$

Thus, for every commutative T -algebra T' , a point of $\mathcal{X}(T')$ is a T' -linear map $q : \mathcal{U} \otimes_T T' \rightarrow T'$ with $q(1) = 1$. These functionals are linear, not multiplicative; the space $\mathcal{U}$ is not an algebra. The coordinates $z_i = q(f_i)$ identify $\mathcal{X}$ with $\mathbf{A}_T^2$, and q_0 is its origin. The *evaluation kernel* is the rank-two direct summand

$$H_0 = \ker q_0 = T\{f_1, f_2\} \subset \mathcal{U}.$$

Proposition 3.2 (The orbit of evaluation). *Let T and the normalized invariant space $\mathcal{U}$ be as in the setup of this section, and define $\mathcal{X}, q_0, H_0$ as above. The dual action of P_T preserves $\mathcal{X}$. In coordinates $z_i = q(f_i)$, its action and its orbit map at q_0 are*

$$v \cdot z = F(v) + \mu_v^2 z, \quad (3.2)$$

$$P_T \longrightarrow \mathcal{X}, \quad g \longmapsto \text{ev}_g|_{\mathcal{U}}. \quad (3.3)$$

The map $q \mapsto \ker q$ identifies $\mathcal{X}$ with the P_T -invariant chart of planes complementary to $T \cdot 1$ in $\text{Gr}_T^{\text{sub}}(2, \mathcal{U})$. Consequently

$$G_T := \text{Stab}_{P_T}(q_0) = \text{Stab}_{P_T}(H_0) \quad (3.4)$$

is a closed subgroup scheme, and

$$F(x, y) = (f_1(x, y), f_2(x, y)), \quad G_T = F^{-1}(q_0) = \text{Spec } T[x, y, \mu^{-1}]/(f_1, f_2). \quad (3.5)$$

Proof. The constant function is fixed by P_T , so the dual action preserves $q(1) = 1$. For the left action on functions,

$$(g \cdot q)(f) = q(g^{-1} \cdot f) = q(f \circ g).$$

It follows that $g \cdot q_0 = \text{ev}_g|_{\mathcal{U}}$. Applying the orbit identity (2.6) to q gives the coordinate action (3.2).

Every functional with $q(1) = 1$ has a kernel complementary to $T \cdot 1$. Conversely, every such complement is the kernel of a unique functional normalized by $q(1) = 1$. This identifies the affine chart, compatibly with the action and arbitrary base change; compare [8, Tag 089R]. It also identifies the two stabilizers. Their equations are $f_1 = f_2 = 0$. Explicitly, it is the fiber product $P_T \times_{\mathcal{X}, q_0} \text{Spec } T$. Elements fixing q_0 are closed under multiplication, identity, and inverse after every base change, so the group operations of P_T restrict to this fiber product. It is closed because q_0 is a closed section of the affine T -scheme $\mathcal{X}$. $\square$

The invariant three-space $\mathcal{U}$ is preserved by all of P_T . The subgroup G_T instead stabilizes the two-dimensional kernel H_0 inside $\mathcal{U}$. Flatness has not yet been asserted. For Artinian invariant lifts the displayed localization is unnecessary because a, b are nilpotent. Proposition 3.6 will show that it is unnecessary on the zero fiber even without that hypothesis.

3.2. A nilpotent perturbation of the Frobenius basis.

Lemma 3.3 (Lifting the four-element basis). *Let T be a commutative ring, let $\mathfrak{a} \subset T$ be a nilpotent ideal, and let $f_1, f_2 \in T[x, y]$. Suppose that, for polynomials $r_1, r_2 \in T[x, y]$ of total degree at most one,*

$$f_1 \equiv x^2 + r_1, \quad f_2 \equiv y^2 + r_2 \pmod{\mathfrak{a}T[x, y]}.$$

The coefficients of r_1, r_2 need not be nilpotent. Then

$$T[z_1, z_2] \longrightarrow T[x, y], \quad z_i \longmapsto f_i,$$

is finite free of degree four, with basis $1, x, y, xy$.

Proof. Let $\mathcal{M} = T[z_1, z_2]\{1, x, y, xy\}$ be the free module on four formal basis symbols, and let $\Phi : \mathcal{M} \rightarrow T[x, y]$ be the proposed basis map, sending $h(z_1, z_2)x^i y^j$ to $h(f_1, f_2)x^i y^j$ for $0 \leq i, j < 2$. Replace f_1, f_2 by $x^2 + r_1, y^2 + r_2$ to obtain Φ_0 . The image under Φ_0 of $z_1^i z_2^j x^\alpha y^\beta$, for $i, j \geq 0$ and $\alpha, \beta \in \{0, 1\}$, is $x^{2i+\alpha} y^{2j+\beta}$ plus terms of smaller total degree. These leading monomials run through every monomial exactly once. Induction on total degree proves surjectivity; the distinct leading monomials prove injectivity by taking the largest total degree in a finite linear relation. Thus Φ_0 is an isomorphism of T -modules.

The T -linear endomorphism $\mathcal{E} = \Phi_0^{-1}(\Phi - \Phi_0)$ maps $\mathfrak{a}^j \mathcal{M}$ into $\mathfrak{a}^{j+1} \mathcal{M}$. If $\mathfrak{a}^r = 0$, then $\mathcal{E}^r = 0$, and

$$\Phi = \Phi_0(1 + \mathcal{E}), \quad (1 + \mathcal{E})^{-1} = 1 - \mathcal{E} + \cdots + (-\mathcal{E})^{r-1}.$$

Thus Φ is bijective. It is $T[z_1, z_2]$ -linear for the original structure map, so it is the required basis isomorphism. $\square$

For a flat group scheme H locally of finite presentation over a scheme Y , a *right H -torsor* is a Y -scheme E' with right H -action that becomes isomorphic to H with its right multiplication action after an fppf covering of Y [8, Definition 39.11.3, Tag 049A]. We use the following sufficient criterion: if $E' \rightarrow Y$ is fppf and surjective and $(e', h) \mapsto (e', e'h)$ is an isomorphism $E' \times_Y H \rightarrow E' \times_Y E'$, then E' is a torsor. Indeed, base change by the covering $E' \rightarrow Y$ supplies the diagonal section, and the displayed isomorphism trivializes the action over that covering.

Theorem 3.4 (Finite flatness and the homogeneous space). *Let $(a, b, \mathcal{U})$ be an invariant lift over $T \in \text{Art}_{\mathbf{Z}_2, k}$, and let $F : P_T \rightarrow \mathcal{X}$ and $G_T = F^{-1}(q_0)$ be as in Proposition 3.2. Then F is finite free of degree four, hence faithfully flat. The group scheme G_T is finite free over T , with*

$$B_T := \mathcal{O}(G_T) = T[x, y]/(f_1, f_2) = T\{1, x, y, xy\}. \quad (3.6)$$

Its special fiber is $\alpha_2 \times \alpha_2$. Moreover, right multiplication makes F a torsor under $G_T \times_T \mathcal{X}$ over $\mathcal{X}$, and

$$P_T/G_T \simeq \mathcal{X} \simeq \mathbf{A}_T^2$$

as fppf sheaves, where the quotient is by right multiplication.

Proof. Apply Lemma 3.3 with $\mathfrak{a} = \mathfrak{m}_T$. The orbit map is finite free of positive rank, hence faithfully flat. Its fiber at the origin has the same free basis. The special-fiber equations are $x^2 = y^2 = 0$ and the inherited group law is addition.

For the torsor assertion, the maps

$$P_T \times_T G_T \xrightarrow{\sim} P_T \times_{\mathcal{X}} P_T, \quad (p, g) \mapsto (p, p * g), \quad (3.7)$$

and $(p, p') \mapsto (p, p^{-1} * p')$ are inverse. Equality of orbit points is exactly the condition that $p^{-1} * p'$ fix q_0 . The map F is also finitely presented: its algebra is obtained by adjoining x, y with the two relations $z_i = f_i(x, y)$. It is therefore an fppf covering. The criterion just proved makes it a torsor. Every point of $\mathcal{X}$ lifts fppf locally to P_T , and (3.7) says that any two such lifts differ by a unique right multiplier in G_T . Since the functor represented by $\mathcal{X}$ is an fppf sheaf [8, Tag 023Q], these local lifting and uniqueness statements identify the sheafification with $\mathcal{X}$. Thus $P_T/G_T \simeq \mathcal{X}$. $\square$

For an affine group scheme with coordinate algebra B , its *Hopf structure* is the dual algebraic form of its group operations. The coproduct $\Delta : B \rightarrow B \otimes_T B$ is pullback by multiplication, the counit $\varepsilon : B \rightarrow T$ is pullback by the identity section, and the antipode $S_B : B \rightarrow B$ is pullback by inversion. The unit is the structural map $u_B : T \rightarrow B$; write $m_B : B \otimes_T B \rightarrow B$ for multiplication in the coordinate algebra. The group axioms say that Δ is coassociative, meaning $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$, that $(\varepsilon \otimes \text{id})\Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id}$, and that

$$m_B(S_B \otimes \text{id})\Delta = m_B(\text{id} \otimes S_B)\Delta = u_B \varepsilon.$$

The coordinate algebra is commutative even when the group scheme is not. We reserve S_B for the antipode; S alone is the coefficient ring $\mathbf{Z}[[a, b]]$.

Corollary 3.5 (The Hopf structure). *For any commutative coefficient ring T and normalized coefficients satisfying (2.8), let $B_T = T[x, y, \mu^{-1}]/(f_1, f_2)$ be the stabilizer algebra of Proposition 3.2. Its structural maps are*

$$\begin{aligned} \Delta(x) &= x \otimes 1 + \mu \otimes x, & \Delta(y) &= y \otimes 1 + \mu \otimes y, \\ \varepsilon(x) &= \varepsilon(y) = 0, & S_{B_T}(x) &= -\mu^{-1}x, & S_{B_T}(y) &= -\mu^{-1}y, \end{aligned} \quad (3.8)$$

where $\mu = 1 + ax + by$, tensor products are over T , and the unit map is $T \rightarrow B_T$. In particular, $\Delta(\mu) = \mu \otimes \mu$. Proposition 3.6 identifies this algebra with $T[x, y]/(f_1, f_2)$ and proves it free of rank four.

Proof. These formulas are dual to the multiplication, identity, and inverse in (2.2). They descend to B_T because Proposition 3.2 has proved that the zero fiber is a subgroup. $\square$

Proposition 3.6 (Finite flatness on the full invariant locus). *Let T be any commutative ring with elements $a, b, \lambda_1, \dots, \lambda_6$ satisfying (2.8), and let $F = (f_1, f_2)$ be the normalized quadratic map (2.4). No Artinian or nilpotence hypothesis on T or these elements is required. Then the polynomial map*

$$F : \mathbf{A}_T^2 \longrightarrow \mathbf{A}_T^2$$

is finite free of degree four, with basis $1, x, y, xy$. Its zero fiber lies in $P_{a,b}$ and is a finite free group scheme of rank four.

More precisely, put $\gamma(z) = 1 + a^2 z_1 + b^2 z_2$ on the value plane and let

$$\mathcal{X}^\circ = \operatorname{Spec} T[z_1, z_2, \gamma(z)^{-1}].$$

This is $P_{a,b}$ -invariant, and $P_{a,b} = F^{-1}(\mathcal{X}^\circ)$. The restricted orbit map $P_{a,b} \rightarrow \mathcal{X}^\circ$ is finite faithfully flat of degree four and is a right torsor under its origin stabilizer; in particular $P_{a,b}/G_T \simeq \mathcal{X}^\circ$ as fppf sheaves. Consequently the construction gives a finite free group scheme already over D itself, and over the algebraic ring $\mathbf{Z}[a, b, \lambda_1, \dots, \lambda_6]/I$ before completion.

Proof. Corollary 2.7 gives $4 = 0$ in T . The two mixed coefficients generate a square-zero ideal, since

$$\begin{aligned} \lambda_3^2 &= (b\lambda_1)(a\lambda_2) = (a\lambda_1)(b\lambda_2) = 0, \\ \lambda_6^2 &= (a\lambda_5)(b\lambda_4) = (a\lambda_4)(b\lambda_5) = 0, \\ \lambda_3\lambda_6 &= (b\lambda_1)(a\lambda_5) = (a\lambda_1)(b\lambda_5) = 4 = 0. \end{aligned}$$

Apply Lemma 3.3 with $\mathbf{a} = (\lambda_3, \lambda_6)$ and with the two linear parts as r_1, r_2 . This proves finite freeness on the entire affine plane. In the proof of that lemma the error operator now has square zero; its inverse is simply $1 - \mathcal{E}$.

