

Derived invariance of the signature and hodge numbers of fourfolds

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Abstract

We prove that a Fourier–Mukai equivalence between smooth proper complex varieties of common even dimension preserves the topological signature. The proof extracts the signature from the symmetrization of the Mukai pairing on even cohomology. Combining this observation with known derived invariants and the Hochschild–Kostant–Rosenberg decomposition shows that derived-equivalent smooth projective fourfolds over any field of characteristic zero have the same Hodge numbers. This argument and text was produced by ChatGPT 5.6 Sol.

1 Introduction

Let X and Y be smooth projective varieties. Orlov conjectured that an equivalence of bounded derived categories should induce an isomorphism of rational Chow motives [6]. Over a field of characteristic zero one consequently expects all Hodge numbers to be derived invariants. The claim is false in positive characteristic [2].

The main observation here is that the topological signature is already encoded by the Euler pairing.

Theorem 1.1. *Let X and Y be smooth proper complex varieties of the same even dimension. If a Fourier–Mukai kernel induces an equivalence*

$$\mathrm{Perf}(X) \simeq \mathrm{Perf}(Y),$$

then their topological signatures agree:

$$\sigma(X) = \sigma(Y).$$

Projectivity is not used in Theorem 1.1; properness supplies compact oriented manifolds and proper cohomological pushforwards. The fourfold consequence is the following.

Theorem 1.2. *Let k be a field of characteristic zero and let X and Y be smooth projective fourfolds over k . If there is a k -linear Fourier–Mukai equivalence $\mathrm{Perf}(X) \simeq \mathrm{Perf}(Y)$, then*

$$h^{p,q}(X) = h^{p,q}(Y) \quad (0 \leq p, q \leq 4).$$

2 The cohomological Mukai pairing

Let X have complex dimension n . For $\alpha = \sum_{j=0}^n \alpha_{2j} \in H^{\text{ev}}(X; \mathbf{Q})$, put

$$\alpha^\vee = \sum_{j=0}^n (-1)^j \alpha_{2j}.$$

For $A \in \text{Perf}(X)$ its Mukai vector is $v_X(A) = \text{ch}(A)\sqrt{\text{td}(X)}$, and the cohomological Mukai pairing is

$$P_X(\alpha, \beta) = \int_X e^{c_1(X)/2} \alpha^\vee \beta. \quad (2.1)$$

This is the cohomological realization of the generalized Mukai pairing under the modified HKR isomorphism; the relevant reference is Căldăraru [3], not the later abstract categorical treatment. Riemann–Roch identifies it with the Euler pairing:

$$P_X(v_X(A), v_X(B)) = \sum_i (-1)^i \dim_{\mathbf{C}} \text{Ext}_X^i(A, B).$$

For a kernel $E \in \text{Perf}(X \times Y)$ define

$$T_E(\alpha) = p_{Y*} \left(p_X^* \alpha \, \text{ch}(E) \sqrt{p_X^* \text{td}(X) p_Y^* \text{td}(Y)} \right).$$

Grothendieck–Riemann–Roch shows that these transforms compose according to convolution and that the transform for the right-adjoint kernel is adjoint with respect to (2.1); see also [4]. The nonprojective Riemann–Roch input needed for smooth proper varieties is available in [5]. Hence:

Proposition 2.1. *If E induces a Fourier–Mukai equivalence between smooth proper complex varieties of the same dimension, then T_E is an isometry for the cohomological Mukai pairings.*

3 Recovering the signature

Lemma 3.1. *Let X be smooth proper of complex dimension $2d$. On $H^{\text{ev}}(X; \mathbf{R})$ set*

$$S_X(\alpha, \beta) = P_X(\alpha, \beta) + P_X(\beta, \alpha).$$

Then S_X is nondegenerate and

$$\text{sig}(S_X) = (-1)^d \sigma(X).$$

Proof. Introduce the symmetric form

$$Q_X(\alpha, \beta) = \int_X \alpha^\vee \beta.$$

Since $(-)^\vee$ is the identity on the top-degree cohomology of a $2d$ -fold, graded commutativity gives

$$S_X(\alpha, \beta) = 2 \int_X \cosh\left(\frac{c_1(X)}{2}\right) \alpha^\vee \beta. \quad (3.1)$$

The class $\cosh(c_1(X)/2)$ has a unique square root B_X with constant term one. Its components have degrees divisible by four, so $B_X^\vee = B_X$. Thus

$$S_X(\alpha, \beta) = 2Q_X(B_X \alpha, B_X \beta).$$

Multiplication by B_X is a real automorphism, and the positive scalar two does not change signature.

For $j < d$, Poincaré duality makes the restriction of Q_X to $H^{2j}(X; \mathbf{R}) \oplus H^{4d-2j}(X; \mathbf{R})$ hyperbolic. These summands contribute zero signature. On $H^{2d}(X; \mathbf{R})$ the involution contributes $(-1)^d$, so Q_X restricts to $(-1)^d$ times the middle intersection form. The displayed formula follows. $\square$

Proof of Theorem 1.1. Proposition 2.1 identifies S_X and S_Y . Their signatures agree, and Lemma 3.1 recovers the two topological signatures with the same dimension-dependent sign. $\square$

4 The fourfold Hodge diamond

We first work over $\mathbf{C}$. Hodge symmetry and Serre duality give

$$h^{p,q} = h^{q,p} = h^{4-p,4-q}.$$

Abuaf's theorem implies derived invariance of $H^*(X, \mathcal{O}_X)$ in dimension at most four [1]; hence all boundary Hodge numbers agree.

Derived invariance of Hochschild homology and HKR give, for every i ,

$$\dim_{\mathbf{C}} \mathrm{HH}_i(X) = \sum_j h^{i+j,j}(X). \quad (4.1)$$

For $i \neq 0$, the diagonal sums, the boundary values, and the two symmetries determine all off-diagonal entries. On the main diagonal only $h^{1,1}$, $h^{2,2}$ and $h^{3,3} = h^{1,1}$ remain. The case $i = 0$ of (4.1) shows that

$$2h^{1,1}(X) + h^{2,2}(X) = 2h^{1,1}(Y) + h^{2,2}(Y). \quad (4.2)$$

The Hodge-theoretic signature formula [7] is

$$\sigma(X) = \sum_{p,q} (-1)^q h^{p,q}(X). \quad (4.3)$$

After the already known entries cancel, Theorem 1.1 gives the independent equation

$$h^{2,2}(X) - 2h^{1,1}(X) = h^{2,2}(Y) - 2h^{1,1}(Y). \quad (4.4)$$

Equations (4.2) and (4.4) determine both remaining numbers. This proves Theorem 1.2 over $\mathbf{C}$.

For an arbitrary characteristic-zero field, the varieties, mutually inverse kernels, and convolution isomorphisms descend to a finitely generated subfield $k_0 \subset k$. Choose an embedding $k_0 \hookrightarrow \mathbf{C}$. The complex result applies after base change, while coherent cohomology commutes with field extension. The dimensions over k , k_0 , and $\mathbf{C}$ therefore agree, proving the general case.

Contributors, affiliations, and support

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