

INCLUSIONS BETWEEN p -BOUNDED CRYSTALLINE LOCI IN DIMENSION TWO

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ABSTRACT. Let p be an odd prime and $K/\mathbf{Q}_p$ a finite unramified extension of degree $f > 1$. Let $\mathcal{Z}(\underline{r})$ be the reduced special fiber of the Emerton–Gee stack of two-dimensional crystalline representations of $\mathrm{Gal}(\overline{K}/K)$ of Hodge type $\underline{r}$. We study the collection of stacks $\mathcal{Z}(\underline{r})$ as $\underline{r}$ varies over p -bounded Hodge types, as a set partially ordered under inclusion. We prove that aside from two degenerate cases, simple inclusions can be classified in terms of three operations on Hodge types, two of which have standard automorphic interpretations. We also prove, with one exception, that inclusions can be detected at the level of an inclusion of closed points (equivalently, semisimple mod p Galois representations). GPT-5.5 Pro was used extensively in the course of this work.

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1. INTRODUCTION

1.1. Main results and reductions. Let p be a prime number and $K/\mathbf{Q}_p$ a finite extension with residue field k . Emerton and Gee [EG] have constructed a moduli stack $\mathcal{X}_2$ of rank 2 (φ, Γ) -modules over K that can be conceived of as a moduli stack of rank 2 representations of G_K , the absolute Galois group of K . Write $\mathcal{X}_{2,\mathrm{red}}$ for the underlying reduced substack of $\mathcal{X}_2$; it is an algebraic stack over $\overline{\mathbf{F}}_p$ of dimension $[K : \mathbf{Q}_p]$. The stack $\mathcal{X}_2$ has deep connections to the representation theory of $\mathrm{GL}_2(K)$. For example, by [EG, Thm. 1.2.1] the irreducible components of $\mathcal{X}_{2,\mathrm{red}}$ are in bijection with *Serre weights*, the isomorphism classes of irreducible $\overline{\mathbf{F}}_p$ -representations of $\mathrm{GL}_2(k)$.

Assume henceforth that p is odd and $K/\mathbf{Q}_p$ is unramified of degree $f > 1$, and set $q = p^f$.

This note is a complement to the forthcoming paper [KLS]. Its purpose is to describe, as a set partially ordered under inclusion, a certain collection of closed substacks of $\mathcal{X}_{2,\mathrm{red}}$ satisfying a natural p -adic Hodge-theoretic condition. To explain precisely which closed substacks we will consider, we first recall what we mean by a

Hodge type. Since $K/\mathbf{Q}_p$ is unramified we can (and do) identify the embeddings $K \hookrightarrow \overline{\mathbf{Q}}_p$ with embeddings $k \hookrightarrow \overline{\mathbf{F}}_p$. Fix an embedding $\kappa_0 : k \hookrightarrow \overline{\mathbf{F}}_p$ and recursively label the remaining embeddings by elements of $\mathbf{Z}/f\mathbf{Z}$ by taking $\kappa_{i+1}^p = \kappa_i$.

Definition 1.1. A *Hodge type* of rank 2 is a tuple of integers $\underline{r} = \{r_{i,j}\}_{\kappa_i : K \hookrightarrow \overline{\mathbf{Q}}_p, 1 \leq j \leq 2}$ with $r_{i,1} \geq r_{i,2}$ for all i . We say that the Hodge type $\underline{r}$ is

- *p*-bounded if $r_{i,1} - r_{i,2} \leq p$ for all i ,
- *regular* if $r_{i,1} - r_{i,2} > 0$ for all i .

If $\underline{r}$ is a Hodge type and $\lambda \in \mathbf{Z}^{\mathbf{Z}/f\mathbf{Z}}$, we let $\underline{r} + \lambda$ denote the translation of $\underline{r}$ by λ , i.e., the Hodge type $(r_{i,1} + \lambda_i, r_{i,2} + \lambda_i)$.

Write ur, BT, cr, and St for the Hodge types with $(r_{i,1}, r_{i,2}) = (r, 0)$ for all i , for $r = 0, 1, p-1, p$ respectively. These stand for unramified, Barsotti–Tate, critical, and Steinberg. We say that a Hodge type is *scalar*, *Barsotti–Tate*, *critical*, or *Steinberg* if it is the translation of ur, BT, cr, or St by some λ .

For each Hodge type $\underline{r}$ of rank 2, there is a closed substack $\mathcal{X}_2^{\text{crys}, \underline{r}} \subset \mathcal{X}_2$ corresponding to crystalline representations with κ_i -labeled Hodge–Tate weights $\{r_{i,1}, r_{i,2}\}$ for each i . Its underlying reduced substack $\mathcal{Z}(\underline{r}) := \mathcal{X}_{2,\text{red}}^{\text{crys}, \underline{r}}$ is equidimensional of codimension $\#\{i : r_{i,1} = r_{i,2}\}$. If $\Lambda \subset \mathbf{Z}^{\mathbf{Z}/f\mathbf{Z}}$ is the sublattice consisting of tuples (λ_i) such that $\sum_i p^{-i} \lambda_i \equiv 0 \pmod{q-1}$, then it is easy to see that $\mathcal{Z}(\underline{r}) = \mathcal{Z}(\underline{r} + \lambda)$ for any $\underline{r}$ and $\lambda \in \Lambda$. Indeed, on Galois representations this amounts to twisting by a crystalline character whose Hodge–Tate weights are $(\lambda_i)_i$ and whose reduction mod p is trivial.

If $\underline{r}$ is regular, p -bounded, and non-Steinberg, then $\mathcal{Z}(\underline{r})$ is a single irreducible component of $\mathcal{X}_{2,\text{red}}$. In our normalizations (following the normalizations of [CEGS2, BBH⁺, KLS]) it is the component $\mathcal{X}(\sigma(\underline{r}))$ corresponding to the Serre weight

$$\sigma(\underline{r}) := \otimes_{i=0}^{f-1} (\det^{1-r_{i,1}} \text{Sym}^{r_{i,1}-r_{i,2}-1} k^2) \otimes_{k, \kappa_i} \overline{\mathbf{F}}_p.$$

The map $\underline{r} \mapsto \sigma(\underline{r})$ is many-to-one: indeed, its fibers are equivalence classes under translation by Λ .

If instead $\underline{r}$ is Steinberg, it is proved in [EG, CEGS2] that $\mathcal{Z}(\underline{r})$ is the union of two irreducible components. To be precise, if $\underline{r} = \text{St} + \lambda$ then one of those components is $\mathcal{Z}(\text{BT} + \lambda)$, while by [EG, Lem. 8.6.4] all the $\overline{\mathbf{F}}_p$ -points of the other component are twists of an extension of the trivial character by the cyclotomic character. This latter component is $\mathcal{X}(\sigma(\underline{r}))$ where $\sigma(\underline{r})$ is the Steinberg Serre weight $\otimes_{i=0}^{f-1} (\det^{1-r_{i,1}} \text{Sym}^{p-1} k^2) \otimes_{k, \kappa_i} \overline{\mathbf{F}}_p$, so that $\mathcal{Z}(\text{St} + \lambda) = \mathcal{Z}(\text{BT} + \lambda) \cup \mathcal{X}(\sigma(\underline{r}))$.

As $\underline{r}$ ranges over all p -bounded and regular Hodge types, $\mathcal{X}(\sigma(\underline{r}))$ ranges over the irreducible components of $\mathcal{X}_{2,\text{red}}$, while $\mathcal{Z}(\underline{r})$ ranges over the *Breuil–Mézard cycles* for $\text{GL}_2(K)$ as in [EG, Conj. 1.7.2] and the discussion following it. This means that $\mathcal{Z}(\underline{r})$ is supported at a representation $\bar{\rho} : G_K \rightarrow \text{GL}_2(\overline{\mathbf{F}}_p)$ if and only if $\sigma(\underline{r})$ is a member of the set of Serre weights associated to $\bar{\rho}$ in the weight part of Serre’s conjecture.

In this article we will be concerned with the closed substacks $\mathcal{Z}(\underline{r})$ as $\underline{r}$ ranges over all p -bounded (but not necessarily regular) Hodge types. Irregular Hodge types can be thought of as corresponding to “phantom Serre weights” in which some or all of the symmetric powers are allowed to be -1 . The loci $\mathcal{Z}(\underline{r})$ for irregular $\underline{r}$ play a role in the local study of Hilbert modular forms of partial weight one, but as these stacks have positive codimension in $\mathcal{X}_{2,\text{red}}$, they are somewhat more difficult

to study than the irreducible components. However, it is proved in [BBH⁺] that if $\underline{r}$ is p -bounded and non-Steinberg then $\mathcal{Z}(\underline{r})$ is irreducible.

Definition 1.2. Let Z be the set of closed substacks $\mathcal{Z}(\underline{r})$ as $\underline{r}$ ranges over all p -bounded Hodge types.

The set Z is partially ordered under inclusion (i.e. under closed immersion), and our specific goal in this article is to describe this structure. To state our main results, we need to introduce several operations on Hodge types.

Definition 1.3. Let $f \geq 2$. For each $j \in \mathbf{Z}/f\mathbf{Z}$, we define operators μ_j , θ_j , and ν_j on p -bounded Hodge types $\underline{r}$ by setting:

$$\mu_j(\underline{r})_i := \begin{cases} (r_{i,1} - 1, r_{i,2}) & \text{if } i = j, \\ (r_{i,1} + p, r_{i,2}) & \text{if } i = j + 1, \\ (r_{i,1}, r_{i,2}) & \text{otherwise.} \end{cases}$$

$$\theta_j(\underline{r})_i := \begin{cases} (r_{i,1}, r_{i,2} - 1) & \text{if } i = j, \\ (r_{i,1} + p, r_{i,2}) & \text{if } i = j + 1, \\ (r_{i,1}, r_{i,2}) & \text{otherwise.} \end{cases}$$

$$\nu_j(\underline{r})_i := \begin{cases} (r_{i,1}, r_{i,2} - 1) & \text{if } i = j, \\ (r_{i,2} + p, r_{i,1}) & \text{if } i = j + 1, \\ (r_{i,1}, r_{i,2}) & \text{otherwise.} \end{cases}$$

The difference between θ_j and ν_j lies in the subscripts at index $i = j + 1$. We say that $\underline{r}$ is irregular at j if $r_{j,1} = r_{j,2}$. The operator $f_j \in \{\mu_j, \theta_j, \nu_j\}$ is *valid* (for $\underline{r}$) if

- $f_j = \mu_j$, $\underline{r}$ is irregular at $j + 1$, and $r_{j,1} - r_{j,2} > 0$;
- $f_j = \theta_j$, $\underline{r}$ is irregular at $j + 1$, and $r_{j,1} - r_{j,2} < p$;
- $f_j = \nu_j$ and $\underline{r}$ is irregular at j .

If f_j is valid for $\underline{r}$ then evidently $f_j(\underline{r})$ is again a p -bounded Hodge type.

The definitions of the operators μ_j and θ_j are motivated by the theory of mod p automorphic forms: the weights of mod p automorphic forms transform by μ_j under partial Hasse invariants, and by θ_j under partial theta operators. The operator ν_j does not have as straightforward an automorphic explanation, but can be related to restriction to the vanishing locus of a partial Hasse invariant; see [Yan] or [KLS] for much more.

Remark 1.4. We caution the reader that the indexing of the operators μ_j and θ_j differs by 1 from the indexing in [BBH⁺]. We have made this change, both here and in [KLS], so that μ_j corresponds to multiplication by the partial Hasse invariant Ha_j rather than Ha_{j-1} , and similarly for θ_j and the partial theta operator Θ_j rather than Θ_{j-1} . There is an added advantage that μ_j , θ_j , and ν_j all affect Hodge–Tate weights in the same pair of embeddings; the disadvantage is that μ_j and θ_j affect an irregular embedding at index $j + 1$ rather than j .

The following proposition is [BBH⁺, Cor. 5.3].

Proposition 1.5. *If $\underline{r}$ is a p -bounded Hodge type and the operation $f_j \in \{\mu_j, \theta_j, \nu_j\}$ is valid, then $\mathcal{Z}(\underline{r}) \subseteq \mathcal{Z}(f_j(\underline{r}))$.*

To simplify notation, write $u_i := r_{i,1} - r_{i,2}$. If $(u_j, u_{j+1}) = (1, 0)$, it is not hard to see that $\nu_j(\mu_j(r)) \in \underline{r} + \Lambda$. Similarly if $(u_j, u_{j+1}) = (0, p)$ then $\mu_j(\nu_j(r)) \in \underline{r} + \Lambda$. Thus if $(u_j, u_{j+1}) = (1, 0)$ we have $\mathcal{Z}(\mu_j(r)) = \mathcal{Z}(\underline{r})$, while if $(u_j, u_{j+1}) = (0, p)$ we have $\mathcal{Z}(\nu_j(r)) = \mathcal{Z}(\underline{r})$. In these situations we say that the operations μ_j and ν_j are invertible.