Modulo 2, the diagonal and mixed relations give

$$a^2 f_1 + b^2 f_2 \equiv a^2 x^2 + b^2 y^2, \quad \mu_v^2 \equiv \gamma(F(v)).$$

For example, $b^2 \lambda_4 = b(a\lambda_5) = 2a$ and $a^2 \lambda_2 = a(b\lambda_1) = 2b$; the other coefficients are handled by the same relations. Since $(2)^2 = 0$, two elements congruent modulo 2 are either both units or both nonunits. Hence μ_v is a unit exactly when $\gamma(F(v))$ is a unit, proving $P_{a,b} = F^{-1}(\mathcal{X}^\circ)$. Restriction is therefore a base change of the finite free map already proved. The origin belongs to $\mathcal{X}^\circ$, so its zero fiber lies in $P_{a,b}$ and is the stabilizer.

Finally, the action formula gives $\gamma(v \cdot z) \equiv \mu_v^2 \gamma(z) \pmod{2}$. Thus the open $\mathcal{X}^\circ$ is invariant. The torsor argument of Theorem 3.4 applies to this restricted orbit map. $\square$

For an Artinian invariant lift, a, b are nilpotent and γ is a unit everywhere, so $\mathcal{X}^\circ = \mathcal{X}$. The original Artinian construction is the case where this open value plane is the entire plane. The finite truncation A_4 will be used to exhibit a finite counterexample base, not to obtain flatness.

Remark 3.7 (Why flatness needs a proof). An arbitrary stabilizer need not be flat, even when the group and the space acted on are smooth. Here flatness follows from a basis of the entire orbit map, not merely from the rank of its special fiber. The orbit map itself is not smooth, since its special fiber is Frobenius.

4. POWER MORPHISMS AND THE UNIVERSAL COUNTEREXAMPLE

Throughout this section, T may be any commutative D -algebra. Proposition 3.6 gives the finite free group scheme $G_T = \operatorname{Spec} B_T$, where $B_T = T[x, y]/(f_1, f_2)$. We compute its powers using only the defining relations, without an Artinian hypothesis, and then prove nonvanishing on the finite quotient A_4 .

4.1. Arithmetic consequences of invariance.

Lemma 4.1 (Consequences of invariance). *In the ring D of Theorem 2.6, and hence after base change to every commutative D -algebra, one has $2\lambda_i = 0$ for $1 \leq i \leq 6$, and*

$$\begin{aligned} a^2\lambda_1 &= b^2\lambda_4 = 2a, & a^2\lambda_2 &= b^2\lambda_5 = 2b, \\ a^2\lambda_3 &= b^2\lambda_6 = 2ab. \end{aligned} \tag{4.1}$$

Consequently the universal polynomials f_1, f_2 satisfy the following identities in $D[X, Y]$:

$$2f_1 = 2X^2, \quad 2f_2 = 2Y^2, \quad a^2f_1 + b^2f_2 = a^2X^2 + b^2Y^2. \tag{4.2}$$

Proof. The adjugate of L is $\text{adj}(L) = \begin{pmatrix} \lambda_5 & -\lambda_2 \\ -\lambda_4 & \lambda_1 \end{pmatrix}$, and $\det L = \lambda_1\lambda_5 - \lambda_2\lambda_4$. It satisfies $\text{adj}(L)L = (\det L)I_2$ over every commutative ring. Multiplying $L \begin{pmatrix} b \\ -a \end{pmatrix} = 0$ by $\text{adj}(L)$ gives $a \det L = b \det L = 0$. Therefore

$$2 \text{adj}(L) = \text{adj}(L)L \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = (\det L) \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = 0.$$

Its entries are $\lambda_5, -\lambda_2, -\lambda_4, \lambda_1$, so their doubles vanish. The same holds for $\lambda_3 = a\lambda_2$ and $\lambda_6 = a\lambda_5$.

For (4.1), the diagonal equations give $a^2\lambda_1 = 2a$ and $b^2\lambda_5 = 2b$, while the mixed equations give

$$b^2\lambda_4 = b(a\lambda_5) = 2a, \quad a^2\lambda_2 = a(b\lambda_1) = 2b.$$

The last two identities follow from $\lambda_3 = b\lambda_1$ and $\lambda_6 = a\lambda_5$. The first two assertions of (4.2) follow from $2\lambda_i = 0$. In its third assertion, the extra coefficients of X, Y, XY are $4a, 4b, 4ab$, which vanish by Corollary 2.7. $\square$

Lemma 4.2 (The scaling error on the stabilizer). *Let T be a commutative D -algebra, put $B_T = T[x, y]/(f_1, f_2)$, and let $v = (x, y)$ be the universal point of the group scheme G_T from Proposition 3.6. Set $\theta = ax + by \in B_T$. Then, in B_T and B_T^2 , respectively,*

$$2x^2 = 2y^2 = 0, \quad \theta^2 = 2abxy, \quad \theta^2v = 0. \tag{4.3}$$

In particular $\theta^3 = 0$.

Proof. In B_T the functions f_1, f_2 vanish. Lemma 4.1 gives $2x^2 = 2y^2 = 0$ and $a^2x^2 + b^2y^2 = 0$, so $\theta^2 = 2abxy$. Hence

$$\theta^2x = ab(2x^2)y = 0, \quad \theta^2y = abx(2y^2) = 0.$$

Applying ℓ to $\theta^2v = 0$ gives $\theta^3 = 0$. $\square$

The equality needed for iteration is $\theta^2v = 0$. It does not require θ^2 itself to vanish; in fact $2ab \neq 0$ in D , and hence $\theta^2 = 2abxy \neq 0$ in B_D by Proposition A.4 below. We call $\theta = \mu - 1$ the *scaling error*. For $g_v(Z) = v + \mu_v Z$, v is the *displacement* and μ_v is the scalar dilation; a *pure translation* is the case $\mu_v = 1$. A *pure dilation* has zero displacement. The *target displacement* of v is $F(v)$ in the action $z \mapsto F(v) + \mu_v^2 z$. The *residual dilation* is the scalar action left on $\mathcal{X}$ when $v \in G_T$, so that $F(v) = 0$.

4.2. All powers and the residual dilation.

Proposition 4.3 (Power formula). *Let T be a commutative D -algebra, set $B_T = T[x, y]/(f_1, f_2)$, and let $v = (x, y) \in G_T(B_T)$ with $\theta = ax + by$. For every integer $n \geq 1$, the power morphism satisfies*

$$[n](v) = nv + \binom{n}{2}\theta v. \tag{4.4}$$

The addition and scalar multiplication on the right are those of the coordinate module B_T^2 , not the group law on G_T . In particular,

$$[4](x, y) = (2bxy, 2axy), \quad [8] = e. \quad (4.5)$$

The values of $[4]$ act as pure translations in P_T , and

$$[4](v) = (\mu_v^2 - 1)v. \quad (4.6)$$

For every commutative T -algebra T' , not necessarily Artinian, these identities hold after base change and

$$[4]_{G_T \times_T T'} = e \iff 2a = 2b = 0 \text{ in } T'.$$

Proof. Affine composition gives $[n](v) = \sum_{j=0}^{n-1} (1 + \theta)^j v$. Lemma 4.2 reduces each summand to $v + j\theta v$, proving (4.4). Since $4 = 0$, the fourth power is $2\theta v$. Its pure-square terms vanish by $2x^2 = 2y^2 = 0$, giving (4.5). For $n = 8$, both coefficients 8 and $\binom{8}{2} = 28$ vanish.

The scalar correction of the fourth-power displacement is

$$a(2bxy) + b(2axy) = 4abxy = 0,$$

so this value is a pure translation. Moreover

$$(\mu_v^2 - 1)v = (2\theta + \theta^2)v = 2\theta v,$$

which proves (4.6). Finally xy is a free basis element of B_T . Thus its two coefficients vanish exactly when $2a = 2b = 0$. The basis remains free after every base change. $\square$

For a point v of G_T , the action on the value plane $\mathcal{X}$ is $z \mapsto \mu_v^2 z$. Equation (4.6) therefore relates the residual dilation on the target to the displacement left after four source transformations. It is an identity of polynomial maps, not a claim that $[4]$ itself is a translation automorphism.

Corollary 4.4 (Commutativity). *For every commutative D -algebra T , the associated group scheme G_T is commutative if and only if $a = b = 0$ in T . The same criterion holds after replacing T by any commutative T -algebra.*

Proof. For independent points $v = (x, y)$ and $v' = (x', y')$, the difference between the coordinate vectors of $v * v'$ and $v' * v$ is

$$(b(yx' - xy'), a(xy' - yx')).$$

The monomials $x \otimes y$ and $y \otimes x$ are distinct basis elements of $B_T \otimes_T B_T$. Their coefficients vanish exactly when $a = b = 0$. $\square$

Remark 4.5 (The matrix calculation). The faithful affine representation sends v to

$$I_3 + N_v = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 + \theta & 0 \\ y & 0 & 1 + \theta \end{pmatrix},$$

which defines the matrix N_v . Lemma 4.2 gives $N_v^3 = 0$: its possible cubic entries are θ^3 and the entries of $\theta^2 v$. Hence $(I_3 + N_v)^4 = I_3 + 2N_v^2$. The coefficient $\binom{4}{2} = 6 \equiv 2 \pmod{4}$ is the surviving contribution in (4.5).

4.3. Nonvanishing on the third-order neighbourhood. The maximal ideal of D is

$$\mathfrak{n} = (a, b, \lambda_1, \lambda_2, \lambda_4, \lambda_5),$$

since $2 = a\lambda_1$ and the eliminated coefficients are products of these generators. Put

$$A_4 = D/\mathfrak{n}^4. \quad (4.7)$$

For every $T \in \text{Art}_{\mathbf{Z}_2, k}$ with $\mathfrak{m}_T^4 = 0$, a local map $D \rightarrow T$ kills $\mathfrak{n}^4$ and therefore factors uniquely through A_4 ; hence A_4 represents the restriction of $\mathcal{D}$ to this full subcategory of test rings.

This is a finite local ring. By the *j*th-order neighbourhood of the closed point we mean $\text{Spec}(D/\mathfrak{n}^{j+1})$; thus A_4 is the third-order neighbourhood, although its subscript is four. A *nilpotent thickening* is a closed immersion defined by a nilpotent ideal. The relations place 2 in $\mathfrak{n}^2$ and $2a, 2b$ in $\mathfrak{n}^3$.

Lemma 4.6 (Nonvanishing of the power coefficients). *Let $\mathfrak{n}$ be the maximal ideal of D and $A_4 = D/\mathfrak{n}^4$. Then $2a \neq 0$ and $2b \neq 0$ in A_4 .*

Proof. Work first in

$$T_0 = (\mathbf{Z}/4)[a, b, \lambda_1, \lambda_2, \lambda_4, \lambda_5]/\mathfrak{n}^4,$$

where the denominator is the fourth power of the ideal generated by the six displayed variables. To identify the quotient, use (2.13): $2 = a\lambda_1$, so those variables generate $\mathfrak{n}$; moreover 4 already belongs to the relation ideal by (2.16). Modulo their fourth power, every power series reduces uniquely to a polynomial of degree at most three before imposing the six relations. Thus quotienting T_0 by those relations gives exactly $D/\mathfrak{n}^4 = A_4$.

The monomials of degree at most three form a free $\mathbf{Z}/4$ -basis of T_0 . Define a $\mathbf{Z}/4$ -linear functional $\varphi_b : T_0 \rightarrow \mathbf{Z}/4$ by

$$\varphi_b(b) = 1, \quad \varphi_b(ab\lambda_1) = \varphi_b(a^2\lambda_2) = \varphi_b(b^2\lambda_5) = 2, \quad (4.8)$$

and set all other monomial values to zero. The supported cubic monomials are the three expressions for $2b$ furnished by the defining relations.

We check the whole relation ideal. For monomial multipliers of degree zero or one, the only products with nonzero values on individual terms are

$$(a\lambda_1 - 2)b, \quad (b\lambda_5 - 2)b, \quad (a\lambda_2 - b\lambda_1)a.$$

The two term values in each product are 2 and cancel. All other products miss the support. For a multiplier m of degree at least two, its product with every quadratic relation term vanishes in T_0 . Only $-2m$ from the first two relations can remain, and its value is zero because $\varphi_b(m)$ is either zero or two.

Thus φ_b descends to A_4 and sends $2b$ to $2 \neq 0$. These are additive functionals, not ring homomorphisms. Let ι denote the involution

$$(a, b, \lambda_1, \lambda_2, \lambda_4, \lambda_5) \mapsto (b, a, \lambda_5, \lambda_4, \lambda_2, \lambda_1) \quad (4.9)$$

which preserves the relations and the truncation and exchanges $2a$ with $2b$. Consequently $\varphi_a = \varphi_b \circ \iota$ descends as well and satisfies $\varphi_a(2a) = 2 \neq 0$. $\square$

Theorem 4.7 (The universal counterexample). *Let D be the ring of Theorem 2.6, let $\mathfrak{n}$ be its maximal ideal, and put $A_4 = D/\mathfrak{n}^4$. Reduce the tautological polynomials f_1, f_2 to A_4 and set*

$$B_{A_4} = A_4[x, y]/(f_1, f_2), \quad G_{A_4} = \text{Spec } B_{A_4},$$

with the Hopf structure of Corollary 3.5. Then B_{A_4} is free on $1, x, y, xy$, and G_{A_4} has exponent exactly eight. More precisely,

$$[4](x, y) = (2bxy, 2axy) \neq e, \quad [8] = e,$$

and the universal point $(x, y) \in G_{A_4}(B_{A_4})$ has order eight. On the corresponding group over $D/\mathfrak{n}^3$, the fourth-power morphism is the constant identity morphism. The untruncated stabilizer $G_D = \text{Spec } D[x, y]/(f_1, f_2)$ of Proposition 3.6 is also finite free of rank four and has exponent eight.

