

GRADED ABSOLUTE INTEGRAL CLOSURES AND PICARD GROUPS

ABSTRACT. We deduce from Bhatt’s graded Cohen–Macaulay and vanishing results that an integer-graded absolute integral closure of a weighted polynomial ring over $\mathbb{Z}_p$, modulo p , is free with basis degrees smaller than the sum of the weights. A direct p -adic expansion turns this degree bound into a uniform solution of the cusp relation modulo nilpotents. Consequently every ring R admits a faithfully flat algebra with seminormal reduction, and every invertible $R[T]$ -module becomes free after a faithfully flat extension of R , answering a question of Drinfeld.

1. THE PICARD STATEMENT

Vladimir Drinfeld asked whether every invertible $R[T]$ -module becomes trivial after a faithfully flat extension of the coefficient ring R ; see [BMV26, §3.1, Question following Corollary 3.4]. The p -nilpotent case is proved in [BMV26, Corollary 3.4]. We establish the following stronger statement.

Theorem 1.1. *For every commutative ring R , there is a faithfully flat R -algebra S whose reduction is seminormal. In particular,*

$$\mathrm{Pic}(S) \xrightarrow{\sim} \mathrm{Pic}(S[T]).$$

Consequently, every invertible $R[T]$ -module becomes free after a faithfully flat extension of R .

The trivializing extension cannot in general be chosen ind-fppf. Gabber’s counterexample, recorded by Rosengarten [Ros23, Remark 2.2.16] and recalled in [BMV26, §3.1], gives an invertible $R[T]$ -module that does not become free fppf locally on R . The same holds ind-fppf locally: a trivialization over a filtered colimit of fppf algebras descends to a finite stage, since invertible modules are finitely presented [Stacks, Tag 05LI].

The proof uses a bounded graded basis theorem over $\mathbb{Z}_p$, separately for each prime p . Only its reduction modulo p is needed; completion is handled by a single convergent expansion. The passage from seminormality to the Picard statement is the Traverso–Swan theorem [Tr70, Sw80], in the form [Wei13, Theorem I.3.11].

For a nonnegatively integer-graded characteristic-zero domain B , write $B^{+, \mathrm{gr}}$ for the union of its finite nonnegatively integer-graded domain extensions inside a fixed algebraic closure of $\mathrm{Frac}(B)$. Degree-zero extensions are allowed. These rings form a directed system: the Euler derivation extends uniquely to the algebraic closure, so the gradings agree on intersections and on composita. In particular,

$$(1.1) \quad B^{+, \mathrm{gr}} = H^{+, \mathrm{gr}} \quad \text{if } B \subset H \text{ is finite and graded.}$$

This follows by transitivity of finiteness and taking composita. All closures below use compatible fields. We set $M(a)_j = M_{a+j}$ and write $\widehat{M}$ for p -adic completion when p is fixed. Numbering for [Bh21] refers to arXiv:2008.08070v2.

2. BOUNDED GRADED BASES

Theorem 2.1 (Consequence of Bhatt’s graded results). *Fix a prime p , and set*

$$P = \mathbb{Z}_p[x_1, \dots, x_r], \quad r \geq 2, \quad \deg(x_i) = w_i \in \mathbb{Z}_{>0}, \quad C = P^{+, \mathrm{gr}}.$$

There is a nonempty homogeneous basis giving an isomorphism

$$(2.1) \quad C/pC \simeq \bigoplus_{\lambda \in \Lambda} (P/pP)(-s_\lambda), \quad 0 \leq s_\lambda < \sum_i w_i.$$

Any homogeneous lifts $u_\lambda \in C$ of this basis induce an isomorphism of completed P -modules

$$(2.2) \quad \left(\bigoplus_{\lambda \in \Lambda} P(-s_\lambda) \right)^\wedge \xrightarrow{\sim} \widehat{C}, \quad e_\lambda \mapsto u_\lambda.$$

In particular, reduction of this isomorphism gives compatible homogeneous bases modulo every power of p .