Remark 1.6. One could alternatively define a set $X = \{\mathcal{X}(\underline{r}) : p\text{-bounded } \underline{r}\}$ by setting $\mathcal{X}(\underline{r}) = \mathcal{Z}(\underline{r})$ if $\underline{r}$ is non-Steinberg (equivalently, if $\mathcal{Z}(\underline{r})$ is irreducible) and setting $\mathcal{X}(\underline{r}) = \mathcal{X}(\sigma(\underline{r}))$ in the Steinberg case. This would seem to have the advantage that all the stacks in X are irreducible.

The reason we choose to work with Z rather than X is that Proposition 1.5 is no longer quite correct if $\mathcal{Z}$ is replaced with $\mathcal{X}$. If $\underline{r}$ is the Hodge type such that $\theta_j(\underline{r}) = \text{St}$, then $\mathcal{X}(\underline{r}) \not\subseteq \mathcal{X}(\text{St})$, as can be seen by checking that there are irreducible representations $\bar{\rho}$ in the support of $\mathcal{X}(\underline{r})$ but none in that of $\mathcal{X}(\text{St})$. In fact if $\theta_j(\underline{r}) = \text{St}$ then we have

$$(1.7) \quad (\nu_{j+f-1} \circ \cdots \circ \nu_{j+2} \circ \nu_{j+1})(\underline{r}) \in \text{BT} + \Lambda,$$

where the composition has $f - 1$ factors, so that indeed $\mathcal{Z}(\underline{r}) \subseteq \mathcal{Z}(\text{St})$, but with image in the component $\mathcal{X}(\text{BT})$ rather than $\mathcal{X}(\text{St})$.

In any case, since X is very nearly the same as Z , it is possible to reformulate all of our results in terms of X rather than Z if one prefers.

Definition 1.8. Let $\sim$ denote the equivalence relation on p -bounded Hodge types generated by $\underline{r} \sim \mu_j(\underline{r})$ for valid invertible μ_j , by $\underline{r} \sim \nu_j(\underline{r})$ for valid invertible ν_j , and by translation by Λ .

Our main results are as follows.

Theorem 1.9. *We have $\mathcal{Z}(\underline{r}) = \mathcal{Z}(\underline{r}')$ if and only if $\underline{r} \sim \underline{r}'$.*

If $\mathcal{Z} \subseteq \mathcal{Z}'$ is an inclusion of elements of Z , let us say that the inclusion is *simple* if $\mathcal{Z} \subsetneq \mathcal{Z}'$ and furthermore there does not exist $\mathcal{Y} \in Z$ with $\mathcal{Z} \subsetneq \mathcal{Y} \subsetneq \mathcal{Z}'$.

Theorem 1.10. *Suppose $\mathcal{Z} \subsetneq \mathcal{Z}'$ is a simple inclusion in Z . Then either:*

- *there exists a p -bounded Hodge type $\underline{r}$ and a valid operation $f_j \in \{\mu_j, \theta_j, \nu_j\}$ such that $\mathcal{Z} = \mathcal{Z}(\underline{r})$ and $\mathcal{Z}' = \mathcal{Z}(f_j(\underline{r}))$,*
- *we have $\mathcal{Z} = \mathcal{Z}(\text{ur} + \lambda)$ and $\mathcal{Z}' = \mathcal{Z}(\text{cr} + \lambda)$ for some λ , or*
- *we have $\mathcal{Z} = \mathcal{Z}(\text{BT} + \lambda)$ and $\mathcal{Z}' = \mathcal{Z}(\text{St} + \lambda)$ for some λ .*

Note that in the second case there is indeed an inclusion $\mathcal{Z}(\text{ur} + \lambda) \subseteq \mathcal{Z}(\text{cr} + \lambda)$: unramified two-dimensional representations of G_K have crystalline lifts with all labeled Hodge–Tate weights equal to $(p - 1, 0)$.

A consequence of Theorem 1.10, and the reason we became interested in these statements in the first place, is that if one wants to use induction on (co)dimension to prove theorems about the stacks $\mathcal{Z}(\underline{r})$, as we do in [KLS], then the only “moves” that one has available in general are the operations μ_j , θ_j , and ν_j . Philosophically this suggests that there is indeed something fundamental about the operation ν_j , alongside μ_j and θ_j with their evident automorphic interpretations.

As a consequence of Theorems 1.9 and 1.10 we have the following:

Corollary 1.11. *If $\mathcal{Z} \subseteq \mathcal{Z}'$ is an inclusion of elements of Z , then either*

- *we have $\mathcal{Z} = \mathcal{Z}(\text{ur} + \lambda)$ and $\mathcal{Z}' = \mathcal{Z}(\text{cr} + \lambda)$ for some λ , or*

- there is a chain

$$\mathcal{Z} = \mathcal{Z}_0 \subsetneq \cdots \subsetneq \mathcal{Z}_n = \mathcal{Z}'$$

of simple inclusions for some $n \geq 0$, and there exist p -bounded Hodge types $\underline{r}_0, \dots, \underline{r}_n$ such that $\mathcal{Z}_m = \mathcal{Z}(\underline{r}_m)$ for all $0 \leq m \leq n$, and for every $1 \leq m \leq n$ we have either:

- $\mathcal{Z}_m = \mathcal{Z}(f^{(m)}(\underline{r}_{m-1}))$ for some valid $f^{(m)}$, and $f^{(m)}(\underline{r}_{m-1}) \sim \underline{r}_m$, or
- $m = n$ and the inclusion $\mathcal{Z}_{n-1} \subsetneq \mathcal{Z}_n$ is some $\mathcal{Z}(\text{BT} + \lambda) \subsetneq \mathcal{Z}(\text{St} + \lambda)$.

Proof. The existence of a chain $\mathcal{Z} = \mathcal{Z}_0 \subsetneq \cdots \subsetneq \mathcal{Z}_n = \mathcal{Z}'$ of simple inclusions follows for dimension reasons. If one of these inclusions has the form $\mathcal{Z}(\text{ur} + \lambda) \subsetneq \mathcal{Z}(\text{cr} + \lambda)$, then $n = 1$: for dimension reasons there are no proper inclusions into $\mathcal{Z}(\text{ur} + \lambda)$ or out of $\mathcal{Z}(\text{cr} + \lambda)$. Similarly any inclusion in the chain that is of the form $\mathcal{Z}(\text{BT} + \lambda) \subsetneq \mathcal{Z}(\text{St} + \lambda)$ must be last in the chain. Then Theorems 1.9 and 1.10 imply that the chain has the asserted form. $\square$

It remains to describe precisely which non-invertible valid operations give simple inclusions; or equivalently, which ones do not. Note that on dimension grounds, the latter would require either reducing the number of irregular embeddings by at least two, or increasing the number of irreducible components. This can occur only if $f_j(\underline{r})$ is Steinberg, or if $(u_j, u_{j+1}) = (0, 0)$ and $f_j = \theta_j$ or ν_j . (Note that in the latter situation θ_j and ν_j are equal.) The actual behavior is as follows.

Theorem 1.12. *Each of the inclusions listed in Theorem 1.10 is simple except for the ones of the form $\mathcal{Z}(\underline{r}) \subseteq \mathcal{Z}(f_j(\underline{r}))$ with either f_j invertible (in which case the inclusion is an equality) or else of the form*

- $f_j \in \{\theta_j, \nu_j\}$, $(u_j, u_{j+1}) = (0, 0)$, and either $u_{j-1} = 1$ or $u_{j+2} = p$; or
- $f_j = \theta_j$ and $\theta_j(\underline{r})$ is Steinberg,

in which case the inclusion is proper but not simple.

In the final case, if $\theta_j(\underline{r}) = \text{St} + \lambda$ we have already seen that $\mathcal{Z}(\underline{r}) \subsetneq \mathcal{Z}(\text{BT} + \lambda) \subsetneq \mathcal{Z}(\text{St} + \lambda)$. In the first case, if for example $u_{j-1} = 1$, then $\theta_j(\underline{r}) \sim \nu_{j-1}(\mu_j(\mu_{j-1}(\underline{r})))$, and $\mathcal{Z}(\mu_j(\mu_{j-1}(\underline{r})))$ is intermediate between $\mathcal{Z}(\underline{r})$ and $\mathcal{Z}(\theta_j(\underline{r}))$. The other cases of non-simplicity are similar.

It is possible to reduce each of Theorems 1.9, 1.10, and 1.12 to statements that are essentially combinatorial, as we now explain.

Definition 1.13. If $\underline{r}$ is a Hodge type, define $\mathcal{Z}_{\text{ss}}(\underline{r})$ to be the set of semisimple representations $\bar{\rho} : G_K \rightarrow \text{GL}_2(\overline{\mathbf{F}}_p)$ on which $\mathcal{Z}(\underline{r})$ is supported, i.e., the set of semisimple $\bar{\rho}$ having a crystalline lift of Hodge type $\underline{r}$; equivalently, $\mathcal{Z}_{\text{ss}}(\underline{r})$ is the collection of closed points of $\mathcal{Z}(\underline{r})$.

Write $\mathcal{Z}_{\text{ss}}$ for the set $\{\mathcal{Z}_{\text{ss}}(\underline{r}) : p\text{-bounded } \underline{r}\}$.

Since the semisimple representations at which $\mathcal{X}(\text{St})$ is supported also lie in the support of $\mathcal{X}(\text{BT})$, we have $\mathcal{Z}_{\text{ss}}(\text{BT} + \lambda) = \mathcal{Z}_{\text{ss}}(\text{St} + \lambda)$. The main combinatorial claim is that versions of Theorems 1.9, 1.10, and 1.12 hold with $\mathcal{Z}$ replaced by $\mathcal{Z}_{\text{ss}}$, suitably modified to account for the equality $\mathcal{Z}_{\text{ss}}(\text{BT} + \lambda) = \mathcal{Z}_{\text{ss}}(\text{St} + \lambda)$.

Theorem 1.14. *We have the following.*

- (i) *We have $\mathcal{Z}_{\text{ss}}(\underline{r}) = \mathcal{Z}_{\text{ss}}(\underline{r}')$ if and only if either $\underline{r} \sim \underline{r}'$ or else there exists λ such that either $(\underline{r} \sim \text{BT} + \lambda \text{ and } \underline{r}' \sim \text{St} + \lambda)$ or $(\underline{r}' \sim \text{BT} + \lambda \text{ and } \underline{r} \sim \text{St} + \lambda)$.*

- (ii) Suppose $\mathcal{Z}_{ss} \subsetneq \mathcal{Z}'_{ss}$ is a simple inclusion in Z_{ss} . Then either:
- there exists a p -bounded Hodge type $\underline{r}$ and a valid operation $f_j \in \{\mu_j, \theta_j, \nu_j\}$ such that $\mathcal{Z}_{ss} = \mathcal{Z}_{ss}(\underline{r})$ and $\mathcal{Z}'_{ss} = \mathcal{Z}_{ss}(f_j(\underline{r}))$, or
 - we have $\mathcal{Z}_{ss} = \mathcal{Z}_{ss}(\underline{ur} + \lambda)$ and $\mathcal{Z}'_{ss} = \mathcal{Z}_{ss}(\underline{cr} + \lambda)$ for some λ .
- (iii) Each of the inclusions listed in (ii) is simple except for the ones of the form $\mathcal{Z}_{ss}(\underline{r}) \subsetneq \mathcal{Z}_{ss}(f_j(\underline{r}))$ with $f_j \in \{\theta_j, \nu_j\}$, $(u_j, u_{j+1}) = (0, 0)$ and either $u_{j-1} = 1$ or $u_{j+2} = p$.

For p -bounded Hodge types $\underline{r}$, the results of [GLS] give an explicit description of $\mathcal{Z}_{ss}(\underline{r})$, thus reducing Theorem 1.14 to an explicit but very complicated combinatorial problem whose proof will occupy the body of the paper. For now, however, we prove the following.