Proof. Finite freeness is Theorem 3.4, the power formulas are Proposition 4.3, and their nonvanishing follows from Lemma 4.6. The universal point has nontrivial fourth power and trivial eighth power, hence order eight. On $D/\mathfrak{n}^3$, both coefficients $2a, 2b \in \mathfrak{n}^3$ vanish. For $T = D$, finite freeness follows from Proposition 3.6, and the power formulas follow directly from Proposition 4.3, which is stated for every commutative D -algebra. The fourth power over D is nonzero because its specialization to A_4 is nonzero. $\square$

This proves Theorem A. No values of the λ_i were prescribed. The third-order neighbourhood is the first in this particular filtration detecting the obstruction; no minimality among all coefficient rings is asserted.

Remark 4.8 (Points over the base ring). Every point of $G_{A_4}(A_4)$ is killed by four. Its coordinates reduce to zero in k , so lie in $\mathfrak{m}_{A_4}$, whereas $2a, 2b \in \mathfrak{m}_{A_4}^3$ annihilate that ideal. The nonzero morphism is detected by the universal point over B_{A_4} , not by a point over A_4 .

5. THE EXPLICIT TWO-GENERATOR SPECIALIZATION

We now prescribe an orbit derivative in the universal family. Its compatibility ring is

$$R = \mathbf{Z}[a, b]/(a^2b - 2, a^3, b^3). \quad (5.1)$$

This section proves Theorem B. Appendix B collects the complete coordinate and representation formulas.

5.1. The coefficient ring.

Lemma 5.1 (The coefficient ring). *The ring $R = \mathbf{Z}[a, b]/(a^2b - 2, a^3, b^3)$ is local Artinian with residue field $\mathbf{F}_2$. Writing $\mathfrak{m} = (a, b)$ and $\kappa = 2b$, one has*

$$4 = 0, \quad 2a = 2b^2 = 0, \quad \mathfrak{m}^5 = 0, \quad \mathfrak{m}^4 = (\kappa), \quad \kappa\mathfrak{m} = 0,$$

and $\kappa \neq 0$.

Proof. The relations give $2a = a^3b = 0$, $2b^2 = a^2b^3 = 0$, and $4 = (a^2b)^2 = 0$. Thus R is finite over $\mathbf{Z}/4$, $(a, b)^5 = 0$, and $(a, b)^4 = (a^2b^2) = (\kappa)$. Since $R/(a, b) = k$, the ring is local Artinian.

For nonvanishing, let $T_1 = \mathbf{Z}[a, b]/(a^3, b^3)$, which is free over $\mathbf{Z}$ on the nine monomials a^ib^j , $0 \leq i, j \leq 2$. If $2b = 0$ in R , then $2b = (2 - a^2b)h$ for some $h \in T_1$. For a basis monomial m , write $h_m \in \mathbf{Z}$ for its coefficient in h . Comparing the coefficients of b and a^2b^2 gives

$$2h_b = 2, \quad 2h_{a^2b^2} = h_b. \quad (5.2)$$

The first forces $h_b = 1$, contradicting the second over $\mathbf{Z}$. Finally, $\kappa\mathfrak{m} = 0$ follows from $\mathfrak{m}^4 = (\kappa)$ and $\mathfrak{m}^5 = 0$. $\square$

5.2. The prescribed derivative.

Consider the triangular prescription

$$A_1 = abq_1, \quad A_2 = b^2q_1 + a^2q_2, \quad L = \begin{pmatrix} ab & b^2 \\ 0 & a^2 \end{pmatrix}. \quad (5.3)$$

This is a choice, not a canonical section. It makes $a\lambda_1 = b\lambda_5$ and $a\lambda_2 = b\lambda_1$ hold identically before imposing further relations.

Proposition 5.2 (The compatibility ring). *Fix $T \in \text{Art}_{\mathbf{Z}_2, k}$ and $a, b \in \mathfrak{m}_T$. An invariant lift whose normalized derivative is the matrix (5.3) exists if and only if*

$$a^2b = 2, \quad a^3 = 0, \quad b^3 = 0 \quad \text{in } T.$$

When it exists, the embedded invariant space is unique. Equivalently, the functor of invariant lifts with this derivative prescription is represented on $\text{Art}_{\mathbf{Z}_2, k}$ by the finite local quotient

$$R \simeq D/(\lambda_1 - ab, \lambda_2 - b^2, \lambda_4, \lambda_5 - a^2). \quad (5.4)$$

The specialization of all six coefficients is

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (ab, b^2, ab^2, 0, a^2, 0), \quad (5.5)$$

and its invariant space is

$$\begin{aligned} \mathcal{U}_R = R\{1, F_1, F_2\}, \quad F_1(X, Y) &= X^2 + abX + b^2(1 + aX)Y, \\ F_2(X, Y) &= Y^2 + a^2Y. \end{aligned} \quad (5.6)$$

Proof. The three vector equations (2.10) have errors

$$\begin{aligned} aA_1 - 2q_1 &= (a^2b - 2)q_1, \\ bA_2 - 2q_2 &= b^3q_1 + (a^2b - 2)q_2, \\ bA_1 - aA_2 &= -a^3q_2. \end{aligned}$$

Their coefficients vanish exactly when the three defining relations of R hold. The mixed coefficient is then forced to be $C_{12} = ab^2q_1$, giving (5.5) and (5.6). Theorem 2.6 proves the representing property. The formal and polynomial presentations coincide because a, b are nilpotent and $4 = 0$. $\square$

We write $F = (F_1, F_2)^t$ for the specialized orbit map, using the same letter as in the universal family. One can also check its invariance directly over S . If $\delta = 2 - a^2b$, then

$$J_F - (1 + aX + bY)J_F(0) = \begin{pmatrix} \delta X & -b^3Y \\ 0 & \delta Y - a^3X \end{pmatrix}. \quad (5.7)$$

Corollary 2.5 identifies the same invariance locus. The derivative is the chosen datum; the mixed coefficients and the base relations are its consequences.

5.3. The finite group and its fourth power.

Theorem 5.3 (The explicit Hopf algebra). *Let R be as in (5.1), and set*

$$B_R = R[x, y]/(x^2 + abx + b^2(1 + ax)y, y^2 + a^2y), \quad G_R = \text{Spec } B_R. \quad (5.8)$$

Let F_1, F_2 be the polynomials in (5.6). The plane $H_0 = R\{F_1, F_2\}$ inside $\mathcal{U}_R = R\{1, F_1, F_2\}$ has stabilizer G_R under the induced action of P_R . The coordinate algebra B_R is free over R on $1, x, y, xy$. It is a Hopf algebra with

$$\begin{aligned} \Delta(x) &= x \otimes 1 + \mu \otimes x, & \Delta(y) &= y \otimes 1 + \mu \otimes y, \\ \varepsilon(x) &= \varepsilon(y) = 0, & S_B(x) &= -\mu^{-1}x, \quad S_B(y) = -\mu^{-1}y, & \mu &= 1 + ax + by. \end{aligned} \quad (5.9)$$

*Here all tensor products are over R , $\varepsilon : B_R \rightarrow R$ fixes R , and the unit map is $R \rightarrow B_R$. For every commutative R -algebra R' , the solutions of the defining equations in $(R')^2$ have multiplication $v * v' = v + \mu_v v'$, identity 0, and inverse $-\mu_v^{-1}v$. The group is noncommutative, has special fiber $\alpha_2 \times \alpha_2$ over $R/(a, b)$, and satisfies*

$$[4](x, y) = (2bxy, 0) \neq e, \quad [8] = e. \quad (5.10)$$

Proof. Group structure. Proposition 5.2 gives the invariant space. Its evaluation kernel is $R\{F_1, F_2\}$, so Proposition 3.2 identifies its stabilizer with (5.8). The Hopf maps are the restrictions of (3.8).

Rank. Theorem 3.4 applies. Alternatively, over arbitrary target coordinates z_1, z_2 , first adjoin y with $y^2 + a^2y = z_2$, and then x with

$$x^2 + (ab + ab^2y)x + b^2y = z_1.$$

These are successive monic quadratic extensions. They exhibit the entire orbit map as finite free of degree four, and its origin fiber has the same basis.

Powers and nonvanishing. Proposition 4.3 and $2a = 0$ give (5.10). The first coordinate is nonzero because $2b \neq 0$ by Lemma 5.1 and xy is a free basis element. The universal point therefore has order eight. Since $b \neq 0$, Corollary 4.4 proves noncommutativity. The special-fiber equations and group law are $x^2 = y^2 = 0$ and addition. $\square$

Remark 5.4 (The elementary power calculation). On this stabilizer, Lemma 4.2 gives $\theta^2 = 2abxy$ for $\theta = ax + by$; since $2a = 0$ in R , this square vanishes. Thus the fourth-power displacement is simply

$$(1 + (1 + \theta) + (1 + 2\theta) + (1 + 3\theta))v = 2\theta v = (\kappa xy, 0).$$

The second coordinate vanishes by $y^2 = -a^2y$ and $\kappa a^2 = 0$.

Corollary 5.5 (Comparison with the universal truncation). *The specialization $D \twoheadrightarrow R$ of Proposition 5.2 factors through $D/\mathfrak{n}^5$, but not through $A_4 = D/\mathfrak{n}^4$, where $\mathfrak{n}$ is the maximal ideal of D .*

Proof. The image of $\mathfrak{n}$ is (a, b) . Lemma 5.1 gives $(a, b)^5 = 0$ but $(a, b)^4 = (2b) \neq 0$. $\square$

The universal derivative parameters become quadratic in a, b under this specialization. The class $2b = ab\lambda_1$ lies in $\mathfrak{n}^3 \setminus \mathfrak{n}^4$ and becomes $a^2b^2 \in \mathfrak{m}^4 \setminus \mathfrak{m}^5$. Thus its *maximal-ideal order*, meaning the largest filtration index containing it, changes from three to four; this is not its additive order. The two bases are different restrictions of the same family. Their lengths and Hilbert functions are proved in Appendix A.

Comparison with the Mathlib formalization. The module `Counterexamples.GrothendieckPower` in Mathlib [10] formalizes the presentation in [9]. To distinguish its coordinate Hopf algebra from B_R , write

$$B_{\text{ML}} = R[U, V]/(U^2 - abU + b^2V, V^2 - a^2V), \quad \Lambda = (1 + aU)(1 + bV).$$

Its base is written as $\mathbf{Z}[a, b]/(a^3, b^3, a^2b + 2)$ in the Lean source. This is canonically the same ring R : both defining ideals contain 4, and $a^2b + 2$ differs from $a^2b - 2$ by 4. The formalized coproduct is

$$\Delta_{\text{ML}}(U) = U \otimes 1 + \Lambda \otimes U, \quad \Delta_{\text{ML}}(V) = V \otimes \Lambda + 1 \otimes V.$$

In contrast, both coordinates of B_R have the same left scalar factor μ in (5.9), and its first quadratic relation has an additional mixed term. Thus the two displayed Hopf-algebra presentations must not be identified merely by renaming U, V as x, y .

In the namespace `Counterexample.GrothendieckPower`, the declaration `counterexample` states finite freeness of rank four and nontriviality of the fourth-power map for B_{ML} ; `powerMap_eight` states triviality of the eighth power. The declaration `id_pow_affineGroupScheme_four_ne_one` gives the fourth-power statement for the corresponding group object in the opposite category of commutative R -algebras. We cite these as a formal verification of that counterexample, not of the invariant-space construction in this article. We neither establish an isomorphism between $\text{Spec } B_{\text{ML}}$ and G_R nor assert that they are nonisomorphic. No transfer of formal verification to B_R or to the universal family over D is claimed.

6. THE INVARIANT QUOTIENT AS A CYCLIC MODULE

Let T be any commutative D -algebra, with the invariant space $\mathcal{U} = T\{1, f_1, f_2\}$. The module and character calculations below hold over D and after arbitrary base change. They identify the same construction with a deformation of a left ideal; the power calculation and nonvanishing proofs do not depend on this interpretation.

6.1. The ambient algebra and its characters. Let $E = Te_1 \oplus Te_2$, with its displayed frame, and define $\ell : E \rightarrow T$ by $\ell(e_1) = a$, $\ell(e_2) = b$. A *nonunital algebra* has an associative bilinear multiplication without a specified identity. Its *unitization* adjoins an identity by forming the direct sum with the scalar ring.