Proof. First suppose $w_i = 1$ for all i , and write the variables as y_i . We explain the two inputs from Bhatt. Put $V = \mathbb{Z}_p$, $X_0 = \mathbf{P}_V^{r-1}$, and let $X^+ \rightarrow X_0$ be an absolute integral closure, with $\mathcal{L}$ the pullback of $\mathcal{O}_{X_0}(1)$. Then

$$(2.3) \quad C = \bigoplus_{m \geq 0} H^0(X^+, \mathcal{L}^{\otimes m}).$$

For a finite graded domain extension $P \subset H$, the quotient $H/(y_1, \dots, y_r)H$ is a finite graded V -module, hence is bounded in degree. Its positive-degree ideal is therefore nilpotent, so the opens $D_+(y_i)$ cover $Y = \text{Proj } H$. On each chart y_i is an invertible element of degree one. Homogeneous P -module generators h_j of H give generators $h_j/y_i^{\deg h_j}$ of $(H[y_i^{-1}])_0$ over $(P[y_i^{-1}])_0$. Thus $\pi : Y \rightarrow X_0$ is finite and surjective. The same trivializations give $\widehat{H(m)} = \pi^* \mathcal{O}_{X_0}(m)$ and an injection of H into its section ring [Stacks, Tags 01M7 and 01MT]. Conversely, the section ring of a finite surjective integral cover of X_0 is a finite graded P -algebra, by the projection formula and [Stacks, Lemma 30.14.1(5), Tag 01YS]. These section rings are cofinal. Their embeddings use the algebraic closure of $K(X_0)$ inside the chosen closure of $\text{Frac}(P)$ and the generic trivialization y_1 . Normalizing the covers preserves finiteness, since V is excellent. Taking their filtered colimit gives (2.3); global sections commute with this limit, as can be checked on the finite standard affine cover. No generation-in-degree-one hypothesis on H is needed.

Let $\overline{V}$ be an absolute integral closure of V . Replacing X_0 by $X_0 \times_V \overline{V}$ does not change X^+ . In [Bh21, Lemma 6.6 and Notation 6.7], the ring in (2.3) is $R^{+, \text{gr}}$, the integer-degree summand of the rationally graded ring $R^{+, \text{GR}}$. By [Bh21, Theorem 6.24 and Corollary 6.25], $R^{+, \text{GR}}/p$ is Cohen–Macaulay over the regular ring P/pP . Here regularity of systems of parameters and their powers, followed by the Koszul direct-limit description of local cohomology, gives cohomological Cohen–Macaulayness. Thus [Bh21, Lemma 2.9] gives flatness. Projection onto integer degrees is P -linear and remains split modulo p , so C/pC is flat. Moreover, [Bh21, Proposition 6.12] gives $H^q(X_{p=0}^+, \mathcal{L}^{\otimes m}) = 0$ for $q > 0$ and $m \geq 0$. The affine-cover Čech comparison [Stacks, Tag 01XD], applied to the degree- m part of the localization complex on the y_i , identifies

$$H_{(y_1, \dots, y_r)}^r(C/pC)_m = H^{r-1}(X_{p=0}^+, \mathcal{L}^{\otimes m}).$$

Indeed, on $U_i \subset X^+$ above $D_+(y_i)$ one has $(C[y_i^{-1}])_m = \Gamma(U_i, \mathcal{L}^{\otimes m})$, and similarly on intersections. These opens are affine, so their sections commute with reduction modulo p . No such assertion for global sections is assumed. As $r \geq 2$, we have obtained

$$(2.4) \quad C/pC \text{ is } P/pP\text{-flat}, \quad (H_{(y_1, \dots, y_r)}^r(C/pC))_{\geq 0} = 0.$$

A nonnegatively graded flat module M over a positively graded polynomial ring Q over a field is graded free, without a finite-generation hypothesis. To see this, lift a homogeneous basis of M/Q_+M to obtain $F \rightarrow M$ with F graded free. Induction on degree makes this map surjective. If K is its kernel, flatness gives $K/Q_+K = 0$, so the same induction gives $K = 0$.