Proposition 1.15. *Assume parts (i) and (ii) of Theorem 1.14. Then we have:*

- (i) *If $\underline{r}$ and $\underline{r}'$ are p -bounded Hodge types, then $\mathcal{Z}(\underline{r}) \subseteq \mathcal{Z}(\underline{r}')$ if and only if $\mathcal{Z}_{ss}(\underline{r}) \subseteq \mathcal{Z}_{ss}(\underline{r}')$, except when there exists λ such that $\underline{r} \sim \text{St} + \lambda$ and $\underline{r}' \sim \text{BT} + \lambda$.*
- (ii) *Theorems 1.9 and 1.10 hold.*
- (iii) *Further assuming Theorem 1.14(iii), Theorem 1.12 holds.*

Remark 1.16. We emphasize the following interpretation of Proposition 1.15(i): for p -bounded Hodge types $\underline{r}, \underline{r}'$, inclusions between the stacks $\mathcal{Z}(\underline{r})$ and $\mathcal{Z}(\underline{r}')$ may be detected at the level of an inclusion between their closed points, with the exception that $\mathcal{Z}_{ss}(\text{St} + \lambda) = \mathcal{Z}_{ss}(\text{BT} + \lambda)$ while $\mathcal{Z}(\text{St} + \lambda) \not\subseteq \mathcal{Z}(\text{BT} + \lambda)$.

Proof. The only-if direction of (i) is immediate. For the reverse, assume that $\mathcal{Z}_{ss}(\underline{r}) \subseteq \mathcal{Z}_{ss}(\underline{r}')$, and that there does not exist λ such that $\underline{r} \sim \text{St} + \lambda$ and $\underline{r}' \sim \text{BT} + \lambda$.

If $\mathcal{Z}_{ss}(\underline{r}) = \mathcal{Z}_{ss}(\underline{r}')$, then Theorem 1.14(i) implies that either $\underline{r} \sim \underline{r}'$ or else $\underline{r} \sim \text{BT} + \lambda$ and $\underline{r}' \sim \text{St} + \lambda$ for some λ . In either case $\mathcal{Z}(\underline{r}) \subseteq \mathcal{Z}(\underline{r}')$.

Otherwise, since the set Z_{ss} is finite it is possible to write

$$\mathcal{Z}_{ss}(\underline{r}) = \mathcal{Z}_{ss,0} \subsetneq \mathcal{Z}_{ss,1} \subsetneq \cdots \subsetneq \mathcal{Z}_{ss,n} = \mathcal{Z}_{ss}(\underline{r}')$$

for some $n > 0$, with each $\mathcal{Z}_{ss,m} \subsetneq \mathcal{Z}_{ss,m+1}$ a simple inclusion in Z_{ss} . By Theorem 1.14(ii), we can write this inclusion as $\mathcal{Z}_{ss}(\underline{r}_m) \subsetneq \mathcal{Z}_{ss}(\underline{r}'_m)$ for Hodge types $\underline{r}_m, \underline{r}'_m$ satisfying one of the two listed possibilities. In each of those cases we know that $\mathcal{Z}(\underline{r}_m) \subseteq \mathcal{Z}(\underline{r}'_m)$.

Regular Hodge types give maximal elements of Z_{ss} (e.g. by Theorem 1.14(ii), because $\underline{r}$ is never regular in the first bullet point, and similarly $\underline{ur} + \lambda$ is not regular in the second bullet point). Thus none of $\underline{r}$, $\underline{r}_m$, or $\underline{r}'_m$ is either Barsotti–Tate or Steinberg except possibly $\underline{r}'_{n-1}$. By Theorem 1.14(i), $\mathcal{Z}_{ss}(\underline{r}'_m) = \mathcal{Z}_{ss}(\underline{r}_{m+1})$ implies that $\underline{r}'_m \sim \underline{r}_{m+1}$, and therefore $\mathcal{Z}(\underline{r}'_m) = \mathcal{Z}(\underline{r}_{m+1})$. Similarly $\mathcal{Z}(\underline{r}) = \mathcal{Z}(\underline{r}_0)$. If $\underline{r}'_{n-1}$ (hence also $\underline{r}'$) is either Barsotti–Tate or Steinberg, then by changing $\underline{r}_{n-1}$ it is possible to arrange that $\underline{r}'_{n-1} \sim \underline{r}'$: this is because if $\theta_j(\underline{s}) = \text{St} + \lambda$, then there exists $\underline{s}' \sim \underline{s}$ such that $\nu_{j+f-1}(\underline{s}') \sim \text{BT} + \lambda$ (namely $\underline{s}' \sim (\nu_{j+f-2} \circ \cdots \circ \nu_{j+2} \circ \nu_{j+1})(\underline{s})$, as in (1.7)). We obtain

$$\mathcal{Z}(\underline{r}) = \mathcal{Z}(\underline{r}_0) \subseteq \mathcal{Z}(\underline{r}'_0) = \mathcal{Z}(\underline{r}_1) \subseteq \cdots \subseteq \mathcal{Z}(\underline{r}'_{n-1}) = \mathcal{Z}(\underline{r}')$$

as desired.

Theorem 1.14(i) implies that there exists a well-defined map from Z_{ss} to $Z \setminus \{\mathcal{Z}(\text{St} + \lambda) : \lambda \in \mathbf{Z}^{\mathbf{Z}/f\mathbf{Z}}\}$ sending $\mathcal{Z}_{\text{ss}}(\underline{r})$ to $\mathcal{Z}(\underline{r})$, except that $\mathcal{Z}_{\text{ss}}(\text{St} + \lambda) = \mathcal{Z}_{\text{ss}}(\text{BT} + \lambda)$ is sent to $\mathcal{Z}(\text{BT} + \lambda)$.

By (i) of this Proposition, this is an isomorphism of partially ordered sets under inclusion. By the same argument with St and BT swapped, there is also a well-defined isomorphism from Z_{ss} to $Z \setminus \{\mathcal{Z}(\text{BT} + \lambda) : \lambda \in \mathbf{Z}^{\mathbf{Z}/f\mathbf{Z}}\}$. This establishes Theorems 1.9, 1.10, and 1.12 for all statements not involving both Barsotti–Tate and Steinberg weights (under the additional assumption of Theorem 1.14(iii), in the case of Theorem 1.12).

The claims involving both Barsotti–Tate and Steinberg weights are easily checked separately. We have $\mathcal{Z}(\text{BT} + \lambda) \neq \mathcal{Z}(\text{St} + \lambda')$ and $\text{BT} + \lambda \not\sim \text{St} + \lambda'$, giving Theorem 1.9. The inclusion $\mathcal{Z}(\text{BT} + \lambda) \subseteq \mathcal{Z}(\text{St} + \lambda)$ is indeed simple, necessitating the third bullet point in Theorem 1.10, and there are no inclusions between $\mathcal{Z}(\text{BT} + \lambda)$ and $\mathcal{Z}(\text{St} + \lambda')$ if $\lambda - \lambda' \notin \Lambda$. For Theorem 1.12, a statement involving both Barsotti–Tate and Steinberg weights could only have $\text{St} + \lambda = f_j(\underline{r})$, with the Barsotti–Tate weight appearing in the middle of a non-simple inclusion; and indeed we have already seen that if $\theta_j(\underline{r}) = \text{St} + \lambda$, then $\mathcal{Z}(\underline{r}) \subsetneq \mathcal{Z}(\text{BT} + \lambda) \subsetneq \mathcal{Z}(\text{St} + \lambda)$, giving the second bullet point of Theorem 1.12 and completing the proof. $\square$

The rest of the paper is devoted to the proof of Theorem 1.14.

1.2. Statement of AI use. Given that for a valid operation f_j , the inclusion $\mathcal{Z}(\underline{r}) \subseteq \mathcal{Z}(f_j(\underline{r}))$ is typically a codimension one inclusion between irreducible stacks, it was natural to wonder whether something like Theorem 1.10 might hold. In the course of our forthcoming work in [KLS], we had previously used calculations with sets of semisimple points to make guesses about the extent to which formulas like $\mathcal{Z}(\mu_j(\underline{r})) \cap \mathcal{Z}(\theta_j(\underline{r})) = \mathcal{Z}(\underline{r})$ might hold. This prompted us to consider the analogue of Theorem 1.10 for semisimple point sets and to realize that the one could be reduced to the other. It appeared to us that the combinatorics required to prove Theorem 1.14 might be rather complicated, but also potentially amenable to current large language models.

Indeed the first proof of Theorem 1.14 was found by GPT-5.5 Pro, by a naive process of giving the problem to the LLM and then prompting it to continue working whenever it stopped. After roughly 30 continuations, the LLM claimed to have a proof. The argument was then turned into a manuscript by an iterative process using two LLM conversations, one the “author” and the other an adversarial “reviewer”. The resulting manuscript was, to us, essentially alien — full of jargon and barely recognizable as related to the original problem. Attempts to formalize the manuscript in Lean using Codex eventually stalled.

We then abandoned the initial manuscript and restarted the process from the beginning, this time also prompting the LLM that it was permitted to introduce new notation, but not new terminology. The new manuscript, which ultimately grew to 45 pages, was clearly recognizable as addressing the problem under consideration, and seemed plausibly to be a proof, although the writing was still very difficult for us to follow. We then formalized the manuscript in Lean via Codex using GPT-5.5 with xhigh reasoning effort. This surfaced several errors in the manuscript, which were corrected with input from GPT-5.5 Pro; but the formalization was eventually completed, convincing us that indeed we had a proof.

It then fell to us to understand the argument. It was immediately clear that the manuscript was organized in what seemed to us to be an unnatural and unnecessarily complicated way. In the process of reorganizing the argument, and trying to understand the reorganized argument, we found a number of simplifications to the statements, the proofs, and the exposition. The exposition below is entirely written by us. Many of the lemma and theorem statements are different from those that can be found in the LLM-generated manuscript. Nevertheless, the underlying logic is still in essence the one that was found by the LLM.

The artifacts that we have described here, including several versions of LLM-generated manuscripts, and the Lean formalization of one of those early versions, are available at <https://github.com/davidsavitt/KLSposet-artifacts>. A formalization of the combinatorial parts of the present manuscript (i.e. the results about the sets $\mathcal{Z}_{ss}(\underline{r})$, but not those concerning the Emerton–Gee stacks) is available at <https://github.com/davidsavitt/KLSposet>.

1.3. Notation and further preliminaries.

1.3.1. Phantom Serre weights. To each Hodge type $\underline{r}$ we may associate the tuple $(u_0, \dots, u_{f-1}; t)$ with $u_i = r_{i,1} - r_{i,2}$ and $t = \sum_{i \in \mathbf{Z}/f\mathbf{Z}} p^{-i} r_{i,2} \in \mathbf{Z}/(q-1)\mathbf{Z}$. This tuple depends only on the class of $\underline{r}$ under translation by Λ , and it is not difficult to see that this defines a bijection from equivalence classes of p -bounded Hodge types up to translation by Λ , to tuples $(u_0, \dots, u_{f-1}; t)$ with $t \in \mathbf{Z}/(q-1)\mathbf{Z}$ and each $u_i \in [0, p]$. We will refer to the tuples $(u_0, \dots, u_{f-1}; t)$ as *phantom Serre weights*. For brevity we often write $\sigma = (u_i; t)$ instead of $(u_0, \dots, u_{f-1}; t)$, and we write $(ur; t)$, $(BT; t)$, $(cr; t)$, and $(St; t)$ for $(u, \dots, u; t)$ with $u = 0, 1, p-1, p$ respectively.

The operations μ_j , θ_j , and ν_j descend to phantom Serre weights. Concretely, we have

$$\begin{aligned} \mu_j : \quad & u_j \mapsto u_j - 1, \quad u_{j+1} \mapsto u_{j+1} + p, \quad t \mapsto t, \\ \theta_j : \quad & u_j \mapsto u_j + 1, \quad u_{j+1} \mapsto u_{j+1} + p, \quad t \mapsto t - p^{-j}, \\ \nu_j : \quad & u_j \mapsto u_j + 1, \quad u_{j+1} \mapsto p - u_{j+1}, \quad t \mapsto t - p^{-j} + u_{j+1} p^{-(j+1)}. \end{aligned}$$

All other u_i 's are fixed. The operations are valid when

- $u_j > 0$ and $u_{j+1} = 0$ for μ_j ;
- $u_j < p$ and $u_{j+1} = 0$ for θ_j ;
- $u_j = 0$ for ν_j .

Now if $(u_j, u_{j+1}) = (1, 0)$ then $\nu_j(\mu_j(\sigma)) = \sigma$, while if $(u_j, u_{j+1}) = (0, p)$ then $\mu_j(\nu_j(\sigma)) = \sigma$, so on phantom Serre weights μ_j and ν_j are literally inverse to one another. The equivalence relation $\sim$ also descends to phantom Serre weights, and we denote it by $\sim$ again.