Definition 6.1 (The scalar-action algebra). Give E the nonunital multiplication $vw = \ell(v)w$. Its unitization is

$$C = T \oplus E, \quad (s, v)(t, w) = (st, sw + tv + \ell(v)w). \quad (6.1)$$

Write $e_0 = 1 \in C$ and $M = C$ for its underlying free rank-three module.

Proposition 6.2 (Two characters and the affine action). *For $C = T \oplus E$ in Definition 6.1, the multiplication is associative. The maps*

$$h(s, v) = s, \quad \chi(s, v) = s + \ell(v) \quad (6.2)$$

are unital T -algebra homomorphisms (characters), and

$$zw = \chi(z)w \quad (z \in C, w \in E). \quad (6.3)$$

The augmentation is $h : C \rightarrow T$, with augmentation ideal $\ker h = E$. The augmentation-one unit group $\{g \in C^\times : h(g) = 1\}$ is P_T , under $v \mapsto 1 + v$. Here $C^\times$ denotes the unit-group functor $T' \mapsto (C \otimes_T T')^\times$. Its left regular representation, acting by left multiplication on $M = C$, is the faithful affine representation

$$\rho_M(v) = \begin{pmatrix} 1 & 0 & 0 \\ x & \mu_v & 0 \\ y & 0 & \mu_v \end{pmatrix}. \quad (6.4)$$

Proof. Both triple products on E equal $\ell(u)\ell(v)w$, proving associativity. The formula for h is an augmentation, and $\ell(vw) = \ell(v)\ell(w)$ proves that χ is multiplicative. The multiplication in (6.1) gives (6.3).

For augmentation-one elements,

$$(1 + v)(1 + w) = 1 + v + (1 + \ell(v))w.$$

An element $1 + v$ is a unit if and only if $\mu_v = \chi(1 + v)$ is a unit: necessity follows from χ , and sufficiency from the two-sided inverse $1 - \mu_v^{-1}v$. This identifies the augmentation-one unit group with P_T after every base change, without a nilpotence hypothesis. Left multiplication sends e_0 to $e_0 + v$ and acts on E by μ_v , giving the matrix. Its first column recovers x, y , and its matrix pattern together with $\mu_v = 1 + ax + by$ defines a closed subgroup of $\mathrm{GL}_{3,T}$, proving faithfulness. $\square$

Faithfulness here is a closed-immersion assertion, as in the conventions of Section 1, not just injectivity on the points of one test ring.

6.2. The divided square and the quotient. Let $\mathfrak{S}_2$ be the permutation group on two letters, acting on $C \otimes_T C$ by interchanging the tensor factors. A superscript $(-)^{\mathfrak{S}_2}$ means the submodule fixed by this action. Define the associative algebra

$$\mathcal{A} = (C \otimes_T C)^{\mathfrak{S}_2}. \quad (6.5)$$

Multiplication in the tensor-product algebra is $(u \otimes v)(u' \otimes v') = uu' \otimes vv'$. The transposition is therefore an algebra automorphism even when C is noncommutative. Its invariant tensors have the free basis

$$d_i = e_i \otimes e_i \quad (0 \leq i \leq 2), \quad c_{ij} = e_i \otimes e_j + e_j \otimes e_i \quad (0 \leq i < j \leq 2). \quad (6.6)$$

Indeed, coefficients on exchanged tensor pairs must agree, while diagonal coefficients are arbitrary. The unit is d_0 , and this basis is compatible with every base change.

For a finite free module V , the *symmetric square* $\text{Sym}^2(V)$ is the quotient of $V \otimes_T V$ by the relations $u \otimes v - v \otimes u$. The *divided square* used here is

$$\Gamma^2(M) = (M \otimes_T M)^{\mathfrak{S}_2} \simeq (\text{Sym}^2(M^\vee))^\vee.$$

Thus its underlying module is $\mathcal{A}$. In characteristic two the invariant tensors should not be confused with the quotient defining the symmetric square; no averaging by $1/2$ is available or needed. This concrete finite-free description is all we use from divided powers; see [7, Prop. IV.5, p. 272] for the finite-free symmetric-tensor identification.

We give $M^\vee$ the contragredient action $(g \cdot \zeta)(m) = \zeta(\rho_M(g)^{-1}m)$ and give $\text{Sym}^2(M^\vee)$ its induced action. To make *homogenization* explicit, let $\zeta_0, \zeta_1, \zeta_2$ be the basis dual to e_0, e_1, e_2 . Replace $1, X, Y, X^2, XY, Y^2$ by $\zeta_0^2, \zeta_0\zeta_1, \zeta_0\zeta_2, \zeta_1^2, \zeta_1\zeta_2, \zeta_2^2$, respectively. On the affine hyperplane $\zeta_0 = 1$ in M , the action is $Z \mapsto g \cdot Z$ and the induced action on polynomial functions is inverse pullback. Thus this homogenization identifies $\mathcal{V}_T$ with $\text{Sym}^2(M^\vee)$ equivariantly for the convention used in Section 2. The dual pairing makes $d_0, c_{01}, c_{02}, d_1, c_{12}, d_2$ dual to $1, X, Y, X^2, XY, Y^2$ in that order. The algebra multiplication on $\mathcal{A}$ is inherited from $C \otimes C$; it is not the degree-raising multiplication in a graded divided-power algebra.

Dualizing $\mathcal{U} \subset \mathcal{V}_T$ gives

$$\pi : \mathcal{A} \twoheadrightarrow W = \mathcal{U}^\vee. \quad (6.7)$$

Write $q_i = \pi(d_i)$. These are the basis dual to $1, f_1, f_2$; in particular $q_0 = \text{ev}_0$, while q_1, q_2 vanish on 1 and are the value-coordinate vectors used earlier. In the source basis $(d_0, d_1, d_2, c_{01}, c_{02}, c_{12})$ and target basis (q_0, q_1, q_2) ,

$$\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & 1 & \lambda_4 & \lambda_5 & \lambda_6 \end{pmatrix}. \quad (6.8)$$

Its kernel is $K = \ker \pi$, the submodule of $\mathcal{A} = \mathcal{V}_T^\vee$ consisting of the functionals that vanish on $\mathcal{U}$. It is the direct summand with basis

$$\begin{aligned} \xi_{01} &= c_{01} - \lambda_1 d_1 - \lambda_4 d_2, \\ \xi_{02} &= c_{02} - \lambda_2 d_1 - \lambda_5 d_2, \\ \xi_{12} &= c_{12} - \lambda_3 d_1 - \lambda_6 d_2. \end{aligned} \quad (6.9)$$

Proposition 6.3 (Invariant kernels are left ideals). *Under the representation $\Gamma^2(M)$, a T -module direct summand $\mathcal{K} \subset \mathcal{A}$ is P_T -invariant if and only if $\mathcal{A}\mathcal{K} \subseteq \mathcal{K}$. In particular, $K = \ker \pi$ is a left ideal. The quotient $W = \mathcal{A}/K$ is cyclic with vector $q_0 = \bar{1}$, and its stabilizer is*

$$G_T(T') = \{g \in P_T(T') : \nu(g) - 1 \in K \otimes_T T'\}, \quad \nu(g) = g \otimes g, \quad (6.10)$$

for every commutative T -algebra T' . Here $\nu(g)$ is viewed in $\mathcal{A} \otimes_T T'$.

Proof. Here $\mathcal{A}^\times(T') = (\mathcal{A} \otimes_T T')^\times$. The map $\nu : P_T \rightarrow \mathcal{A}^\times$ is multiplicative. Left multiplication by $\nu(g)$ is the action of g on $\Gamma^2(M)$. For the universal point $g = 1 + xe_1 + ye_2$ over $T[x, y, \mu_v^{-1}]$,

$$\nu(g) = d_0 + xc_{01} + yc_{02} + x^2d_1 + xyc_{12} + y^2d_2. \quad (6.11)$$

If $\mathcal{K}$ is invariant, then for $\eta \in \mathcal{K}$ the image of $\nu(g)\eta$ vanishes in $(\mathcal{A}/\mathcal{K})[x, y, \mu_v^{-1}]$. The localization map from $(\mathcal{A}/\mathcal{K})[x, y]$ is injective by the constant-term argument in Proposition 2.4; hence the same polynomial already vanishes before localization. Comparing coefficients proves that left multiplication by each of the six basis tensors preserves $\mathcal{K}$. This is the left-ideal condition. Conversely a left ideal is preserved by $\nu(g)$ and its inverse after every base change.

The quotient of the left regular module is cyclic with vector $\bar{1}$. The equality $\nu(g)\bar{1} = \bar{1}$ is precisely the condition in (6.10). This vector is evaluation at the origin, so its stabilizer is G_T by Proposition 3.2. $\square$

6.3. A scalar criterion for two-sidedness. The characters of C give two characters of $\mathcal{A}$, using $T \otimes_T T \simeq T$:

$$\beta = (h \otimes h)|_{\mathcal{A}}, \quad \tau = (\chi \otimes \chi)|_{\mathcal{A}} : \mathcal{A} \longrightarrow T. \quad (6.12)$$

Lemma 6.4 (The quotient module). *For a commutative D -algebra T and the quotient $W = \mathcal{A}/K = \mathcal{U}^\vee$ defined above, let $\varepsilon_W(q) = q(1)$. This is the augmentation of the quotient representation, distinct from the Hopf counit on B_T . There is an exact sequence of $\mathcal{A}$ -modules*

$$0 \longrightarrow W_1 = Tq_1 \oplus Tq_2 \longrightarrow W \xrightarrow{\varepsilon_W} T \longrightarrow 0, \quad (6.13)$$

where the action on W_1 is through τ and the action on T is through β . Its underlying T -module splitting sends 1 to q_0 . On P_T , the corresponding representation is

$$\rho_W(v) = \begin{pmatrix} 1 & 0 & 0 \\ f_1(v) & \mu_v^2 & 0 \\ f_2(v) & 0 & \mu_v^2 \end{pmatrix}. \quad (6.14)$$

Proof. Equation (6.3) implies that left multiplication by $z \in \mathcal{A}$ on each d_i , $i = 1, 2$, is multiplication by $\tau(z)$. This gives the claimed action on W_1 . The character β vanishes on K , and descends to the quotient map ε_W with $\varepsilon_W(q_0) = 1$ and $\varepsilon_W(q_1) = \varepsilon_W(q_2) = 0$. Thus the action on its target is through β . On P_T the two characters become 1 and μ_v^2 . Projecting (6.11) gives the first column of (6.14). $\square$

A *splitting* is a right inverse of ε_W ; the displayed splitting sends 1 to q_0 . It is not generally equivariant. Its failure to be equivariant is the *translation cocycle* F : this name refers precisely to the identity $F(v * w) = F(v) + \mu_v^2 F(w)$ for the scalar action μ^2 .

A subset of an algebra *annihilates* a module if each of its elements acts as zero on that module. A quotient with the prescribed projection is an *algebra quotient* only if its kernel is two-sided; a left ideal alone gives a left-module quotient.

Lemma 6.5 (Cyclic quotient criterion). *Let T be a commutative ring, let $\mathcal{B}$ be a unital T -algebra, and let $K \subset \mathcal{B}$ be a left ideal. Set $W = \mathcal{B}/K$ and $q_0 = \bar{1}$. Suppose $W = Tq_0 \oplus W_1$ as a T -module, where W_1 is a $\mathcal{B}$ -submodule that is free of positive rank over T , and the action on W_1 is scalar multiplication through a character $\tau : \mathcal{B} \rightarrow T$. Then*

$$K \text{ is two-sided} \iff \tau(K) = 0.$$

Proof. The condition $K\mathcal{B} \subset K$ says exactly that K annihilates W . Every $\eta \in K$ already annihilates q_0 , and acts on W_1 by the scalar $\tau(\eta)$. Since W_1 is free of positive rank, this operator is zero exactly when $\tau(\eta) = 0$. $\square$

The notation $\tau(K)$ means the image of K under the T -linear map τ . It is an ideal of T , because this image is a T -submodule of T . It measures the *obstruction to two-sidedness*: the quotient ring $T/\tau(K)$ is the base on which all these scalar values vanish.