Apply this observation to C/pC . A free summand with generator in degree s contributes the nonzero class $1/(y_1 \cdots y_r)$ in degree $s - r$ to top local cohomology. The vanishing in (2.4) therefore forces $s < r$. The basis is nonempty because $P \subset C$ is integral and $C/pC \neq 0$ by lying over.

For arbitrary weights, adjoin y_i with $y_i^{w_i} = x_i$ and $\deg(y_i) = 1$. The ring $P' = V[y_1, \dots, y_r]$ is finite free over P , and (1.1) identifies their graded closures. Restrict the basis over P'/pP' to P/pP , using the monomials $y_1^{a_1} \cdots y_r^{a_r}$ with $0 \leq a_i < w_i$. The resulting basis degrees are at most

$$(r-1) + \sum_i (w_i - 1) = \sum_i w_i - 1,$$

which proves (2.1).

Finally, set $F = \bigoplus_\lambda P(-s_\lambda)$. The map $F \rightarrow C$ defined by the chosen lifts is an isomorphism after derived reduction modulo p , since both modules are p -torsion-free. Its derived completion is an isomorphism: the cofiber is derived complete and has zero reduction modulo p , so derived Nakayama applies [Stacks, Tag 0G1U]. By [Bh21, Lemma 5.7], applied to the one-element sequence (p) , these derived completions are the ordinary completions and remain p -torsion-free. This proves (2.2) and its reductions modulo p^n . $\square$

Corollary 2.2. *In Theorem 2.1, put $W = \sum_i w_i$ and $B = \widehat{C}$. Then B is flat over P , $B/pB = C/pC$, and*

$$(2.5) \quad \text{im}(C_{\geq W} \longrightarrow B) \subseteq (x_1, \dots, x_r)B.$$

Proof. Since C is p -torsion-free and C/pC is flat over P/pP , completed flatness follows from [Bh21, Lemmas 5.7 and 5.15]. The identity $B/pB = C/pC$ is the usual reduction property of derived completion [Stacks, §15.93, Tag 091N].

By (2.1), $C/(p, x_1, \dots, x_r)C$ has no components in degrees $\geq W$. Given $z_0 \in C_m$, $m \geq W$, choose successively

$$z_j = \sum_i x_i a_{i,j} + pz_{j+1}, \quad a_{i,j} \in C_{m-w_i}, \quad z_{j+1} \in C_m.$$

For every $N \geq 1$ the exact identity

$$z_0 = \sum_i x_i \left(\sum_{j=0}^{N-1} p^j a_{i,j} \right) + p^N z_N$$

holds in C . In B , the remainder tends to zero and each coefficient series converges. Thus $z_0 = \sum_i x_i (\sum_{j \geq 0} p^j a_{i,j})$, proving (2.5) without any closedness assertion for the parameter ideal. Only the degree bound modulo p is used. $\square$

3. THE UNIVERSAL CUSP

Set $A = \mathbb{Z}[a, b]$ and $\Delta = b^2 - a^3$.

Proposition 3.1. *There is a faithfully flat A -algebra E containing an element c such that*

$$(3.1) \quad (c^2 - a)^6 \in \Delta E, \quad (c^3 - b)^6 \in \Delta E.$$

Proof. For a prime p , put

$$A_p = \mathbb{Z}_p[a, b], \quad P_p = \mathbb{Z}_p[a, t], \quad D_p = \mathbb{Z}_p[a, b, t]/(b^2 - a^3 - t^6),$$