1.3.2. Fundamental characters. Take $\pi = (-p)^{1/(q-1)}$ and for $g \in G_K$ we set $h(g) = g(\pi)/\pi \in \mu_{q-1}(K)$. Identifying $\mu_{q-1}(K)$ with $k^\times$, we define fundamental characters ω_i of level f by setting $\omega_i = \kappa_i \circ h : I_K \rightarrow \overline{\mathbf{F}}_p^\times$ for each $i \in \mathbf{Z}/f\mathbf{Z}$ (cf. [CEGS1, Lem. 1.4.1] and the discussion following). Fix ω'_0 to be either of the two fundamental characters of level $2f$ such that $(\omega'_0)^{q+1} = \omega_0$, and recursively define ω'_i for $i \in \mathbf{Z}/2f\mathbf{Z}$ by $(\omega'_{i+1})^p = \omega'_i$.

1.3.3. *The sets $\mathcal{Z}_{\text{ss}}(\sigma)$.* We now describe the sets $\mathcal{Z}_{\text{ss}}(\underline{r})$ explicitly.

Definition 1.17. We say that a set J is a *profile* if either

- $J \subset \mathbf{Z}/f\mathbf{Z}$ is any subset; or
- $J \subset \mathbf{Z}/2f\mathbf{Z}$, and for all i we have $i \in J$ if and only if $i + f \notin J$.

In the former case we say that J has niveau 1, and in the latter case we say that J has niveau 2.

If $\sigma = (u_i; t)$, then for each profile $J \subset \mathbf{Z}/f\mathbf{Z}$ we define

$$\bar{\rho}(\sigma, J) = \omega_0^t \otimes \left(\prod_{i \in J} \omega_i^{u_i} \oplus \prod_{i \in J^c} \omega_i^{u_i} \right),$$

while for each profile $J \subset \mathbf{Z}/2f\mathbf{Z}$ we define

$$\bar{\rho}(\sigma, J) = \omega_0^t \otimes \left(\prod_{i \in J} (\omega'_i)^{u_i} \oplus \prod_{i \in J^c} (\omega'_i)^{u_i} \right),$$

with the notation u_i extended to $i \in \mathbf{Z}/2f\mathbf{Z}$ periodically modulo f .

Theorem 1.18 ([GLS, CEGS2]). *If σ is the phantom Serre weight associated to $\underline{r}$, then we have*

$$\mathcal{Z}_{\text{ss}}(\underline{r}) = \{\text{semisimple } \bar{\rho} \text{ such that } \bar{\rho}|_{I_K} \cong \bar{\rho}(\sigma, J)^\vee \text{ for some profile } J\}.$$

The dual comes from our normalization for Hodge–Tate weights: following [CEGS2, BBH⁺] we chose the normalization so that the cyclotomic character has all Hodge–Tate weights equal to -1 .

Definition 1.19. If σ is a phantom Serre weight, we define

$$\mathcal{Z}_{\text{ss}}(\sigma) = \{\bar{\rho}(\sigma, J) : J \text{ any profile}\}.$$

Theorem 1.14 thus descends to an equivalent statement about phantom Serre weights and the finite sets $\mathcal{Z}_{\text{ss}}(\sigma)$. This is the language in which we will phrase the proof of Theorem 1.14.

Remark 1.20. Suppose $\bar{\rho}$ is semisimple and non-scalar on inertia. If J is a profile and $\bar{\rho}|_{I_K} \cong \bar{\rho}(\sigma, J)^\vee$ then either $\bar{\rho}$ is split and J has niveau 1, or else $\bar{\rho}$ is irreducible and J has niveau 2.

If on the other hand $\bar{\rho}$ is scalar on inertia, then it can happen that $\bar{\rho}|_{I_K} \cong \bar{\rho}(\sigma, J)^\vee$ with J of niveau 2. But for each such σ , there is also a witness $\bar{\rho}|_{I_K} \cong \bar{\rho}(\sigma, J')^\vee$ with J' of niveau 1; see the discussion in the last paragraph of [GHS, Example 7.1.7].

If J is a profile, let δ_J be the indicator function of J , and set $\varepsilon_i^J = (-1)^{\delta_J(i)}$.

2. PINNINGS

2.1. The carry equation. The following key lemma is also proved in [KLS], but we include a proof here for the sake of completeness.

Lemma 2.1. *Let $\sigma = (u_i; t)$ and $\sigma' = (u'_i; t')$ be two phantom Serre weights. Suppose that J, J' are profiles of the same niveau such that $\bar{\rho}(\sigma, J) = \bar{\rho}(\sigma', J')$, with equality rather than isomorphism meaning that the two characters of $\bar{\rho}(\sigma, J)$ and of $\bar{\rho}(\sigma', J')$ are in the same order. Set $f' = f$ if J has niveau 1, and $f' = 2f$ otherwise.*

(i) There are unique integers (x_i) such that

$$\varepsilon_i^{J'} u'_i - \varepsilon_i^J u_i = px_{i-1} - x_i$$

for all $i \in \mathbf{Z}/f'\mathbf{Z}$. Moreover $x_{i+f} = -x_i$ if $f' = 2f$.

(ii) The parities of the x_i 's depend only on σ, σ' and in particular are independent of the profiles J, J' .

(iii) We have $x_i \in [-2, 2]$ for all i , unless $p = 3$, $u_i = u'_i = 3$ for all i , $t' - t = (q-1)/2$, and $\{J, J'\} = \{\emptyset, \mathbf{Z}/f\mathbf{Z}\}$.

The integers x_i in the lemma can be thought of as carries in the base p subtraction $\sum_i \varepsilon_i^{J'} u'_i p^{f'-i} - \sum_i \varepsilon_i^J u_i p^{f'-i}$; hence the name of the subsection.

Proof. The equality $\bar{\rho}(\sigma, J) = \bar{\rho}(\sigma', J')$ implies that there are congruences

$$(2.2) \quad \sum_{i \in J} p^{f'-i} u_i \equiv T + \sum_{i \in J'} p^{f'-i} u'_i \pmod{p^{f'} - 1}$$

and

$$(2.3) \quad \sum_{i \notin J} p^{f'-i} u_i \equiv T + \sum_{i \notin J'} p^{f'-i} u'_i \pmod{p^{f'} - 1}$$

where $T = t' - t$ if $f' = f$ and $T = (q+1)(t' - t)$ if $f' = 2f$.

Taking the difference between equations (2.2) and (2.3) and rearranging gives $\sum_{i=0}^{f'-1} p^{f'-i} y_i \equiv 0 \pmod{p^{f'} - 1}$ where

$$y_i = (-1)^{\delta_{J'}(i)} u'_i - (-1)^{\delta_J(i)} u_i.$$

The lattice Λ' of vectors $(y_i) \in \mathbf{Z}^{\mathbf{Z}/f'\mathbf{Z}}$ such that $\sum_{i=0}^{f'-1} p^{f'-i} y_i \equiv 0 \pmod{p^{f'} - 1}$ has a basis consisting of the vectors $v_j = (v_{j,i})$ with $v_{j,j} = -1$, $v_{j,j+1} = p$, and $v_{j,i} = 0$ otherwise. It follows immediately that there exist unique integers x_i with $y_i = px_{i-1} - x_i$.

Suppose $f' = 2f$. The sequences u_i, u'_i are periodic modulo f , while $i \in J$ if and only if $i+f \notin J$, and similarly for J' . It follows that the tuple (x'_i) with $x'_i := -x_{i+f}$ also satisfies the conditions of the lemma. By uniqueness it follows that $x_i = -x_{i+f}$. This proves (i).

Let $M = \max_j |x_j|$ and suppose that $|x_{i-1}| = M$. Then

$$pM = |px_{i-1}| = |y_i + x_i| \leq 2p + M.$$

If $p > 3$ then $M \leq 2p/(p-1) < 3$ by the hypothesis on p , and $x_i \in [-2, 2]$ for all i .

If $p = 3$, then the same conclusion still holds unless $|x_{i-1}| = |x_i| = 3$ and $|y_i| = 6$; then iteratively we obtain $|y_i| = 6$ for all i , and $u_i = u'_i = 3$ for all i . Finally either $x_i = 3$ for all i , in which case $\varepsilon_i^{J'} - \varepsilon_i^J = 2$ for all i and $J = \mathbf{Z}/f\mathbf{Z}$, $J' = \emptyset$; or else $x_i = -3$ for all i , $\varepsilon_i^{J'} - \varepsilon_i^J = -2$ for all i , and $J = \emptyset$, $J' = \mathbf{Z}/f\mathbf{Z}$. If $J = \mathbf{Z}/f\mathbf{Z}$ and $J' = \emptyset$, then (2.2) gives $T \equiv \sum_{i=0}^{f'-1} 3^{f'-i} \cdot 3 \equiv (q-1)/2 \pmod{q-1}$. The case $J = \emptyset$, $J' = \mathbf{Z}/f\mathbf{Z}$ gives the negative of the same congruence, hence again $T = (q-1)/2$. This gives (iii).

Finally suppose that (K, K') is a pair of profiles of the same niveau with $\bar{\rho}(\sigma, K) = \bar{\rho}(\sigma', K')$, leading to parameters $\tilde{f}'$, $\tilde{y}_i$ and $\tilde{x}_i$ in place of f' , y_i and x_i . Suppose first that $\tilde{f}' = f'$. Since $(-1)^{\delta_{J'}(i)} = 1 - 2\delta_J(i)$ the difference $z_i = \frac{1}{2}(y_i - \tilde{y}_i)$ is equal to

$$(\delta_{K'}(i) - \delta_{J'}(i))u'_i - (\delta_K(i) - \delta_J(i))u_i.$$

Taking the difference between (2.2) for (J, J') and for (K, K') shows that $(z_i) \in \Lambda'$. Writing $z_i = pw_{i-1} - w_i$ for integers w_i we find that $x_i - \tilde{x}_i = 2w_i$ and therefore $x_i \equiv \tilde{x}_i \pmod{2}$.

If instead $\tilde{f}' \neq f'$, suppose without loss of generality that $f' = f$ and $\tilde{f}' = 2f$. Then exactly the same argument in the previous paragraph goes through after multiplying (2.2) for (J, J') by $q + 1$ (and f -periodically extending u_i, u'_i, x_i, J, J' to $\mathbf{Z}/2f\mathbf{Z}$) to obtain a congruence modulo $p^{2f} - 1$. $\square$

Remark 2.4. In the exceptional case of Lemma 2.1(iii), one can furthermore check that there are no J, J' such that $\bar{\rho}(\sigma, J) = \bar{\rho}(\sigma', J')$ other than $\{J, J'\} = \{\emptyset, \mathbf{Z}/f\mathbf{Z}\}$. Indeed, in niveau 1, the equality of $\sum_{i \in J} p^{-i} u_i$ and $\sum_{i \in J'} p^{-i} u'_i$ modulo $q - 1$ says that two subsets of $1, 3, \dots, 3^{f-1}$ have sums that differ by $(q - 1)/2$ modulo $q - 1$. Since each such subset sum lies between 0 and $(q - 1)/2$, this forces the two subsets to be $\emptyset$ and $\mathbf{Z}/f\mathbf{Z}$. In niveau 2 the argument is essentially the same, except that since $\emptyset, \mathbf{Z}/2f\mathbf{Z}$ are not profiles in niveau 2, the conclusion is simply that there are no solutions J, J' .

In particular $\mathcal{Z}_{\text{ss}}(\sigma) \not\subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ and vice-versa. For this reason, in proving Theorem 1.14 we will generally never be in the exceptional case of part (iii) of the lemma, and will always have $x_i \in [-2, 2]$. In arguments where the exceptional case is trivially excluded, e.g. because of a hypothesis that $\bar{\rho}(\sigma, J) = \bar{\rho}(\sigma', J')$ for some $J \notin \{\emptyset, \mathbf{Z}/f\mathbf{Z}\}$, or that $u_i \neq p$ for some i , we will pass over this without further mention.

Remark 2.5. If $x_i = 0$ for all i , then Lemma 2.1 immediately implies $u_i = u'_i$ for all i , and that J, J' agree on all indices i such that $u_i \neq 0$. Then (2.2) implies that $t = t'$, and therefore $\sigma = \sigma'$.

As a first application of the lemma, we analyze the situation where $\bar{\rho}(\sigma, J)$ is scalar for a niveau 2 profile J .

Corollary 2.6. *Suppose that J is a profile such that $\bar{\rho}(\sigma, J)$ is scalar. Then there exist unique integers (y_i) such that*

$$\varepsilon_i^J u_i = py_{i-1} - y_i,$$

and if J has niveau 2 then $y_{i+f} = -y_i$ for all $i \in \mathbf{Z}/2f\mathbf{Z}$. Furthermore $|y_i| \leq 1$ for all i .