Proposition 6.6 (The obstruction to two-sidedness). *For a commutative D -algebra T , let $K = \ker(\pi : \mathcal{A} \rightarrow \mathcal{U}^\vee)$ and $\tau = (\chi \otimes \chi)|_{\mathcal{A}}$. Then, as ideals of T ,*

$$\tau(K) = (2a, 2b), \quad K \text{ is two-sided} \iff 2a = 2b = 0. \quad (6.15)$$

Proof. Under the perfect pairing $\mathcal{A} = \mathcal{V}_T^\vee$, the character τ is the quadratic function $\sigma(X, Y) = (1 + aX + bY)^2$. Indeed, its values on the tensor basis are exactly the coefficients of that polynomial:

$$\tau(d_0) = 1, \quad \tau(d_1) = a^2, \quad \tau(d_2) = b^2, \quad \tau(c_{01}) = 2a, \quad \tau(c_{02}) = 2b, \quad \tau(c_{12}) = 2ab. \quad (6.16)$$

Using Lemma 4.1 and $-2 = 2$ since $4 = 0$, one obtains

$$\begin{aligned} \tau(\xi_{01}) &= 2a - a^2\lambda_1 - b^2\lambda_4 = 2a, \\ \tau(\xi_{02}) &= 2b - a^2\lambda_2 - b^2\lambda_5 = 2b, \\ \tau(\xi_{12}) &= 2ab - a^2\lambda_3 - b^2\lambda_6 = 2ab. \end{aligned} \quad (6.17)$$

These values generate $(2a, 2b)$. Apply Lemmas 6.4 and 6.5. Equivalently, since K is the annihilator of the direct summand $\mathcal{U}$, the condition $\tau(K) = 0$ says exactly that σ belongs to $\mathcal{U}$. This gives a polynomial interpretation of the character test. In the Artinian setting of Section 7, Proposition 7.3 also identifies it with right-translation invariance. $\square$

Remark 6.7 (The special-fiber ideal). In this remark only, take an Artinian point $D \rightarrow T$ and reduce along $T \rightarrow k$. Put $C_0 = C \otimes_T k = k \oplus k^2$ and $\mathcal{A}_0 = \mathcal{A} \otimes_T k$. A bar denotes the image of a tensor or a quotient vector on this special fiber. The augmentation ideal of C_0 is *square-zero*, meaning that any product of two of its elements is zero, and the reductions of h and χ coincide. Put $C_0^{(1)} = C_0 \otimes_{k, F_k} k$, where $F_k(c) = c^2$; this is the *Frobenius twist*, namely scalar extension along Frobenius. Since $k = \mathbf{F}_2$, the map F_k is the identity and this twist is canonically C_0 as a k -algebra [8, Tag 0CC6].

The reduced mixed tensors span the two-sided ideal

$$K_0 = K \otimes_T k = k\{\bar{c}_{01}, \bar{c}_{02}, \bar{c}_{12}\} \subset \mathcal{A}_0, \quad \mathcal{A}_0/K_0 \simeq C_0^{(1)}.$$

Indeed, the quotient has unit $\bar{q}_0$ and square-zero span $\bar{q}_1, \bar{q}_2$, and (6.11) projects to $\bar{q}_0 + x^2\bar{q}_1 + y^2\bar{q}_2$. Over a deformation base, it is the corrected generators ξ_{ij} , not the mixed tensors themselves, that span the invariant left ideal K . The prescribed projection is an algebra quotient precisely on the locus in (6.15).

7. NORMALITY AND THE COMMON OBSTRUCTION

Continue over an Artinian point $D \rightarrow T$. Define the semidirect product $Q = \mathbf{G}_{aT}^2 \rtimes \mathbf{G}_{mT}$ by

$$(z, t)(z', t') = (z + tz', tt'), \quad (z, t)^{-1} = (-t^{-1}z, t^{-1}),$$

where z, z' are two-component vectors and t, t' are units. It acts by $(z, t)Z = z + tZ$. Its identity is $(0, 1)$, and $\mathcal{T} = \{(0, t)\} \simeq \mathbf{G}_{mT}$ is the pure-dilation subgroup. The orbit identity gives a homomorphism

$$\rho : P_T \longrightarrow Q, \quad v \longmapsto (F(v), \mu_v^2).$$

The evaluation stabilizer and the homogeneous space fit into

$$\begin{array}{ccc} G_T & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow \\ P_T & \xrightarrow{\rho} & Q, \end{array} \quad P_T/G_T \simeq Q/\mathcal{T} \simeq \mathcal{X}. \quad (7.1)$$

The square is *Cartesian*, meaning that $G_T = P_T \times_Q \mathcal{T}$ is its fiber product; the upper arrow sends $v \in G_T$ to $(0, \mu_v^2)$. All three ambient groups are smooth; finite flatness of G_T was proved separately in Theorem 3.4.

7.1. Normality and the residual character.

Lemma 7.1 (Normal stabilizers). *Let S_0 be a scheme, let an S_0 -group scheme P act on an S_0 -scheme X , and let $x_0 : S_0 \rightarrow X$ be a section. Assume the orbit map $p \mapsto p \cdot x_0$ is faithfully flat and locally of finite presentation. Then $G = \text{Stab}_P(x_0)$ is normal in P if and only if its action on X is trivial, that is, the action morphism $G \times_{S_0} X \rightarrow X$ is the projection.*

Proof. If G is normal, every point of X is fppf locally $p \cdot x_0$, and for $g \in G$,

$$g(p \cdot x_0) = p((p^{-1}gp) \cdot x_0) = p \cdot x_0.$$

Equality descends, so the action is trivial. Conversely, if G acts trivially, every conjugate pgp^{-1} fixes x_0 and lies in G . Applying the same argument to p^{-1} gives equality of the conjugate subgroups, after every base change. $\square$

Proposition 7.2 (The normality locus). *For an Artinian point $D \rightarrow T$, the evaluation stabilizer $G_T = \text{Stab}_{P_T}(q_0)$ satisfies*

$$G_T \triangleleft P_T \iff 2a = 2b = 0 \text{ in } T.$$

Proof. On G_T , the translation part of the target action vanishes, so the residual action is multiplication by μ^2 . In its free coordinate basis, Lemma 4.2 gives

$$\mu^2 - 1 = 2ax + 2by + 2abxy. \quad (7.2)$$

This is zero exactly when $2a = 2b = 0$. The orbit map is fppf surjective by Theorem 3.4, so Lemma 7.1 applies. $\square$

7.2. Right translation and the square of the scalar character. For $v \in P_T$, let R_v denote right translation, $R_v(w) = w * v$. Since $R_v(w) = v + w + \ell(w)v$ is affine in w , its pullback preserves quadratic polynomials and acts by $f \mapsto f \circ R_v$. By *right-translation invariance* we mean preservation of $\mathcal{U}$ by all these pullbacks after every base change. The space $\mathcal{U}$ is already invariant under left translation. Put

$$\sigma(X, Y) = (1 + aX + bY)^2 \in \mathcal{V}_T.$$

This polynomial is the square of the scalar character, viewed as a function on the source.

Let $P_T^\sharp$ be the scalar-affine group with coefficients a^2, b^2 . Its underlying scheme is the value plane, with identity 0 and law

$$z \diamond z' = z + (1 + a^2 z_1 + b^2 z_2) z'. \quad (7.3)$$

The scalar is invertible because a, b are nilpotent. This formula uses coefficientwise squares, not a Frobenius endomorphism of the base.

Proposition 7.3 (Right translation and the quotient-group locus). *Let $D \rightarrow T$ be an Artinian point with invariant space $\mathcal{U} = T\{1, f_1, f_2\}$. The following conditions are equivalent:*

- (1) $\mathcal{U}$ is invariant under right translations of P_T ;
- (2) $\sigma = (1 + aX + bY)^2$ belongs to $\mathcal{U}$;
- (3) $2a = 2b = 0$ in T ;
- (4) the orbit map $F : P_T \rightarrow P_T^\#$ is a group homomorphism.

When they hold, F is finite faithfully flat with kernel G_T . If they do not hold, no group-scheme law on $\mathcal{X}$ with identity q_0 makes F a homomorphism.

Proof. The orbit identity, with its two inputs interchanged, reads

$$F(w * v) = F(w) + \sigma(w)F(v).$$

Thus (2) implies (1). Conversely, pass to $\mathcal{V}_T/\mathcal{U}$ and regard $v = (x, y)$ as independent polynomial variables. The coefficient of x^2 in $f_1(v)$ is one. Right-translation invariance therefore forces the class of σ in $\mathcal{V}_T/\mathcal{U}$ to vanish. This proves (1) $\Leftrightarrow$ (2).

The constant and pure-square coefficients of σ are $1, a^2, b^2$. Hence its only possible expression in the normalized space is $1 + a^2 f_1 + b^2 f_2$. Lemma 4.1 gives the exact discrepancy

$$\mu_v^2 - (1 + a^2 f_1(v) + b^2 f_2(v)) = 2ax + 2by + 2abxy. \quad (7.4)$$

The classes of X, Y, XY form a free basis of $\mathcal{V}_T/\mathcal{U}$, so (2) is equivalent to (3). When they hold, $\mu_v^2 = 1 + a^2 f_1(v) + b^2 f_2(v)$, and the orbit identity is exactly the homomorphism identity for (7.3). Thus (3) implies (4). Conversely, any group homomorphism has normal kernel. Proposition 7.2 therefore proves (4) $\Rightarrow$ (3), as well as the assertion excluding every other target group law. Finite faithful flatness was proved in Theorem 3.4. $\square$

The quotient $\mathcal{V}_T/\mathcal{U}$ records precisely which quadratic functions were not retained. The class of σ has coordinates $(2a, 2b, 2ab)$ in its basis X, Y, XY . Their ideal is $(2a, 2b)$, since the third coefficient is a multiple of either of the first two. This is the same obstruction as $\tau(K)$ because τ and σ are dual descriptions of the same functional.

Remark 7.4 (The quotient algebra). On the same locus, Proposition 6.6 makes $W = \mathcal{A}/K$ an algebra with unit q_0 . Define the linear form $\ell^\# : W_1 \rightarrow T$ by $\ell^\#(q_1) = a^2$ and $\ell^\#(q_2) = b^2$. Its products satisfy

$$q_i q_j = \ell^\#(q_i) q_j, \quad \ell^\#(q_1) = a^2, \quad \ell^\#(q_2) = b^2 \quad (i, j = 1, 2),$$

because $d_i d_j = \chi(e_i)^2 d_j$. Its augmentation-one unit group is exactly $P_T^\#$. The homomorphism on units induced by $\pi\nu$ is F .

7.3. The actual kernel. Write $N = \ker \rho$ for the scheme-theoretic kernel, the fiber of ρ over $(0, 1) \in Q$. Thus $N(T')$ consists of elements acting as the identity on the whole value plane; the stabilizer $G_T(T')$ only requires them to fix its origin. The notation N here is a group scheme, not the matrix N_v of Remark 4.5.

Proposition 7.5 (Flatness of the kernel). *Let $D \rightarrow T$ be an Artinian point and let $N = \ker \rho$, where $\rho(v) = (F(v), \mu_v^2)$. This is a finite closed subgroup scheme of G_T , with*

$$\mathcal{O}(N) = B_T/(\mu^2 - 1).$$

The group scheme N is flat over T if and only if $2a = 2b = 0$. It is the largest closed subgroup scheme of P_T that is normal in P_T and contained in G_T ; no flatness is imposed on subgroups in this maximality assertion.

Proof. The kernel condition requires both $F = 0$ and $\mu^2 = 1$, giving its algebra. The extra equation disappears on the special fiber. If N were flat, its finite algebra would be free over the local Artinian ring T , with rank four determined by its special fiber. The surjection from $B_T \simeq T^4$ would then be a surjection of free modules of the same rank, hence an isomorphism. Thus flatness forces $\mu^2 - 1 = 0$ in B_T , equivalently $2a = 2b = 0$ by (7.2). The converse is immediate, since then $N = G_T$.

The affine action of Q is faithful, so N is the kernel of the action on $\mathcal{X}$. A normal subgroup of P_T contained in the origin stabilizer fixes the whole homogeneous space, by the transitivity argument in Lemma 7.1. It therefore lies in N , proving maximality. $\square$

Theorem 7.6 (The common obstruction). *Let $D \rightarrow T$ be an Artinian point. Let $G_T = \text{Stab}_{P_T}(q_0)$ be the evaluation stabilizer, $K = \ker(\mathcal{A} \rightarrow \mathcal{U}^\vee)$ its invariant left ideal, and $N = \ker(\rho : P_T \rightarrow Q)$ the kernel of the affine representation. The following conditions are equivalent:*

- (1) $2a = 2b = 0$ in T ;
- (2) the fourth-power morphism $[4]_{G_T} : G_T \rightarrow G_T$ is constant at the identity;
- (3) G_T is normal in P_T ;
- (4) the invariant left ideal $K \subset \mathcal{A}$ is two-sided;
- (5) the actual kernel $N = \ker \rho$ is flat over T ;
- (6) the scheme $\mathcal{X}$ admits a T -group scheme structure with identity q_0 for which the orbit map $F : P_T \rightarrow \mathcal{X}$ is a homomorphism.

These conditions are also equivalent to $\sigma = (1 + aX + bY)^2 \in \mathcal{U}$, or to invariance of $\mathcal{U}$ under right translations. They hold exactly when $D \rightarrow T$ factors through $D/(2a, 2b)$. When they hold, the target group can be taken to be $P_T^\sharp$.