with $\deg(a, b, t) = (2, 3, 1)$. The ring D_p is a domain: $a^3 + t^6$ has odd pole order at $a = \infty$ in $\mathbb{Q}_p(t)(a)$ and hence is not a square. The decompositions $D_p = P_p \oplus bP_p = \bigoplus_{j=0}^5 A_p t^j$ show that both polynomial inclusions are finite free. Thus (1.1) gives

$$C_p := D_p^{+, \text{gr}} = A_p^{+, \text{gr}} = P_p^{+, \text{gr}}.$$

Let $B_p = \widehat{C_p}$. Applied to P_p , the degree estimate (2.5) gives $b \in (a, t)B_p$, since $\deg b = 3 = \deg a + \deg t$. Write $b = ac_p + td_p$. The same corollary makes B_p flat over P_p , so multiplication by a is injective on B_p/tB_p . In that quotient,

$$a^2(c_p^2 - a) = b^2 - a^3 = 0.$$

Hence $c_p^2 = a$ and $c_p^3 = ac_p = b$ modulo t , giving

$$(3.2) \quad c_p^2 - a, c_p^3 - b \in tB_p.$$

Applied to A_p , Corollary 2.2 shows that B_p is A_p -flat and that B_p/pB_p is a nonzero free A_p/pA_p -module. Since $\mathbb{Z}_p$ is flat over $\mathbb{Z}$ [Stacks, Tag 00MB], the base change $A_p = A \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is A -flat, and so is every B_p . Set

$$E = \prod_{p \text{ prime}} B_p.$$

Products of flat modules over the Noetherian ring A are flat [Stacks, Tag 05CZ]. Every maximal ideal of A has positive residue characteristic [Stacks, Tag 00GB]. If this characteristic is p , then $A_p/pA_p = A/pA$ and

$$B_p/\mathfrak{m}B_p = (B_p/pB_p) \otimes_{A/pA} (A/\mathfrak{m}) \neq 0$$

by freeness of B_p/pB_p . Projection to this factor gives $E/\mathfrak{m}E \neq 0$. Thus E is faithfully flat. Finally take $c = (c_p)_p$; equations (3.2) and $t^6 = \Delta$ give (3.1) componentwise. $\square$

4. SEMINORMAL REDUCTION AND PICARD GROUPS

Proof of Theorem 1.1. Starting with $R_0 = R$, form R_{i+1} by tensoring, over R_i , the base changes of $A \rightarrow E$ for every map $A \rightarrow R_i$, equivalently for every pair $(a, b) \in R_i^2$. Infinite tensor products mean filtered colimits of finite tensor products. All these maps are faithfully flat, as is $R \rightarrow S := \varinjlim_i R_i$: flatness passes to filtered colimits, and a finite equation putting 1 in the extension of a proper ideal would already occur at a finite stage.

A pair $\bar{a}, \bar{b} \in S_{\text{red}}$ with $\bar{b}^2 = \bar{a}^3$ lifts to $a, b \in S$ coming from some R_i . At the next stage, (3.1) supplies an element c . Choose $N \geq 1$ with $(b^2 - a^3)^N = 0$ in S ; then $(c^2 - a)^{6N} = (c^3 - b)^{6N} = 0$ in S . Thus the image of c is the required root, and S_{red} is seminormal by the square-and-cube criterion [Wei13, Section I.3].

The Traverso–Swan theorem [Sw80, Theorem 1], in the formulation [Wei13, Theorem I.3.11], now gives $\text{Pic}(S) \simeq \text{Pic}(S[T])$; this formulation applies to arbitrary rings in terms of their reductions. If L is an invertible $R[T]$ -module, its pullback to $S[T]$ is consequently extended from $M = (L/TL) \otimes_R S$. Choose a finite principal open cover $D(f_j)$ of $\text{Spec } S$ trivializing M . Then $R' = \prod_j S_{f_j}$ is faithfully flat over R , and $L \otimes_{R[T]} R'[T] \simeq R'[T]$. $\square$

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