Proof. Since $\bar{\rho}(\sigma, J)$ is scalar, it is possible to choose $t' \in \mathbf{Z}/(q - 1)\mathbf{Z}$ such that $\bar{\rho}(\sigma, J) = \bar{\rho}(\text{ur}; t'), J')$ for some (indeed any) profile J' of the same niveau as J . Applying Lemma 2.1 to the pair $(\text{ur}; t')$ and σ (in that order), we obtain unique integers y_i such that

$$\varepsilon_i^J u_i = py_{i-1} - y_i$$

for all i , and $y_{i+f} = -y_i$ if J has niveau 2, giving the first part of the corollary. The bound $|y_i| \leq 1$ follows by the same argument as for the bound in Lemma 2.1(iii). $\square$

We saw in Remark 1.20 that if J has niveau 2 and $\bar{\rho}(\sigma, J)$ is scalar, then there also exists a profile J' of niveau 1 such that $\bar{\rho}(\sigma, J') = \bar{\rho}(\sigma, J)$. We now analyze the same situation with the niveaux reversed.

Proposition 2.7. *Suppose that J is a profile of niveau 1 such that $\bar{\rho}(\sigma, J)$ is scalar. Then there also exists a profile J' of niveau 2 such that $\bar{\rho}(\sigma, J') = \bar{\rho}(\sigma, J)$ unless $\sigma = (\text{cr}; t)$ for some t , in which case no such J' exists.*

Proof. Write $\sigma = (u_i; t)$. The nonexistence of J' when $\sigma = (\text{cr}; t)$ is a consequence of Corollary 2.6. Suppose J' of niveau 2 were to exist, and apply the Corollary. Since $u_i = p - 1$ for all i we have $y_{i-1} = y_i \in \{-1, 1\}$ for all i , and recursively the sequence y_i is constant. This contradicts $y_{i+f} = -y_i$.

Now suppose that σ is non-critical and $\bar{\rho}(\sigma, J)$ is scalar, with J of niveau 1. Corollary 2.6 gives integers $y_i \in [-1, 1]$ such that $\varepsilon_i^J u_i = p y_{i-1} - y_i$ for all $i \in \mathbf{Z}/f\mathbf{Z}$. Since σ is non-critical, some y_i must be 0. Without loss of generality suppose that $y_{-1} = 0$. Set $y'_i = y_i$ and $\varepsilon_i^{J'} = \varepsilon_i^J$ for $i \in \mathbf{Z}/2f\mathbf{Z}$ whose least non-negative residue lies in $[0, f-1]$, and $y'_i = -y_i$, $\varepsilon_i^{J'} = -\varepsilon_i^J$ otherwise. Let J' be the niveau 2 profile implicitly defined by the choice of $\varepsilon_i^{J'}$. Now evidently

$$(2.8) \quad \varepsilon_i^{J'} u_i = p y'_{i-1} - y'_i$$

for $i \neq 0, f$ (so that $\{i-1, i\}$ is contained in either $[0, f-1]$ or $[f, 2f-1]$). But since $y'_{-1} = y'_{f-1} = 0$, the equation (2.8) holds for $i = 0, f$ as well, i.e., it holds for all $i \in \mathbf{Z}/2f\mathbf{Z}$.

Multiplying (2.8) by p^{-i} and summing over $\mathbf{Z}/2f\mathbf{Z}$ proves that $\sum_{i \in J'} u_i p^{-i} - \sum_{i \notin J'} u_i p^{-i} \equiv 0 \pmod{q^2 - 1}$, or in other words that $\bar{\rho}(\sigma, J')$ is scalar. However, we need the stronger statement that $\bar{\rho}(\sigma, J') = \bar{\rho}(\sigma, J)$, or equivalently that

$$\sum_{i \in J'} u_i p^{-i} - (q+1) \sum_{i \in J} u_i p^{-i} \equiv 0 \pmod{q^2 - 1}.$$

Since $\delta_J(i) = \frac{1}{2}(1 - \varepsilon_i^J)$ and similarly for J' , the previous display rewrites as

$$\sum_{i=0}^{2f-1} \frac{1}{2}(1 - \varepsilon_i^{J'}) u_i p^{-i} - \sum_{i=0}^{2f-1} \frac{1}{2}(1 - \varepsilon_i^J) u_i p^{-i} \equiv 0 \pmod{q^2 - 1}.$$

From the definition of $\varepsilon_i^{J'}$, the terms with $0 \leq i < f$ in the two sums cancel, and the desired congruence is equivalent to

$$\sum_{i=0}^{f-1} \varepsilon_i^J u_i p^{-i} \equiv 0 \pmod{q^2 - 1}.$$

Substituting $\varepsilon_i^J u_i = p y_{i-1} - y_i$, the terms on the left-hand side involving y_i for $0 \leq i < f-1$ telescope, leaving only the terms involving y_{-1} and y_{f-1} . But $y_{-1} = y_{f-1} = 0$, so the previous congruence holds. $\square$

As another early application of these ideas we show that the inclusion $\mathcal{Z}_{\text{ss}}((\text{ur}; t)) \subseteq \mathcal{Z}_{\text{ss}}((\text{cr}; t))$ is simple. In fact we have the following stronger statement.

Proposition 2.9. *Suppose that $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}((\text{cr}; t))$. Then $\sigma = (\text{ur}; t)$ or $\sigma = (\text{cr}; t)$.*

Proof. The proof will require the following claim. If

$$\sum_{i \in \mathbf{Z}/f\mathbf{Z}} a_i p^{f-i} \equiv \sum_{i \in \mathbf{Z}/f\mathbf{Z}} b_i p^{f-i} \pmod{p^{f-1} + \dots + p + 1}$$

with $a_i \in \{0, 1, 2\}$ and $b_i \in \{0, 1\}$ for all i , then $a_i - b_i$ is constant. To see this, we treat the two sides as the integers $A = \sum_{i=1}^f a_i p^{f-i}$ and $B = \sum_{i=1}^f b_i p^{f-i}$, so that the sums give the base p expansions of A, B respectively. Set $M = p^{f-1} + \dots + p + 1$. Evidently $A - B \in \{-M, 0, M, 2M\}$. If $A - B = -M$ then $a_i = 0$ and $b_i = 1$ for all i . If $A - B = 0$ then by uniqueness of base- p expansions we have $a_i = b_i$ for all i . If $A - B = M$ then $A = \sum_{i=1}^f (b_i + 1) p^{f-i}$. Since $b_i + 1 < p$, this is the base- p

expansion of A , and $a_i = b_i + 1$ for all i . If $A - B = 2M$ then $a_i = 2$ and $b_i = 0$ for all i . In all cases the desired conclusion holds.

Without loss of generality suppose that $t = 0$. Write $\sigma = (u_i; t')$. Write $\bar{\rho}(\sigma, \emptyset) = \bar{\rho}((\text{cr}; 0), K)$ for a profile K , which we can take to have niveau 1 by Remark 1.20. Similarly write $\bar{\rho}(\sigma, \{i\}) = \bar{\rho}((\text{cr}; 0), K_i)$ for a niveau 1 profile K_i for each i . Each $\bar{\rho}((\text{cr}; 0), J)$ with J of niveau 1 has the form $\omega_0^a \oplus \omega_0^b$ with a, b divisible by $p - 1$. Applied to $\bar{\rho}(\sigma, \emptyset)$ and $\bar{\rho}(\sigma, \{i\})$ this shows that t' and all of the u_i are divisible by $p - 1$. In particular $u_i \in \{0, p - 1\}$ for all i . If K is $\mathbf{Z}/f\mathbf{Z}$, we replace it with $\emptyset$ instead.

Now (2.2) shows that $t' \equiv (p - 1) \sum_{i \in K} p^{f-i} \pmod{q-1}$, and another application of (2.2) for the profile $\{j\}$ shows, after dividing by $p - 1$, that

$$p^{f-j} + \sum_{i \in K} p^{f-i} \equiv \sum_{i \in K_j} p^{f-i} \pmod{M}$$

for each j in the set $S := \{i : u_i = p - 1\}$. An application of our claim then shows that $j \notin K$. Therefore $K \cap S = \emptyset$.

The equality $\det \bar{\rho}(\sigma, \emptyset) = \det \bar{\rho}((\text{cr}; 0), K)$ gives $2t' + (p - 1) \sum_{i \in S} p^{f-i} \equiv 0 \pmod{q-1}$, or equivalently

$$\sum_{i \in K} 2 \cdot p^{f-i} + \sum_{i \in S} p^{f-i} \equiv 0 \pmod{M}.$$

Since $K \cap S = \emptyset$ and $K \neq \mathbf{Z}/f\mathbf{Z}$, another application of our claim gives $K = \emptyset$ and $S \in \{\emptyset, \mathbf{Z}/f\mathbf{Z}\}$. Thus $t' = 0$ and $\sigma = (\text{ur}; 0)$ or $\sigma = (\text{cr}; 0)$. $\square$

2.2. Further consequences of the carry equation.

Definition 2.10. Let σ, σ' be a pair of phantom Serre weights.

- (i) We define $\mathcal{P}(\sigma, \sigma')$ to be the set of profiles J such that there exists a profile K_J of the same niveau with $\bar{\rho}(\sigma, J) = \bar{\rho}(\sigma', K_J)$.
- (ii) A *pinning* is a choice, for each $J \in \mathcal{P}(\sigma, \sigma')$, of a profile K_J of the same niveau with $\bar{\rho}(\sigma, J) = \bar{\rho}(\sigma', K_J)$. Given a pinning we write x_i^J for the integers of Lemma 2.1 applied to the pair of profiles J, K_J .

We note the following subtlety in the definition of $\mathcal{P}(\sigma, \sigma')$. If J has niveau 2 and $\bar{\rho}(\sigma, J)$ is scalar, then the existence of some J' such that $\bar{\rho}(\sigma, J) = \bar{\rho}(\sigma', J')$ need not imply that $J \in \mathcal{P}(\sigma, \sigma')$, since it may not be possible to choose J' of niveau 2. By Proposition 2.7, this happens precisely when σ' is critical. To emphasize this, we highlight the case $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$.

Corollary 2.11. *Suppose that $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$. If σ' is non-critical then the set $\mathcal{P}(\sigma, \sigma')$ contains all profiles, while if σ' is critical then $\mathcal{P}(\sigma, \sigma')$ contains all profiles except for any niveau 2 profiles J with $\bar{\rho}(\sigma, J)$ scalar.*

Suppose for the remainder of the section that $\emptyset \in \mathcal{P}(\sigma, \sigma')$, and we fix a pinning. It will be convenient to abbreviate $x_i := x_i^\emptyset$ and $K := K_\emptyset$, and we do so for the remainder of the paper. If $i \in \mathbf{Z}/2f\mathbf{Z}$ then x_i means $x_{i \pmod{f}}$. Note that $\varepsilon_i^\emptyset = 1$ for all i .

If also $J \in \mathcal{P}(\sigma, \sigma')$, we shall need to understand the joint behavior of the integers x_i and x_i^J , beyond what is immediately provided by Lemma 2.1 (that they have the same parity). When jointly analyzing the equations coming from Lemma 2.1(i) for the pair of profiles $\emptyset, J$ it is often better to use the sum and difference of those

equations instead. To that end, for any $J \in \mathcal{P}(\sigma, \sigma')$ we write $s_i^J = \frac{1}{2}(x_i^J + x_i)$ and $d_i^J = \frac{1}{2}(x_i^J - x_i)$, so that $s_i^J, d_i^J \in [-2, 2]$ apart from the exceptional case when $p = 3$. These are integers thanks to Lemma 2.1(ii). Then

$$(2.12) \quad \frac{1}{2}(\varepsilon_i^{K_J} + \varepsilon_i^K)u'_i - \delta_{J^c}(i)u_i = ps_{i-1}^J - s_i^J$$

and

$$(2.13) \quad \frac{1}{2}(\varepsilon_i^{K_J} - \varepsilon_i^K)u'_i + \delta_J(i)u_i = pd_{i-1}^J - d_i^J.$$

Since $\varepsilon_i^{K_J} = \pm \varepsilon_i^K$, we deduce the following.

Lemma 2.14. *Suppose that $\emptyset, J \in \mathcal{P}(\sigma, \sigma')$. Then for each i one of the following alternatives holds: either*

$$-\delta_{J^c}(i)u_i = ps_{i-1}^J - s_i^J \quad \text{or} \quad \delta_J(i)u_i = pd_{i-1}^J - d_i^J.$$

Here are three sample applications of the lemma that will be used in the next section. In what follows we will frequently use the following observations. One of the two alternatives of Lemma 2.14 will always be either $ps_{i-1}^J = s_i^J$ or $pd_{i-1}^J = d_i^J$, depending on whether $i \in J$ or $i \notin J$. Outside the exceptional case, we have $s_{i-1}^J, s_i^J \in [-2, 2]$ but $p \geq 3$, so that $ps_{i-1}^J = s_i^J$ implies $s_{i-1}^J = s_i^J = 0$; and similarly for $pd_{i-1}^J = d_i^J$. Also outside the exceptional case, if $x_i < 0$ then $s_i^J \leq 0$ and $d_i^J \geq 0$: consider that if $x_i = -1$ then $x_i^J = \pm 1$ and if $x_i = -2$ then $x_i^J \in \{-2, 0, 2\}$.