More precisely, let $\mathfrak{o}_4(T)$ be the ideal of T generated by the coefficients of $[4]^*(x)$ and $[4]^*(y)$ in the free basis $1, x, y, xy$ of B_T . Then

$$\mathfrak{o}_4(T) = \tau(K) = (2a, 2b).$$

Proof. Proposition 4.3 proves (1) $\Leftrightarrow$ (2). Propositions 7.2, 6.6, 7.5, and 7.3 prove the equivalence of (1) with (3), (4), (5), and (6), respectively. The factorization condition is exactly the vanishing of $2a, 2b$ in T . The ideal equality follows from (4.5) and (6.15). $\square$

Corollary 7.7 (The character and the fourth power). *For an Artinian point $D \rightarrow T$, identify P_T with the augmentation-one unit group of C . Let T' be any commutative T -algebra and $g = 1 + v \in G_T(T')$. Then, in $C \otimes_T T'$,*

$$g^4 - 1 = \tau(\nu(g) - 1)(g - 1), \tag{7.5}$$

where ν and τ are base changed to T' .

Proof. One has $\tau(\nu(g)) = \chi(g)^2 = \mu_v^2$. Under $g = 1 + v$, the asserted equality is (4.6). $\square$

For a stabilizer element, $\nu(g) - 1$ lies in K . Thus the character detecting failure of two-sidedness is precisely the scalar left by affine iteration. This identifies the obstruction within the family; its nonvanishing still requires the coefficient certificates of Sections 4 and 5.

Remark 7.8 (The ambient bracket). The Lie algebra $\text{Lie}(P_T)$ is its tangent module at the identity, here identified with E : its vectors correspond to elements of the kernel of $P_T(T[\epsilon]/(\epsilon^2)) \rightarrow P_T(T)$, where ϵ is a new variable with square zero. The bracket is induced by group commutators. In the realization by units of C , it is the algebra commutator $[v, w] = vw - wv = \ell(v)w - \ell(w)v$, so $[e_1, e_2] = ae_2 - be_1$. Every alternating bilinear bracket on a framed rank-two module (that is, $[u, u] = 0$ for all u) has this form: if $[e_1, e_2] = ce_1 + de_2$, take $\ell(e_1) = d$ and $\ell(e_2) = -c$. These

values are forced, and the associative algebra C gives an explicit smooth group with that bracket. In this frame,

$$[4](x, y) = 2xy[e_1, e_2],$$

since $-2 = 2$. The formula involves the double of the ambient bracket, not merely its nonvanishing. It is an identity for this family, not a classification of smooth group deformations by Lie algebras.

APPENDIX A. THE PARAMETER SPACE AND THE SIZES OF THE BASES

We record the reduced deformation space and the additive structures of the two coefficient rings. These calculations are not needed for the counterexamples in Sections 4 and 5. The *nilradical* of a commutative ring is its ideal of nilpotent elements; its *reduced quotient* is the quotient by that ideal. We write D_{red} for this quotient of D .

For a finite local ring U with residue field k and maximal ideal $\mathfrak{m}_U$, its *length* $\text{length}_U(U)$ is its composition length as a U -module: the number of simple factors in a composition series. We write $\text{length}(U)$ when the base ring is understood. Its *associated graded ring* is $\bigoplus_{j \geq 0} \mathfrak{m}_U^j / \mathfrak{m}_U^{j+1}$, with $\mathfrak{m}_U^0 = U$, and its *Hilbert function* is the sequence of dimensions of these graded pieces over k , with trailing zeros omitted. The maximal-ideal order of a nonzero element c is the largest j with $c \in \mathfrak{m}_U^j$, not its order in the additive group. A subscript $(-)_{\text{add}}$ denotes the underlying additive group.

A.1. Two reduced branches and a nilpotent thickening. For two local k -algebras with residue maps to k , their ring fiber product over k consists of pairs with the same residue. In the following formula the residue maps take a power series to its constant term. The *Zariski tangent space* of $\text{Spec } D$ at its closed point is $\text{Hom}_k(\mathfrak{n}/\mathfrak{n}^2, k)$. The two *branches* below are the two irreducible components of $\text{Spec } D_{\text{red}}$ through that point.

Proposition A.1 (The reduced parameter space). *For the complete local ring D of Theorem 2.6, its nilradical is $\mathfrak{r} = (2, \lambda_3, \lambda_6)$, with $\mathfrak{r}^2 = 0$, and*

$$D_{\text{red}} \simeq k[[a, b]] \times_k k[[\lambda_1, \lambda_2, \lambda_4, \lambda_5]]. \quad (\text{A.1})$$

The Zariski tangent space at the closed point with residue field $k = \mathbf{F}_2$ has dimension six over k .

Proof. We know $4 = 2\lambda_3 = 2\lambda_6 = 0$. Moreover,

$$\lambda_3^2 = (a\lambda_2)(b\lambda_1) = (a\lambda_1)(b\lambda_2) = 0, \quad \lambda_6^2 = (a\lambda_5)(b\lambda_4) = 0, \quad \lambda_3\lambda_6 = (b\lambda_1)(a\lambda_5) = 4 = 0.$$

Hence $\mathfrak{r}^2 = 0$. Eliminating λ_3, λ_6 and then imposing $2 = \lambda_3 = \lambda_6 = 0$ in (2.13) gives exactly

$$D/\mathfrak{r} \simeq \frac{k[[a, b, \lambda_1, \lambda_2, \lambda_4, \lambda_5]]}{(a\lambda_i, b\lambda_i : i \in \{1, 2, 4, 5\})}.$$

In this quotient every series is uniquely a common constant plus a series of positive degree in a, b and a series of positive degree in the four λ variables. Sending it to those two series with the same constant identifies it with the fiber product (A.1). The latter is a subring of a product of two domains, hence is reduced. Therefore $\mathfrak{r}$ is the nilradical.

Finally, $2 = a\lambda_1 \in \mathfrak{n}^2$, and the six relations in (2.13) impose no linear relation on the six parameter classes. More explicitly,

$$D/\mathfrak{n}^2 \simeq k[a, b, \lambda_1, \lambda_2, \lambda_4, \lambda_5]/(a, b, \lambda_1, \lambda_2, \lambda_4, \lambda_5)^2.$$

This proves $\dim_k(\mathfrak{n}/\mathfrak{n}^2) = 6$. □

On one reduced branch, the acting group varies while the quadratic space stays equal to the literal space of constants and squares. On the other branch, the group stays additive while the linear corrections to the squares vary. The fourth-power obstruction lives in the nilpotent thickening, not on either reduced branch.

A.2. The universal truncation.

Proposition A.2 (The universal truncation). *Let $\mathfrak{n}$ be the maximal ideal of D . The ring $A_4 = D/\mathfrak{n}^4$ has composition length 49, additive group*

$$(A_4)_{\text{add}} \simeq (\mathbf{Z}/4)\{1, a, b\} \oplus (\mathbf{Z}/2)^{43}, \quad (\text{A.2})$$

and maximal-ideal Hilbert function $(1, 6, 16, 26)$.

Proof. Besides the identities already used, the defining equations give

$$2a^2 = ab(a\lambda_5) = b^2(a\lambda_4) = 0, \quad 2b^2 = ab(b\lambda_1) = a^2(b\lambda_2) = 0.$$

The four linear λ parameters annihilate $\mathfrak{r}$. For example,

$$\lambda_1\lambda_3 = \lambda_2(a\lambda_1) = 2\lambda_2 = 0, \quad \lambda_2\lambda_3 = \lambda_1(b\lambda_2) = 0;$$

using $\lambda_3 = a\lambda_2 = b\lambda_1$ gives the other two products, and the involution (4.9) gives those with λ_6 . Multiplication by a, b is

$$a\lambda_3 = 2b, \quad b\lambda_3 = 0, \quad a\lambda_6 = 0, \quad b\lambda_6 = 2a. \quad (\text{A.3})$$

In A_4 , $2a, 2b$ are annihilated by $\mathfrak{n}$, since they lie in $\mathfrak{n}^3$. Thus the image $\mathfrak{r}_{A_4}$ is spanned, as an additive k -vector space, by

$$2, \quad \lambda_3, \quad \lambda_6, \quad 2a, \quad 2b.$$

These five classes are independent. Modulo two, $\lambda_3 = a\lambda_2 = b\lambda_1$ and $\lambda_6 = a\lambda_5 = b\lambda_4$ are independent quadratic classes: the degree-two relation space has basis

$$a\lambda_1, \quad b\lambda_5, \quad a\lambda_4, \quad b\lambda_2, \quad a\lambda_2 - b\lambda_1, \quad a\lambda_5 - b\lambda_4,$$

and leaves exactly those two mixed classes. To separate $2, 2a, 2b$, use φ_b from (4.8), the functional $\varphi_a = \varphi_b \circ \iota$ for the involution (4.9), and the functional φ_0 given by

$$\varphi_0(1) = 1, \quad \varphi_0(a\lambda_1) = \varphi_0(b\lambda_5) = 2,$$

with all other monomials assigned zero. The only generator-level cancellations for φ_0 are $a\lambda_1 - 2$ and $b\lambda_5 - 2$; all their other multiples either miss its support or double a value two. Thus it descends just as in Lemma 4.6. The three functionals take value two on their respective classes $2, 2a, 2b$ and zero on the other two. Therefore $\dim_k \mathfrak{r}_{A_4} = 5$.

By (A.1), the quotient $A_4/\mathfrak{r}_{A_4}$ has a basis consisting of all pure monomials of degrees at most three on the two branches, with their constants identified. Its dimension is

$$1 + (2 + 3 + 4) + (4 + 10 + 20) = 44.$$

Hence $\text{length}(A_4) = 44 + 5 = 49$. Every mixed quadratic parameter monomial reduces to $0, 2, \lambda_3$, or λ_6 , and every mixed cubic to $0, 2a$, or $2b$. It follows that A_4 is generated additively by $1, a, b$ of order four; the seven pure a, b monomials of degrees two and three; the 34 pure monomials in the four remaining λ variables of degrees one through three; and λ_3, λ_6 . The last 43 generators have order two. Their total additive length is $2 \cdot 3 + 43 = 49$, so this generating map is an isomorphism, proving (A.2).

This additive decomposition also describes the filtration. The ideal $\mathfrak{n}^3$ is the additive span of the pure degree-three monomials on the two branches and $2a, 2b$: the relations reduce every mixed degree-three monomial to these classes, and $\mathfrak{n}^4 = 0$. The ideal $\mathfrak{n}^2$ is generated additively by $\mathfrak{n}^3$, the pure degree-two monomials, and $2, \lambda_3, \lambda_6$. Independence follows from the additive decomposition and the independence of the five nilradical classes already proved. Thus $\mathfrak{n}^2/\mathfrak{n}^3$ has dimension $3 + 10 + 3 = 16$, and $\mathfrak{n}^3$ has dimension $4 + 20 + 2 = 26$. Together with the constant class and six linear classes, this proves the Hilbert function $(1, 6, 16, 26)$. $\square$

A.3. The small base has nine composition factors.

Proposition A.3 (The two-generator finite base). *The ring $R = \mathbf{Z}[a, b]/(a^2b - 2, a^3, b^3)$ has additive decomposition*

$$R_{\text{add}} \simeq (\mathbf{Z}/4)\{1, b\} \oplus (\mathbf{Z}/2)\{a, a^2, ab, b^2, ab^2\}. \quad (\text{A.4})$$

It has 512 elements, composition length nine, and maximal-ideal Hilbert function $(1, 2, 3, 2, 1)$. For $\kappa = 2b$ and $\mathfrak{m} = (a, b)$,

$$\mathfrak{m}^4 = (\kappa) \neq 0, \quad \mathfrak{m}^5 = 0, \quad \kappa\mathfrak{m} = 0.$$

Proof. In the free abelian group $T_1 = \mathbf{Z}[a, b]/(a^3, b^3)$, the additive group of R is the cokernel of multiplication by $2 - a^2b$. On the two summands with ordered bases $(1, a^2b)$ and (b, a^2b^2) , this endomorphism has matrix

$$\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}.$$

Each cokernel is cyclic of order four, generated by the first basis vector: the second generator equals twice the first. On each of the five other monomials in (A.4), the endomorphism is multiplication by two. This proves the additive decomposition and the cardinality $4^2 \cdot 2^5 = 512$.

The ideal $\mathfrak{m} = (a, b)$ is nilpotent with residue field $\mathbf{F}_2$, so it is the unique maximal ideal. Its graded pieces have bases

$$\{1\}, \quad \{a, b\}, \quad \{a^2, ab, b^2\}, \quad \{a^2b, ab^2\}, \quad \{a^2b^2\}.$$

Independence follows from (A.4); in particular $a^2b = 2$ and $a^2b^2 = 2b$ are the nonzero doubles in the two order-four summands. The displayed Hilbert function and the remaining assertions follow. $\square$

These sizes distinguish the two restrictions of the same universal family:

base	length	Hilbert function	order of 2 in the filtration
$A_4 = D/\mathfrak{n}^4$	49	$(1, 6, 16, 26)$	2
R	9	$(1, 2, 3, 2, 1)$	3

The last column refers to maximal-ideal order, not additive order. Both rings have characteristic four.

A.4. A nonzero square on the full deformation ring.

Proposition A.4 (The scaling error need not be square-zero). *The element $2ab$ is nonzero in $D/\mathfrak{n}^5$, and hence in D . Consequently the universal stabilizer satisfies $\theta^2 = 2abxy \neq 0$ in B_D , although $\theta^2(x, y) = 0$ in B_D^2 .*

Proof. Put $T_5 = (\mathbf{Z}/4)[a, b, \lambda_1, \lambda_2, \lambda_4, \lambda_5]/(a, b, \lambda_1, \lambda_2, \lambda_4, \lambda_5)^5$. As in Lemma 4.6, its quotient by the six invariance relations is $D/\mathfrak{n}^5$. Define a $\mathbf{Z}/4$ -linear functional $\psi : T_5 \rightarrow \mathbf{Z}/4$ by

$$\psi(ab) = 1,$$

$$\psi(a^2b\lambda_1) = \psi(a^3\lambda_2) = \psi(ab^2\lambda_5) = \psi(b^3\lambda_4) = 2,$$

and zero on all other monomials. Among multiples of the six relations by monomials of degree at most two, the only nonzero individual term values occur in

$$(a\lambda_1 - 2)ab, \quad (b\lambda_5 - 2)ab, \quad (a\lambda_2 - b\lambda_1)a^2, \quad (a\lambda_5 - b\lambda_4)b^2.$$

Their values cancel. For multipliers of degree at least three, all quadratic terms disappear in T_5 , and the remaining terms $-2m$ have zero value because $\psi(m)$ is zero or two. Thus ψ descends and sends $2ab$ to $2 \neq 0$. The assertion about θ follows from Lemma 4.2 and the free basis of B_D . $\square$

APPENDIX B. COMPLETE FORMULAS FOR THE EXAMPLE OVER R

This appendix collects the formulas for the single group scheme in Theorem 5.3. The displayed group operations are functorial in commutative test algebras, and all tensor products and structural maps are over

$$R = \mathbf{Z}[a, b]/(a^2b - 2, a^3, b^3), \quad \kappa = 2b.$$

B.1. The group, the invariant space, and the two representations. In this appendix $v = (x, y)$ and $w = (x', y')$ denote coordinate vectors, whereas $\omega = xy$ denotes a basis element of the finite coordinate algebra. The smooth group has underlying scheme $\mathbf{A}_R^2$ and law

$$\mu_v = 1 + ax + by, \quad v * w = v + \mu_v w, \quad v^{-1} = -\mu_v^{-1}v.$$

Its invariant three-space and two-dimensional evaluation kernel are

$$\begin{aligned} F_1 &= X^2 + abX + b^2(1 + aX)Y, & F_2 &= Y^2 + a^2Y, \\ \mathcal{U}_R &= R\{1, F_1, F_2\}, & H_0 &= R\{F_1, F_2\}. \end{aligned}$$

The stabilizer of this kernel plane is $G_R = \text{Stab}_{P_R}(H_0)$, with

$$B_R = R[x, y]/(F_1(x, y), F_2(x, y)) = R\{1, x, y, \omega\}, \quad \omega = xy. \quad (\text{B.1})$$

For every commutative R -algebra R' , its points are the pairs $(x, y) \in (R')^2$ satisfying $F_1(x, y) = F_2(x, y) = 0$. Their multiplication, identity and inverse are

$$\begin{aligned} (x, y) * (x', y') &= (x + \mu_v x', y + \mu_v y'), \\ e &= (0, 0), \quad (x, y)^{-1} = (-\mu_v^{-1}x, -\mu_v^{-1}y). \end{aligned}$$

The scalar μ_v is a unit over every R' because a, b generate a nilpotent ideal.

In the ordered basis (e_0, e_1, e_2) , the smooth source P_R is faithfully represented by

$$\rho_M(v) = \begin{pmatrix} 1 & 0 & 0 \\ x & \mu_v & 0 \\ y & 0 & \mu_v \end{pmatrix}.$$

Its image is the closed subscheme of $\text{GL}_{3,R}$ cut out by this matrix pattern and by the equation for μ_v . The determinant is μ_v^2 , so invertibility of the matrix is exactly invertibility of μ_v . The first column recovers v , proving that this is a closed embedding.

The quotient representation, which need not be faithful, is

$$\rho_W(v) = \begin{pmatrix} 1 & 0 & 0 \\ F_1(v) & \mu_v^2 & 0 \\ F_2(v) & 0 & \mu_v^2 \end{pmatrix}.$$

It acts on $\{q : q(1) = 1\}$ by $z \mapsto F(v) + \mu_v^2 z$. Its distinguished vector is $q_0 = (1, 0, 0)^t$, and its stabilizer is the same G_R .

B.2. Multiplication and the Hopf structure. The element 1 is the unit of B_R , and the other basis products are

	x	y	ω
x	$abx + b^2y + ab^2\omega$	ω	$\kappa y + ab\omega$
y	ω	a^2y	$a^2\omega$
ω	$\kappa y + ab\omega$	$a^2\omega$	0 .

(B.2)

For example, $x\omega = x^2y = -ab\omega + a^2b^2y = ab\omega + \kappa y$ and $y\omega = -a^2\omega = a^2\omega$; then $\omega^2 = -a^2x\omega = 0$. The signs agree because $2a = 2b^2 = 0$.

In B_R , set $\theta = ax + by$ and $\mu = 1 + \theta$. The relations imply

$$\theta^2 = 0, \quad \theta\omega = 0, \quad \mu^{-1} = 1 - \theta, \quad \mu^2 = 1 + \kappa y, \quad \mu^4 = 1. \quad (\text{B.3})$$

These are identities on the finite stabilizer; $\theta^2 = 0$ is not being asserted on the whole ambient group. The equality $\theta\omega = 0$ follows from the two mixed products in (B.2).

The coproduct $\Delta : B_R \rightarrow B_R \otimes_R B_R$, counit $\varepsilon : B_R \rightarrow R$, and antipode $S_B : B_R \rightarrow B_R$ on the full free basis are

$$\begin{aligned} \Delta(1) &= 1 \otimes 1, \\ \Delta(x) &= x \otimes 1 + \mu \otimes x, \\ \Delta(y) &= y \otimes 1 + \mu \otimes y, \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} \Delta(\omega) &= \omega \otimes 1 + \mu x \otimes y + \mu y \otimes x + \mu^2 \otimes \omega, \\ \varepsilon(1) &= 1, \quad \varepsilon(x) = \varepsilon(y) = \varepsilon(\omega) = 0, \end{aligned} \quad (\text{B.5})$$

$$S_B(1) = 1, \quad S_B(x) = x - b\omega - ab^2y, \quad S_B(y) = y + a\omega, \quad S_B(\omega) = \omega.$$

The unit map is the inclusion $R \rightarrow B_R$. The coproduct of ω is the product of those of x, y , and $\Delta(\mu) = \mu \otimes \mu$. The antipode is equivalently $S_B(x) = -(1 - \theta)x$, $S_B(y) = -(1 - \theta)y$. To verify its expanded form, the quadratic relations give

$$(2 + \theta)x = -b\omega - ab^2y, \quad (2 + \theta)y = a\omega.$$

Together with $4 = 0$ this yields (B.5); also $S_B(\omega) = (1 - \theta)^2\omega = \omega$ by (B.3).

For all $n \geq 1$, the power morphism is

$$[n](x, y) = \left(n + \binom{n}{2}\theta \right) (x, y). \quad (\text{B.6})$$

In particular

$$[2](x, y) = (-b\omega - ab^2y, a\omega), \quad [4](x, y) = (\kappa\omega, 0), \quad [8] = e.$$

The identities follow from affine iteration; no parameter is canceled. Since $\kappa \neq 0$ and ω is a free basis element, the fourth-power morphism is nonzero. Every R -valued point is nevertheless killed by two; Proposition B.2 below computes the entire abstract group $G_R(R)$.

B.3. The divided-square quotient and its left ideal. Here $M = C$ is the standard rank-three representation and $W = \mathcal{U}_R^\vee$. In the tensor bases of Section 6, the quotient $\pi : \Gamma^2(M) \twoheadrightarrow W$ and augmentation $\varepsilon_W : W \rightarrow R$, $q \mapsto q(1)$, have matrices

$$\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & ab & b^2 & ab^2 \\ 0 & 0 & 1 & 0 & a^2 & 0 \end{pmatrix}, \quad \varepsilon_W = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}. \quad (\text{B.7})$$

The source order is $(d_0, d_1, d_2, c_{01}, c_{02}, c_{12})$ and the target order is (q_0, q_1, q_2) . The kernel in $\mathcal{A} = (C \otimes_R C)^{\mathfrak{S}_2}$ is

$$\begin{aligned} K &= R\{\xi_{01}, \xi_{02}, \xi_{12}\}, \\ \xi_{01} &= c_{01} - ab d_1, \quad \xi_{02} = c_{02} - b^2 d_1 - a^2 d_2, \quad \xi_{12} = c_{12} - ab^2 d_1. \end{aligned} \quad (\text{B.8})$$

Here $C = R \oplus E$ has $e_i e_j = \ell(e_i) e_j$ for $i, j = 1, 2$. Invariance proves that K is a left ideal. The scalar character of Section 6 gives

$$\tau(\xi_{01}) = 0, \quad \tau(\xi_{02}) = \kappa, \quad \tau(\xi_{12}) = 0.$$

Thus, for example, $\pi(\xi_{02} d_1) = \kappa q_1 \neq 0$. The quotient is not an algebra quotient with this prescribed map.

B.4. The smaller, nonflat kernel.

Proposition B.1 (The kernel in the explicit example). *Over $R = \mathbf{Z}[a, b]/(a^2b - 2, a^3, b^3)$, let N_R be the kernel of the affine representation $\rho(v) = (F(v), \mu_v^2)$. With $\kappa = 2b$ and $\omega = xy$, its coordinate algebra is*

$$\mathcal{O}(\ker \rho) = B_R/(\kappa y) \simeq R\{1, x\} \oplus (R/(\kappa))\{y, \omega\} \quad (\text{B.9})$$

as an R -module. The group scheme N_R is not flat over R , although its special fiber equals that of G_R , namely $\alpha_2 \times \alpha_2$.

Proof. On G_R , (B.3) gives $\mu^2 - 1 = \kappa y$, so the kernel is defined by the additional equation $\kappa y = 0$. The multiplication table (B.2) gives

$$(\kappa y)B_R = R\kappa y \oplus R\kappa \omega :$$

the products of κy with $1, x, y, \omega$ are $\kappa y, \kappa \omega, 0, 0$, and $\kappa \omega$ is annihilated by x, y . Quotienting the free basis gives (B.9).

The module $R/(\kappa)$ is not flat. Indeed, $\kappa \neq 0$ and $\kappa^2 = 0$, while tensoring $(\kappa) \hookrightarrow R$ with $R/(\kappa)$ gives the zero map from the nonzero module $(\kappa)/(\kappa)^2$. Thus the displayed direct sum is not flat. Finally, κ reduces to zero in the residue field, so the additional equation disappears on the special fiber. $\square$

B.5. Points over the coefficient ring.

Proposition B.2 (The abstract group of R -points). *Let $\mathfrak{m} = (a, b) \subset R$ and $\kappa = 2b$. As a subset of R^2 ,*

$$G_R(R) = \mathfrak{m}^2 \times V_R, \quad V_R = (\mathbf{Z}/2)\{2, ab + b^2, ab^2, \kappa\}.$$

Here V_R is an additive subgroup of R , and $\mathfrak{m}^2 = (\mathbf{Z}/2)\{2, a^2, ab, b^2, ab^2, \kappa\}$. The group law on $G_R(R)$ is ordinary addition, so $G_R(R) \simeq (\mathbf{Z}/2)^{10}$. In particular it has 1,024 points and every point is killed by two.

Proof. A solution $(x, y) \in R^2$ reduces to $(0, 0)$ in the residue field, so $x, y \in \mathfrak{m}$. Modulo $\mathfrak{m}^3$, the equations become $x^2 = y^2 = 0$. The independent classes a^2, b^2 in $\mathfrak{m}^2/\mathfrak{m}^3$ show that both linear classes vanish, hence $x, y \in \mathfrak{m}^2$.