Lemma 2.15. *Let I be the niveau 1 profile $\{i : 0 < u_i < p\}$. If $\emptyset, I \in \mathcal{P}(\sigma, \sigma')$ and $x_i < 0$, then $(u_i, u_{i+1}) \neq (1, p)$.*

Proof. Assume $u_i = 1$, so that $i \in I$. We claim that $s_i^I = 0$. In the first alternative of Lemma 2.14 we have $ps_{i-1}^I = s_i^I$, implying $s_{i-1}^I = s_i^I = 0$. In the second alternative we get $1 = pd_{i-1}^I - d_i^I$. But the hypothesis $x_i < 0$ implies that $d_i^I \in \{0, 1, 2\}$, and so the only possibility is that $p = 3$, $d_{i-1}^I = 1$, and $d_i^I = 2$. The latter forces $x_i = -2$ and $x_i^I = 2$, and so again $s_i^I = 0$.

Now suppose in addition that $u_{i+1} = p$, so that $i+1 \notin I$, and consider Lemma 2.14 applied at $i+1$. Since $s_i^I = 0$ by the previous paragraph, the first alternative becomes $-p = -s_{i+1}^I$, which contradicts the bounds on s_{i+1}^I . The second alternative gives $pd_i^I - d_{i+1}^I = 0$, and therefore $d_i^I = d_{i+1}^I = 0$. But $s_i^I = 0$ implies $d_i^I = |x_i| \geq 1$, and again we obtain a contradiction. $\square$

Lemma 2.16. *Let P be the niveau 2 profile satisfying $P \cap \{0, \dots, f-1\} = \{i : u_i = p\}$. If $\emptyset \in \mathcal{P}(\sigma, \sigma')$, $u_0 = 0$, $(u_{i-1}, u_i) \neq (0, p)$ for all i , and $x_i < 0$ for all i , then $P \notin \mathcal{P}(\sigma, \sigma')$.*

Proof. We argue by contradiction, so let us suppose that $P \in \mathcal{P}(\sigma, \sigma')$. Suppose $1 \leq i \leq f-1$. We claim, first, that if $s_i^P = 0$ then also $s_{i-1}^P = 0$. In the second alternative of Lemma 2.14, the left-hand side is always divisible by p for the profile P , and therefore $d_i^P = 0$. But $d_i^P = s_i^P = 0$ implies $x_i = 0$, contradicting our hypothesis, so it must be the first alternative that holds. In that alternative the right-hand side is divisible by p . But if the left-hand side is divisible by p , then it must be 0 (because if $u_i = p$ then $\delta_{P^c}(i) = 0$). Therefore also $s_{i-1}^P = 0$.

In the converse direction, we claim that if $s_{i-1}^P = 0$ and $u_i \neq p$ then $s_i^P = u_i = 0$, again for $1 \leq i \leq f-1$. As in the previous paragraph, the second alternative gives $d_i^P = 0$. Since $u_i \neq p$ by hypothesis, the left-hand side of this alternative is 0, and

we conclude $d_{i-1}^P = 0$. But $d_{i-1}^P = s_{i-1}^P = 0$ contradicts $x_{i-1} \neq 0$, so again it is the first alternative that holds. Now $s_i^P = \delta_{P^c}(i)u_i = u_i$. But $x_i < 0$ implies $s_i^P \leq 0$, and this combined with $s_i^P = u_i \geq 0$ gives $s_i^P = u_i = 0$.

Finally consider $i = 0$. Since $u_0 = 0$ the alternatives give either $s_{-1}^P = s_0^P = 0$ or $d_{-1}^P = d_0^P = 0$. Also since $u_0 = 0$ the equation $\varepsilon_0^K u'_0 = px_{-1} - x_0$ gives $|x_{-1}| \leq 1$. Therefore $x_{-1} = -1$, and $x_{-1}^P = \pm 1$.

Suppose first that $d_{-1}^P = d_0^P = 0$, so that $x_{-1} = x_{-1}^P = -1$. Then $x_{f-1}^P = -x_{-1}^P = 1$, and $s_{f-1}^P = 0$. Iteratively applying the conclusion in the first paragraph of the proof, we conclude that $s_0^P = 0$. But $s_0^P = d_0^P = 0$ implies $x_0 = 0$, a contradiction. So we must instead have $s_{-1}^P = s_0^P = 0$. Iteratively applying the conclusion in the second paragraph of the proof, which is possible because of the hypothesis $(u_{i-1}, u_i) \neq (0, p)$ for all i , we obtain $s_{f-1}^P = 0$. But $s_{-1}^P = s_{f-1}^P = 0$ is again a contradiction, because $x_{f-1}^P = -x_{-1}^P \neq x_{-1}^P$. $\square$

Lemma 2.17. *Suppose that $\emptyset, J \in \mathcal{P}(\sigma, \sigma')$.*

- (i) *If $x_{i-1}, x_i < 0$, $u_i > 0$, and $s_{i-1}^J = 0$ then $i \in J$.*
- (ii) *If $x_{i-1}, x_i, x_{i+1} < 0$, $u_{i-1}, u_i, u_{i+1} > 0$, and $J \cap \{i-1, i, i+1\} = \{i\}$, then $u_{i-1} = p+1+x_{i-1}$ and $u_i = p-d_i^J$.*

Proof. (i) Suppose $i \notin J$. Then the alternatives of Lemma 2.14 are

$$-u_i = ps_{i-1}^J - s_i^J \quad \text{or} \quad pd_{i-1}^J = d_i^J.$$

The hypotheses $s_{i-1}^J = 0$, $x_{i-1} < 0$ rule out $d_{i-1}^J = 0$, so the second cannot hold. The first alternative then gives $u_i = s_i^J$. But $x_i < 0$ implies $s_i^J \leq 0$, contradicting the hypothesis $u_i > 0$.

(ii) Since $u_i > 0$ and $d_i^J \geq 0$, Lemma 2.14 applied at $i \in J$ gives either

$$(2.18) \quad s_{i-1}^J = s_i^J = 0 \quad \text{or} \quad d_{i-1}^J = 1, \quad u_i = p - d_i^J.$$

Lemma 2.14 applied at $i+1 \notin J$ gives either

$$-u_{i+1} = ps_i^J - s_{i+1}^J \quad \text{or} \quad d_i^J = d_{i+1}^J = 0.$$

Either of these alternatives contradicts the first alternative of (2.18). Indeed if $s_i^J = 0$ then we cannot have $u_{i+1} = s_{i+1}^J$ (because the left-hand side is positive and the right-hand side is not), nor can we have $d_i^J = 0$ (because $x_i \neq 0$). Therefore it is the second alternative in (2.18) that holds. In particular $u_i = p - d_i^J$.

Now apply Lemma 2.14 at $i-1 \notin J$. Since $d_{i-1}^J = 1$, the second alternative cannot hold, and the first alternative gives

$$u_{i-1} = -ps_{i-2}^J + s_{i-1}^J.$$

Since $u_{i-1} > 0$ and $s_{i-1}^J \leq 0$, we must have $s_{i-2}^J = -1$ and $u_{i-1} = p + s_{i-1}^J$. Since $s_{i-1}^J = x_{i-1} + d_{i-1}^J = x_{i-1} + 1$, the lemma follows. $\square$

3. THE SIX ALTERNATIVES

Suppose that $\mathcal{Z}_{ss}(\sigma) \subseteq \mathcal{Z}_{ss}(\sigma')$. If σ' is non-critical then the set $\mathcal{P}(\sigma, \sigma')$ contains all profiles, while if σ' is critical then $\mathcal{P}(\sigma, \sigma')$ contains all profiles except for any niveau 2 profiles J with $\bar{\rho}(\sigma, J)$ scalar.

Assume that $\sigma \neq \sigma'$ and fix a pinning. By Remark 2.5 the following four alternatives are exhaustive and mutually exclusive.

- (+) We have $x_i > 0$ for all i .

- (P) There exists j such that $x_{j-1} \leq 0$ and $x_j > 0$.
- (N) We have $x_i \leq 0$ for all i , and there exists j such that $x_{j-1} = 0$, $x_j < 0$.
- (−) We have $x_i < 0$ for all i .

We refine alternatives (P) and (N) into two subcases each, for a total of six alternatives, no longer mutually exclusive since both subcases may occur for different values of j .

- (P.a) There exists j such that $x_{j-1} \leq 0$ and $x_j > 0$, and $u_j > 0$.
- (P.b) There exists j such that $x_{j-1} \leq 0$ and $x_j > 0$, and $u_j = 0$.
- (N.a) We have $x_i \leq 0$ for all i , and there exist integers $j' \geq j$ such that $x_{j-1} = 0$, $x_j, \dots, x_{j'} < 0$, $u_j = \dots = u_{j'} = 0$, and $u_{j'+1} > 0$.
- (N.b) We have $x_i \leq 0$ for all i , and there exists j such that $x_{j-1} = 0$, $x_j < 0$, but there is no j' as in (N.a) for this j .

We introduce the following condition on pinning.

Definition 3.1. Suppose $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$. We say that a pinning of the pair σ, σ' is *minimal* if $\sum_{i \in \mathbf{Z}/f\mathbf{Z}} |x_i|$ is as small as possible as one varies over all $\tilde{\sigma} \sim \sigma$ and all pinning of $\tilde{\sigma}, \sigma'$.

Note that, according to this definition, a pair σ, σ' with $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ need not have a minimal pinning, but a minimal pinning always exists after possibly replacing σ with another phantom Serre weight $\tilde{\sigma}$ with $\tilde{\sigma} \sim \sigma$. In what follows, when we take a minimal pinning to be given, we mean literally on σ rather than after replacement.

Lemma 3.2. *Let σ, σ' be phantom Serre weights such that $\emptyset \in \mathcal{P}(\sigma, \sigma')$. There is no index j with $x_j = 2$, with $x_j = 1$ and $x_{j+1} < 0$, or with $x_j = -2$ and $x_{j+1} > 0$.*

If furthermore $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ is equipped with a minimal pinning, then:

- (i) *There is no index j with $(u_j, u_{j+1}) = (1, 0)$ and $x_j > 0$.*
- (ii) *There is no index j with $(u_j, u_{j+1}) = (0, p)$ and $x_j < 0$.*

Proof. Lemma 2.1 applied at $J = \emptyset$ gives $u'_{j+1} = |u_{j+1} + px_j - x_{j+1}|$. Taking $x_j = 2$ or $x_j = 1$, $x_{j+1} < 0$ or $x_j = -2$, $x_{j+1} > 0$ gives a contradiction to $u'_{j+1} \in [0, p]$.

For (i), Lemma 2.1 gives

$$\begin{aligned} \varepsilon_j^K u'_j - 1 &= px_{j-1} - x_j \\ \varepsilon_{j+1}^K u'_{j+1} - 0 &= px_j - x_{j+1}. \end{aligned}$$

These equations rearrange to

$$\begin{aligned} \varepsilon_j^K u'_j - 0 &= px_{j-1} - (x_j - 1) \\ \varepsilon_{j+1}^K u'_{j+1} - p &= p(x_j - 1) - x_{j+1} \end{aligned}$$

which, along with $\varepsilon_i^K u'_i - u_i = px_{i-1} - x_i$ for $i \neq j, j+1$ must provide the unique output of Lemma 2.1 for the pair $(\mu_j(\sigma), \sigma')$ with $(J, K_J) = (\emptyset, K)$. (Note that $\bar{\rho}(\mu_j(\sigma), \emptyset) = \bar{\rho}(\sigma, \emptyset) = \bar{\rho}(\sigma', K)$, so the lemma applies.) Since $x_j > 0$, the sum $|x_j - 1| + \sum_{i \neq j} |x_i|$ is strictly smaller than $\sum_i |x_i|$, contradicting minimality of the original pinning. This establishes (i), and the proof of (ii) is similar. $\square$

In preparation for the next section, we establish the following reductions.

Theorem 3.3. *Suppose $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$, equipped with a minimal pinning. Then:*

- *In case (+) we have $\sigma, \sigma' = (\text{ur}; t), (\text{cr}; t)$ or $(\text{BT}; t), (\text{St}; t)$ for some t .*

- In case (P), and for j as in that case, treating j as an integer there exists a least $j' \geq j$ such that $x_j, \dots, x_{j'} > 0$ and $x_{j'+1} \leq 0$. Then $u_{j+1} = \dots = u_{j'+1} = 0$.
- In case (N.b), and for j as in that case, we have $u_{j+1} = 0$.
- In case (–) we have $\sigma, \sigma' = (\text{ur}; t), (\text{cr}; t)$ or $(\text{St}; t), (\text{BT}; t)$ for some t .