The displayed additive basis of $\mathfrak{m}^2$ follows from Proposition A.3. Write

$$y = 2u_0 + a^2u_1 + abu_2 + b^2u_3 + ab^2u_4 + \kappa u_5, \quad u_i \in \{0, 1\}.$$

For every $x \in \mathfrak{m}^2$, direct multiplication in this basis gives

$$F_1(x, y) = \kappa u_1, \quad F_2(x, y) = \kappa(u_2 + u_3).$$

Indeed, if $t \in \{0, 1\}$ is the coefficient of ab in the displayed additive basis expansion of $x \in \mathfrak{m}^2$, then $x^2 = \kappa t = abx$. Thus $x^2 + abx = 2\kappa t = 0$. Also $b^2y = \kappa u_1$, and the mixed term ab^2xy lies in $\mathfrak{m}^7 = 0$. Thus the equations hold exactly when $u_1 = 0$ and $u_2 = u_3$, giving V_R and $2^6 \cdot 2^4$ solutions. Finally, if v, w have coordinates in $\mathfrak{m}^2$, then $\ell(v)w$ has coordinates in $\mathfrak{m}^5 = 0$. Therefore $v * w = v + w$. Each coordinate is killed by two, proving the group assertion. $\square$

This concerns only points over R itself. The nonzero fourth-power morphism is detected over the coordinate algebra B_R , and noncommutativity is detected by two independent universal points over $B_R \otimes_R B_R$.

APPENDIX C. COEFFICIENT AND MULTIPLICATION TABLES

C.1. The two substitution errors. For direct comparison with the six-parameter calculation, put $v = (x, y)$, $Z = (X, Y)^t$, and $\mu_v = 1 + ax + by$. Define the substitution errors by

$$E_i = f_i(x + \mu_v X, y + \mu_v Y) - f_i(x, y) - \mu_v^2 f_i(X, Y).$$

In the localization $\mathbf{Z}[a, b, \lambda_1, \dots, \lambda_6, x, y, X, Y, \mu_v^{-1}]$, before imposing any coefficient relations, the exact errors are

$$\begin{aligned} \mu_v^{-1} E_1 &= ((2 - a\lambda_1)x + (\lambda_3 - b\lambda_1)y)X \\ &\quad + ((\lambda_3 - a\lambda_2)x - b\lambda_2 y)Y, \\ \mu_v^{-1} E_2 &= (-a\lambda_4 x + (\lambda_6 - b\lambda_4)y)X \\ &\quad + ((\lambda_6 - a\lambda_5)x + (2 - b\lambda_5)y)Y. \end{aligned}$$

These are integer polynomial identities after multiplying by μ_v . Their eight coefficients give exactly (2.8). Substituting the specialized coefficients of Section 5, but before imposing the relations on a, b , gives the following identity in $\mathbf{Z}[a, b, x, y, x', y']^2$, with $v = (x, y)$ and $v' = (x', y')$:

$$F(v + \mu_v v') - F(v) - \mu_v^2 F(v') = \mu_v \left(\begin{aligned} &(2 - a^2 b)xx' - b^3 yy' \\ &(2 - a^2 b)yy' - a^3 xy' \end{aligned} \right).$$

This also supplies a direct proof of its action identity independently of the differential shorthand.

C.2. The rank-six algebra. The unit of $\mathcal{A}$ is d_0 . Its other products, with the row as left factor, are

	d_1	d_2	c_{01}	c_{02}	c_{12}
d_1	$a^2 d_1$	$a^2 d_2$	$2ad_1$	ac_{12}	$a^2 c_{12}$
d_2	$b^2 d_1$	$b^2 d_2$	bc_{12}	$2bd_2$	$b^2 c_{12}$
c_{01}	$2ad_1$	$2ad_2$	$ac_{01} + 2d_1$	$ac_{02} + c_{12}$	$2ac_{12}$
c_{02}	$2bd_1$	$2bd_2$	$bc_{01} + c_{12}$	$bc_{02} + 2d_2$	$2bc_{12}$
c_{12}	$2abd_1$	$2abd_2$	$2bd_1 + ac_{12}$	$2ad_2 + bc_{12}$	$2abc_{12}$

(C.1)

The table holds over $\mathbf{Z}[a, b]$, before imposing invariance. Every entry follows by multiplying the tensor factors with the rule $e_i e_j = \ell(e_i) e_j$ for $i, j > 0$.

C.3. All right-ideal defects from the scalar character. Work over any commutative D -algebra T , including D itself, with the objects and results of Section 6. For $\eta \in K$ and $z \in \mathcal{A}$, the class $\pi(\eta z) \in W$ is the *right-ideal defect*: its vanishing for every such pair is exactly closure of K under right multiplication. In the table, each column specifies the right factor, and $\pi(\xi_{ij} \cdot)$ means the map $z \mapsto \pi(\xi_{ij} z)$. The complete defect table on the universal family is

	d_0	d_1	d_2	c_{01}	c_{02}	c_{12}
$\pi(\xi_{01} \cdot)$	0	$2aq_1$	$2aq_2$	0	0	0
$\pi(\xi_{02} \cdot)$	0	$2bq_1$	$2bq_2$	0	0	0
$\pi(\xi_{12} \cdot)$	0	$2abq_1$	$2abq_2$	0	0	0.

(C.2)

There is no separate calculation for its zero columns. Write $\pi(z) = \beta(z)q_0 + z_1 q_1 + z_2 q_2$. For $\eta \in K$, one has $\eta q_0 = 0$, and η acts on W_1 by $\tau(\eta)$. Hence

$$\pi(\eta z) = \tau(\eta)(\pi(z) - \beta(z)q_0) = \tau(\eta)(z_1 q_1 + z_2 q_2). \quad (C.3)$$

Apply (6.17). The unit has no W_1 component. The two diagonal tensors have components q_1, q_2 . The mixed tensors have the components displayed in (6.8), whose coefficients are all killed by two. These observations prove every column of (C.2) at once.

GUIDE TO NOTATION

The same letter denotes a coefficient and its image under a specified base change. A subscript T on a scheme abbreviates base change to $\text{Spec } T$.

Notation	Meaning	Definition or formula
Coefficient rings and deformation data		
$k, \mathbf{Z}_2, S$	$\mathbf{F}_2$, the 2-adic integers, and $\mathbf{Z}[[a, b]]$.	Section 1
T, T'	A coefficient ring and a commutative test T -algebra; Artinian hypotheses are stated explicitly.	p. 4; Definition 2.2
$\mathcal{D}(T)$	Invariant lifts $(a, b, \mathcal{U})$ with the ambient coordinates fixed.	Definition 2.2
$\lambda_1, \dots, \lambda_6$	Coefficients of the two normalized quadratic polynomials.	(2.4)
$D, I, \mathfrak{n}$	Complete representing ring, invariance ideal, and maximal ideal of D .	Theorem 2.6; p. 15
A_4	$D/\mathfrak{n}^4$, the third-order neighbourhood.	(4.7)
$R, \mathfrak{m}, \kappa$	$R = \mathbf{Z}[a, b]/(a^2b - 2, a^3, b^3)$, $\mathfrak{m} = (a, b)$, and $\kappa = 2b = a^2b^2 \neq 0$.	Lemma 5.1
$\varphi_b, \varphi_a, \varphi_0$	Additive functionals detecting $2b, 2a, 2$ in A_4 ; they are not ring maps.	Lemma 4.6; Proposition A.2
ψ	Additive functional on $D/\mathfrak{n}^5$ detecting $2ab$.	Proposition A.4
ι	Involution exchanging the two source directions and their coefficients.	(4.9)
$\mathfrak{r}, D_{\text{red}}$	The nilradical $(2, \lambda_3, \lambda_6)$ and the quotient $D/\mathfrak{r}$.	Proposition A.1
Functions, representations, and stabilizers		
$v = (x, y), Z = (X, Y)^t$	A group-element coordinate vector and a column of polynomial variables.	(1.1)
ℓ, μ, θ	$\ell(x, y) = ax + by$, $\mu = 1 + ax + by$, and $\theta = \mu - 1$.	p. 5; Lemma 4.2
$P = P_{a,b}$	Smooth scalar-affine group, with $v * w = v + \mu_v w$.	p. 5; (2.2)
$\mathcal{V}, \mathcal{U}_0, \mathcal{U}$	All quadratic polynomials; $k\{1, X^2, Y^2\}$; and an invariant rank-three lift containing 1.	§2
f_1, f_2, F	Normalized generators of $\mathcal{U}$ and the orbit-coordinate map $F = (f_1, f_2)^t$. Capital F_i denote their specialization over R .	(2.4), (5.6)
J_F, L, Q_2, B_{Q_2}	Jacobian; $L = J_F(0)$, also viewed as a linear map; the homogeneous quadratic part; and its polarization.	p. 6
A_1, A_2, C_{12}	The columns of L and the vector of mixed quadratic coefficients.	p. 7
$\mathcal{X}, q_0, H_0$	Value plane $q(1) = 1$; evaluation at the identity; and its rank-two kernel inside $\mathcal{U}$.	Definition 3.1
$\mathcal{X}^\circ, \gamma$	The open where $\gamma(z) = 1 + a^2z_1 + b^2z_2$ is invertible, over a general coefficient ring.	Proposition 3.6
σ, R_v	$\sigma(X, Y) = (1 + aX + bY)^2$ and right translation $w \mapsto w * v$.	§7.2

Notation	Meaning	Definition or formula
G_T, B_T	Evaluation stabilizer and its coordinate Hopf algebra $T[x, y]/(f_1, f_2)$.	Proposition 3.2; Proposition 3.6
$[n]_G, e, [n]^*$	Power morphism; identity section or constant identity morphism; and pullback on functions.	Section 1
Δ, ε, S_B	Hopf coproduct, counit, and antipode.	p. 11; (3.8)
ω	The free-basis element $xy \in B_R$, not a second group element.	(B.1)
V_R	Additive subgroup giving the second coordinates of $G_R(R)$.	Proposition B.2
The cyclic-module interpretation		
E, C, M	$E = Te_1 \oplus Te_2$; the unital algebra $C = T \oplus E$; and its underlying rank-three module M .	Definition 6.1
h, χ	Characters of C , given by $h(s, v) = s$ and $\chi(s, v) = s + \ell(v)$.	Proposition 6.2
$\mathcal{A}, \Gamma^2(M)$	The algebra $(C \otimes_T C)^{\mathfrak{S}_2}$, whose underlying module is the divided square.	(6.5); p. 20
d_i, c_{ij}	Diagonal and mixed symmetric tensor basis elements.	(6.6)
π, W, K, ξ_{ij}	Dual quotient $\pi : \mathcal{A} \rightarrow W = \mathcal{U}^\vee$, its kernel, and the three displayed kernel generators.	(6.7)–(6.9)
q_0, q_1, q_2, W_1	Basis dual to $1, f_1, f_2$, and $W_1 = Tq_1 \oplus Tq_2$. These q_1, q_2 identify the earlier value coordinates.	p. 20; Lemma 6.4
$\varepsilon_W, \beta, \tau$	Evaluation $q \mapsto q(1)$ on W ; augmentation character of $\mathcal{A}$; and its scalar-action character.	(6.12), (6.13)
ν, ρ_M, ρ_W	$g \mapsto g \otimes g$; the faithful representation on M ; and the representation on W , a rank-three quotient of $\Gamma^2(M)$.	(6.11), (6.4), (6.14)
$C_0^{(1)}$	Frobenius twist $C_0 \otimes_{k, F_k} k$ of $C_0 = C \otimes_T k$.	Remark 6.7
$Q, \mathcal{T}, \rho$	Scalar-affine group $\mathbf{G}_{aT}^2 \rtimes \mathbf{G}_{mT}$; its dilation subgroup; and $\rho(v) = (F(v), \mu_v^2)$.	§7
$P_T^\sharp, \ell^\sharp$	Target scalar-affine group and its defining linear form with coefficients a^2, b^2 .	(7.3); p. 24
N, N_R	Kernel of ρ , over the general and explicit bases; not the point stabilizer in general.	Proposition 7.5; Appendix B
$\mathfrak{o}_4(T)$	Ideal of coefficients of the fourth-power map; equal to $\tau(K) = (2a, 2b)$.	Theorem 7.6

PROVENANCE AND SCOPE

The text was generated entirely by OpenAI ChatGPT. Anthropic Claude was used for refereeing and corrections. No assertion of complete human checking is made.

The written proofs do not rely on computer algebra. Python and SymPy supplied supplementary exact-arithmetic checks. The Lean/Mathlib formalization [10] concerns the separately presented counterexample discussed at the end of Section 5; no proof-assistant formalization of the scalar-affine family or its invariant-space deformation theory is claimed here.

The snapshot cited in [9] credits a rank-four counterexample to Codex and Claude. This article studies the scalar-affine family (2.2), its embedded deformation problem, and its specialization over R ; no priority or minimality claim is made. The source's ring is the same R : the ideals $(a^3, b^3, a^2b - 2)$ and $(a^3, b^3, a^2b + 2)$ coincide because each contains 4 and their third generators differ by 4. The Mathlib module [10] uses that source presentation. Its displayed group law differs from the scalar-affine law in this article; the comparison in Section 5 explains the scope of the citation.

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