Proof. Throughout the proof we take $\sigma = (u_i; t)$ and $\sigma' = (u'_i; t')$.

The case (+). By hypothesis $x_i > 0$ for all i . Lemma 3.2 tells us that $x_i = 1$ for all i . Lemma 2.1 gives $\varepsilon_i^K u'_i = u_i + (p-1)$. Since $u'_i \in [0, p]$ we have $u_i \in \{0, 1\}$ for all i and $K = \emptyset$. By Lemma 3.2(i) there is no j with $(u_{j-1}, u_j) = (1, 0)$. Therefore either $u_i = 0$ for all i , or $u_i = 1$ for all i . In the first case $u'_i = p-1$ for all i , and (2.2) with $J = J' = \emptyset$ shows that $t' = t$, giving the first option $\sigma, \sigma' = (\text{ur}; t), (\text{cr}; t)$. If instead $u_i = 1$ for all i then in the same manner we obtain $\sigma, \sigma' = (\text{BT}; t), (\text{St}; t)$.

The case (P). By Lemma 3.2 we in fact have $x_j = \dots = x_{j'} = 1$ and $x_{j'+1} = 0$. The equation $\varepsilon_{j'+1}^K u'_{j'+1} = u_{j'+1} + px_{j'} - x_{j'+1} = u_{j'+1} + p$ forces $u_{j'+1} = 0$. Similarly for $j < i \leq j'$ the equation $\varepsilon_i^K u'_i = u_i + px_{i-1} - x_i = u_i + p - 1$ gives $u_i \in \{0, 1\}$. But $(u_i, u_{i+1}) = (1, 0)$ is ruled out by Lemma 3.2(i), and since already $u_{j'+1} = 0$, we iteratively conclude that $u_i = 0$ for $j < i \leq j'$.

The case (N.b). By hypothesis $x_i \leq 0$ for all i , $x_{j-1} = 0$, and $x_j < 0$. If $u_j = 0$, then the hypothesis that we are not in case (N.a) already gives $u_{j+1} = 0$, so we may assume $u_j > 0$. Lemma 2.1 for $J = \emptyset$ gives $u'_j = u_j - x_j$. In particular $u_j \leq p + x_j$ and $j \notin K$.

Lemma 2.14 applied to the profile $\{j\}$ gives either

$$(3.4) \quad ps_{j-1}^{\{j\}} = s_j^{\{j\}} \quad \text{or} \quad u_j = pd_{j-1}^{\{j\}} - d_j^{\{j\}}.$$

In the first alternative $s_{j-1}^{\{j\}} = s_j^{\{j\}} = 0$, so that $x_j^{\{j\}} = d_j^{\{j\}} = -x_j$. In the second alternative since $x_j < 0$ we have $d_j^{\{j\}} \geq 0$, and $u_j > 0$ implies $d_{j-1}^{\{j\}} = 1$, $x_{j-1}^{\{j\}} = 2$, and $u_j = p - d_j^{\{j\}}$. Comparing the latter with $u_j \leq p + x_j$ we have $-x_j \leq d_j^{\{j\}}$. With $x_j < 0$ this is only possible if $d_j^{\{j\}} = -x_j$. Thus in either alternative we obtain $d_j^{\{j\}} = -x_j$ and $s_j^{\{j\}} = 0$.

Lemma 2.14 applied to the profile $\{j\}$ at the index $j+1$ then gives either

$$u_{j+1} = s_{j+1}^{\{j\}} \quad \text{or} \quad d_{j+1}^{\{j\}} = -px_j.$$

The latter is impossible because $x_j \neq 0$ and $|d_{j+1}^{\{j\}}| \leq 2$, so the first alternative must hold.

Assume for the sake of contradiction that $u_{j+1} > 0$. Then $s_{j+1}^{\{j\}} > 0$. Combined with $x_{j+1} \leq 0$ the only possibility is $x_{j+1} = 0$, $u_{j+1} = s_{j+1}^{\{j\}} = 1$. Then also $d_{j+1}^{\{j\}} = 1$. Now consider Lemma 2.14 applied to the profile $\{j\}$ at the index $j+2$. If $f \neq 2$, then $j+2 \notin \{j\}$, and the lemma gives either

$$-u_{j+2} = ps_{j+1}^{\{j\}} - s_{j+2}^{\{j\}} \quad \text{or} \quad pd_{j+1}^{\{j\}} = d_{j+2}^{\{j\}}.$$

But $s_{j+1}^{\{j\}} = d_{j+1}^{\{j\}} = 1$ gives a contradiction to both alternatives.

Finally, suppose $f = 2$. Returning to the profile $\{j\}$, since $s_{j+1}^{\{j\}} = s_{j-1}^{\{j\}} = 1$ the first alternative of (3.4) is ruled out. The second alternative with $d_{j+1}^{\{j\}} = d_{j-1}^{\{j\}} = 1$ gives $u_j = p + x_j$ and $u'_j = p$. Since $\varepsilon_{j+1}^K u'_{j+1} = u_{j+1} + px_j - x_{j+1} = 1 + px_j$,

we can rule out $x_j = -2$. Therefore $x_j = -1$, $u_j = p - 1$, $u'_{j+1} = p - 1$, and $K = \{j + 1\}$. To fix ideas suppose that $j = 0$. Then (2.2) gives $t' - t \equiv p - 1 \pmod{p^2 - 1}$. We conclude that $\sigma = (p - 1, 1; t)$ and $\sigma' = (p, p - 1; t + (p - 1))$. However, one can check explicitly for the niveau 2 profile $J = \{0, 3\} \subset \mathbf{Z}/4\mathbf{Z}$ that $\bar{\rho}(\sigma, J) = \omega_0^t \otimes ((\omega'_0)^{2p-1} \oplus (\omega'_0)^{2p^3-p^2}) \notin \mathcal{Z}_{ss}(\sigma')$, a contradiction. We conclude that $u_{j+1} = 0$.

The case (-). Suppose first that $u_i > 0$ for all i . Consider the singleton profiles $\{i\}$. Lemma 2.17(ii) applies to each of these profiles. Therefore $u_i = p + 1 + x_i$ and $u_i = p - d_i^{\{i\}}$ for all i . Comparing, we find that $d_i^{\{i\}} = -x_i - 1$ for all i .

One possibility is that $x_i = -1$ for all i . Then $u_i = p$ for all i , and

$$\varepsilon_i^K u'_i = u_i + px_{i-1} - x_i = u_i - (p - 1)$$

for all i gives $u'_i = 1$ for all i and $K = \emptyset$. Now (2.2) gives $t' = t$ and we obtain $\sigma = (\text{St}; t)$ and $\sigma' = (\text{BT}; t)$. This is the second option in the theorem.

Another possibility is that $x_i = -2$ for all i . Then $u_i = p - 1$ for all i , and

$$\varepsilon_i^K u'_i = u_i + px_{i-1} - x_i = u_i - 2(p - 1).$$

Therefore $u'_i = (p - 1)$ and $K = \mathbf{Z}/f\mathbf{Z}$. Then (2.2) gives $t' = t$ and we obtain $\sigma = \sigma' = (\text{cr}; t)$. This contradicts the hypothesis that $\sigma \neq \sigma'$.

Now suppose neither $x_i = -1$ for all i nor $x_i = -2$ for all i . Then there exists some j with $x_{j-1} = -2$ and $x_j = -1$. From $x_{j-1} = -2$ we get $d_{j-1}^{\{j-1\}} = 1$, while $x_j = -1$ gives $u_j = p$. Lemma 2.14 applied at $j \notin \{j - 1\}$ gives

$$-u_j = ps_{j-1}^{\{j-1\}} - s_j^{\{j-1\}} \quad \text{or} \quad pd_{j-1}^{\{j-1\}} = d_j^{\{j-1\}}.$$

The second alternative is ruled out because $d_{j-1}^{\{j-1\}} = 1$. Since $u_j = p$, the first alternative yields $s_j^{\{j-1\}} = 0$. Now Lemma 2.17(i) implies $j + 1 \in \{j - 1\}$. We conclude in this case that $f = 2$. Supposing without loss of generality that $x_0 = -1$ we have $\sigma = (p, p - 1; t)$. Lemma 2.1 then gives $(u'_0, u'_1) = (p - 1, 1)$ and $K = \{0\}$, whence $t' = t - (p - 1)$ by (2.2). Therefore $\sigma' = (p - 1, 1; t - (p - 1))$. This is exactly the reverse of the $f = 2$ subcase of (N.b). One can check explicitly for the niveau 2 profile $J = \{0, 1\} \subset \mathbf{Z}/4\mathbf{Z}$ that $\bar{\rho}(\sigma, J) = \omega_0^{t-(p-1)} \otimes ((\omega'_0)^{2p-p^2} \oplus (\omega'_0)^{2p^3-1}) \notin \mathcal{Z}_{ss}(\sigma')$, another contradiction. This completes the case where $u_i > 0$ for all i .

We may therefore suppose that $u_i = 0$ for some i . We aim to prove that $u_i = 0$ for all i . Without loss of generality suppose that $u_0 = 0$. Lemma 3.2 gives $(u_{i-1}, u_i) \neq (0, p)$ for all i , and then Lemma 2.16 gives $P \notin \mathcal{P}(\sigma, \sigma')$. Since $\mathcal{Z}_{ss}(\sigma) \subseteq \mathcal{Z}_{ss}(\sigma')$, the conclusion $P \notin \mathcal{P}(\sigma, \sigma')$ implies that $\bar{\rho}(\sigma, P)$ is scalar. Now Corollary 2.6 gives integers $y_i \in [-1, 1]$ such that

$$\varepsilon_i^P u_i = py_{i-1} - y_i$$

for $0 < i < f$. Since $y_{f-1} = -y_{-1}$ we also have $\varepsilon_0^P u_0 = -py_{f-1} - y_0$. The assumption $u_0 = 0$ forces $y_0 = y_{f-1} = 0$. For $0 < i < f$, by definition $\varepsilon_i^P > 0$ if and

only if $u_i \neq p$, and so we have the following possibilities:

u_i	y_{i-1}	y_i
0	0	0
1	0	-1
$p-1$	1	1
p	-1	0

If some y_i is nonzero, choose the first such $i > 0$. Then $y_{i-1} = 0$, so the table gives $u_i = 1$ and $y_i = -1$. Applying the table at $i+1$ gives $u_{i+1} = p$. But $(u_i, u_{i+1}) = (1, p)$ contradicts Lemma 2.15; therefore $y_i = 0$ for all i , and $u_i = 0$ for all i .

Finally, from $u_i = 0$ for all i we have $\varepsilon_i^K u'_i = px_{i-1} - x_i$ for all i . If some x_{i-1} were -2 , we would obtain a contradiction to $u'_i \in [0, p]$. Therefore $x_i = -1$ and $u'_i = p-1$ for all i , with $K = \mathbf{Z}/f\mathbf{Z}$. It follows that $t' = t$ and $\sigma, \sigma' = (\text{ur}; t), (\text{cr}; t)$, as claimed. $\square$

4. FOUR INCLUSIONS

In this section we prove the following key result.

Theorem 4.1. *Suppose $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$, equipped with a minimal pinning. Assume $\sigma \neq \sigma'$, and that $\{\sigma, \sigma'\} \neq \{(\text{BT}; t), (\text{St}; t)\}$ for all t . Then*

- In case (+) we have $\sigma, \sigma' = (\text{ur}; t), (\text{cr}; t)$ for some t .
- In case (P.a), and for j as in that case, we have $\mathcal{Z}_{\text{ss}}(\mu_j(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$.
- In case (P.b), and for j as in that case, we have $\mathcal{Z}_{\text{ss}}(\theta_j(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$.
- In case (N.a), and for j' as in that case, we have $\mathcal{Z}_{\text{ss}}(\nu_{j'}(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$.
- In case (N.b), and for j as in that case, we have $\mathcal{Z}_{\text{ss}}(\theta_j(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$.
- In case (−) we have $\sigma, \sigma' = (\text{ur}; t), (\text{cr}; t)$ for some t .

Moreover at least one of these cases holds.

Remark 4.2. If $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ is equipped with a minimal pinning, then by Remark 2.5 together with the minimality hypothesis, we have $\sigma = \sigma'$ if and only if $x_i = 0$ for all i . Thus the hypothesis $\sigma \neq \sigma'$ serves only to ensure that at least one of the cases in the theorem holds.

Remark 4.3. If $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ with σ' critical, and $\sigma \neq \sigma'$, then Proposition 2.9 implies that σ is scalar. But then either $x_i = 1$ for all i , or $x_i = -1$ for all i . In particular, if σ' is critical in Theorem 4.1 then it is either (+) or (−) that holds, and not one of the four middle cases.

The same argument shows that if $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ and furthermore $u_j = x_{j'} = 0$ for some j, j' , then σ' is not critical: if it were, $u_j = 0$ implies that σ is scalar, and $x_{j'} = 0$ provides a contradiction.

Remark 4.4. Crucially, each of the operations in the four middle cases of Theorem 4.1 is valid and non-invertible. To begin with, in the applications of μ_j and θ_j above we have $u_{j+1} = 0$ by Theorem 3.3, while $u_{j'} = 0$ in the application of $\nu_{j'}$ by definition. Then:

- In case (P.a) we are given $u_j > 0$, and since $x_j > 0$, $u_{j+1} = 0$ we have $u_j \neq 1$ by Lemma 3.2(i). Therefore μ_j is valid and non-invertible.

- In case (P.b), the operation θ_j is valid because $u_j \neq p$ (indeed we are given $u_j = 0$), and θ_j is always non-invertible.
- In case (N.a), we are given $x_{j'} < 0$, $u_{j'} = 0$. Therefore $u_{j'+1} \neq p$ by Lemma 3.2(ii), and $\nu_{j'}$ is valid and non-invertible.
- In case (N.b), if $u_j = p$ then we would have $u'_j = p + px_{j-1} - x_j > p$. Therefore $u_j \neq p$, and so θ_j is valid and non-invertible.

The cases (+) and (−) of Theorem 4.1 have already been proved in Theorem 3.3. To organize the rest of the proof, each of the other four cases will be established as a separate proposition, stated somewhat more generally than above. Before this, however, we record some features common to all four cases.

Definition 4.5. If σ is a phantom Serre weight and J is a profile of niveau 1, we set

$$\mathcal{L}(\sigma, J) = t + \sum_{i \in J} p^{-i} u_i, \quad \mathcal{R}(\sigma, J) = t + \sum_{i \in J^c} p^{-i} u_i \quad \text{in } \mathbf{Z}/(q-1)\mathbf{Z}.$$

Similarly if J has niveau 2, we set

$$\mathcal{L}(\sigma, J) = (q+1)t + \sum_{i \in J} p^{-i} u_i, \quad \mathcal{R}(\sigma, J) = (q+1)t + \sum_{i \in J^c} p^{-i} u_i \quad \text{in } \mathbf{Z}/(q^2-1)\mathbf{Z}.$$

Thus

$$\bar{\rho}(\sigma, J) = \begin{cases} \omega_0^{\mathcal{L}(\sigma, J)} \oplus \omega_0^{\mathcal{R}(\sigma, J)} & J \text{ of niveau 1} \\ (\omega'_0)^{\mathcal{L}(\sigma, J)} \oplus (\omega'_0)^{\mathcal{R}(\sigma, J)} & J \text{ of niveau 2.} \end{cases}$$

Also set $d(\sigma) := 2t + \sum_{i \in \mathbf{Z}/f\mathbf{Z}} p^{-i} u_i$, so that $\det \bar{\rho}(\sigma, J) = \omega_0^{d(\sigma)}$ for all profiles J . If $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$, then evidently we must have $d(\sigma) = d(\sigma')$.

Definition 4.6. We write

$$\mathcal{L}(\sigma) = \{\mathcal{L}(\sigma, J) : \text{all profiles } J\} \subset \mathbf{Z}/(q-1)\mathbf{Z} \coprod \mathbf{Z}/(q^2-1)\mathbf{Z}$$

and similarly for $\mathcal{R}(\sigma)$.

Remark 4.7. Since $\mathcal{R}(\sigma, J) = d(\sigma) - \mathcal{L}(\sigma, J)$ for all niveau 1 profiles (with an analogous formula for niveau 2 profiles) it follows that if σ' is non-critical then $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ if and only if $d(\sigma) = d(\sigma')$ and $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\sigma')$. If σ' is critical, the ‘if’ direction still holds; the problem with the ‘only-if’ direction is that $\bar{\rho}(\sigma, J)$ might be scalar for some niveau 2 profiles J , and the associated values of $\mathcal{L}(\sigma)$ will not be in the $\mathbf{Z}/(q^2-1)\mathbf{Z}$ part of $\mathcal{L}(\sigma')$.

One checks easily from the definitions that $d(f_i(\sigma)) = d(\sigma)$ for any valid operation f_i . If we are given $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$, then the value $d := d(\sigma)$ is common to all three of $\sigma, \sigma', f_i(\sigma)$. So, if we want to establish that $\mathcal{Z}_{\text{ss}}(f_i(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$, it suffices to prove that $\mathcal{L}(f_i(\sigma)) \subseteq \mathcal{L}(\sigma')$.

Since J is a profile if and only if J^c is a profile and $\mathcal{R}(\sigma, J) = \mathcal{L}(\sigma, J^c)$, we see immediately that $\mathcal{L}(\sigma) = \mathcal{R}(\sigma)$ for any σ . Thus $\mathcal{L}(f_i(\sigma), J) \in \mathcal{L}(\sigma')$ if and only if $\mathcal{R}(f_i(\sigma), J) \in \mathcal{R}(\sigma')$, if and only if $\mathcal{L}(f_i(\sigma), J^c) \in \mathcal{L}(\sigma')$. Thus to prove the inclusion $\mathcal{L}(f_i(\sigma)) \subseteq \mathcal{L}(\sigma')$, it is enough to establish for all J that either $\mathcal{L}(f_i(\sigma), J)$ or $\mathcal{L}(f_i(\sigma), J^c)$ is in $\mathcal{L}(\sigma')$; that is, one only needs to consider one profile out of each pair $\{J, J^c\}$.

In fact, for each f_i there is a family of profiles J for which one sees in one line from the definitions that $\mathcal{L}(f_i(\sigma), J) = \mathcal{L}(\sigma, J')$ for suitable J' . If σ' is non-critical

then $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\sigma')$, and so we conclude $\mathcal{L}(f_i(\sigma), J) \in \mathcal{L}(\sigma')$ for such J .

f_i	condition on J	J'
μ_i	$i \in J, i+1 \in J$	J
θ_i	$i \in J, i+1 \notin J$	J
ν_i	$i \in J, i+1 \notin J$	$J \triangle \{i+1\}$

Here and throughout we will use the following convention. If J has niveau 1 and $S \subset \mathbf{Z}/f\mathbf{Z}$, then the symmetric difference $J \triangle S$ has its usual meaning. However, if J has niveau 2, then S should be viewed as a subset of $\mathbf{Z}/f\mathbf{Z}$ extended f -periodically to a subset of $\mathbf{Z}/2f\mathbf{Z}$, so that $J \triangle S$ is again a profile. Thus in the niveau 2 case the last line of the table should be conventionally interpreted as the usual symmetric difference $J \triangle \{i+1, i+f+1\}$.

Thanks to the discussion in the previous two paragraphs, it remains to establish the following.

- For an operation μ_j : that $\mathcal{L}(\mu_j(\sigma), J) \in \mathcal{L}(\sigma')$ when $j \in J, j+1 \notin J$.
- For an operation θ_j : that $\mathcal{L}(\theta_j(\sigma), J) \in \mathcal{L}(\sigma')$ when $j \in J, j+1 \in J$.
- For an operation $\nu_{j'}$: that $\mathcal{L}(\nu_{j'}(\sigma), J) \in \mathcal{L}(\sigma')$ when $j' \in J, j'+1 \in J$.

We also note the following lemma.

Lemma 4.8. *Fix a pinning of σ, σ' . As J varies over $\mathcal{P}(\sigma, \sigma')$ the product $\varepsilon_i^J x_i^J$ takes at most two values for each i .*

Proof. The exceptional $p = 3$ case can be checked directly, so we can assume we are not in that case. Multiplying the equation $\varepsilon_i^{K_J} u'_i - \varepsilon_i^J u_i = px_{i-1}^J - x_i^J$ by ε_i^J , reducing mod p , and rearranging gives $u_i \pm u'_i \equiv \varepsilon_i^J x_i^J \pmod{p}$. Thus $\varepsilon_i^J x_i^J$ takes at most two different values mod p . But $-2, 0, 2$ are distinct mod p for all odd p , so in fact $\varepsilon_i^J x_i^J$ takes at most two different values in $\mathbf{Z}$. $\square$

We are now ready to proceed with the four cases.

Proposition 4.9. *Suppose $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$, equipped with a pinning. Suppose that $x_{j-1} \leq 0, u_j > 1$, and that, treating j as an integer, there exists $j' \geq j$ with*

$$x_j, x_{j+1}, \dots, x_{j'} > 0, \quad x_{j'+1} = 0, \quad u_{j+1} = \dots = u_{j'+1} = 0.$$

Then $\mathcal{Z}_{\text{ss}}(\mu_j(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$.

In particular this conclusion holds in case (P.a) if $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ is equipped with a minimal pinning.

Proof. If (P.a) holds and $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ is equipped with a minimal pinning, then case (P) of Theorem 3.3 establishes the existence of j' such that $x_j, x_{j+1}, \dots, x_{j'} > 0, x_{j'+1} = 0$, and $u_{j+1} = \dots = u_{j'+1} = 0$. Furthermore $u_j > 1$ by Lemma 3.2(i). Thus the final sentence of the Proposition follows from the first part.

Note that since $x_i \neq 2$ for all i , we have $x_j = x_{j+1} = \dots = x_{j'} = 1$. The last sentence of Remark 4.3 shows that σ' is non-critical. In particular $\mathcal{P}(\sigma, \sigma')$ contains every profile, and $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\sigma')$.

Put $T = \{j+1, j+2, \dots, j'+1\}$. Lemma 2.1 applied at $i \in T$ gives

$$(4.10) \quad \varepsilon_i^{K_J} u'_i = px_{i-1}^J - x_i^J.$$

Taking $J = \emptyset$, our hypotheses on the x_i 's give

$$u'_i = \begin{cases} p-1 & i \in T \setminus \{j'+1\} \\ p & i = j'+1 \end{cases}$$

and $K \cap T = \emptyset$. Feeding this back into (4.10), we obtain

$$x_{i-1}^J = x_i^J = \varepsilon_i^{K_J}$$

for $i \in T \setminus \{j'+1\}$, while $x_{j'+1}^J = 0$ and $\varepsilon_{j'+1}^{K_J} = x_{j'}^J$. That is, either

$$x_j^J = \cdots = x_{j'}^J = 1, \quad T \cap K_J = \emptyset \quad \text{or} \quad x_j^J = \cdots = x_{j'}^J = -1, \quad T \subset K_J,$$

along with $x_{j'+1}^J = 0$.

By the discussion immediately before the statement of the Proposition, it suffices to prove that $\mathcal{L}(\mu_j(\sigma), J) \in \mathcal{L}(\sigma')$ for profiles J such that $j \in J$ and $j+1 \notin J$. Fix such a profile. We have

$$\mathcal{L}(\mu_j(\sigma), J) = \mathcal{L}(\sigma, J) - \begin{cases} p^{-j} & J \text{ of niveau 1} \\ (1-q)p^{-j} & J \text{ of niveau 2.} \end{cases}$$

Set $\lambda = 1$ if J has niveau 1 and $\lambda = 1-q$ if J has niveau 2, so that we can uniformly write $\mathcal{L}(\mu_j(\sigma), J) = \mathcal{L}(\sigma, J) - \lambda \cdot p^{-j}$. This is an equation in $\mathbf{Z}/(q-1)\mathbf{Z}$ if J has niveau 1, and in $\mathbf{Z}/(q^2-1)\mathbf{Z}$ if J has niveau 2.

Observing that $\sum_{i=j+1}^{j'+1} u'_i p^{-i} = p^{-j}$, if $T \subset K_J$ then

$$\mathcal{L}(\mu_j(\sigma), J) = \mathcal{L}(\sigma', K_J \triangle T)$$

as desired, so it remains to consider the case $T \cap K_J = \emptyset$, whence $x_j^J = 1$.

Lemma 2.14 at the index j for the profile J gives the alternatives

$$(4.11) \quad s_{j-1}^J = s_j^J = 0 \quad \text{or} \quad u_j = p d_{j-1}^J - d_j^J.$$

Since $x_j = x_j^J = 1$, we have $s_j^J = 1$ and $d_j^J = 0$. This rules out the first alternative, and the second alternative becomes $u_j = p d_{j-1}^J$. Since $u_j > 1$ we get $u_j = p$ and $d_{j-1}^J = 1$.

Now either $x_{j-1} = -1$ or $x_{j-1} = 0$. (The first part of Lemma 3.2 rules out $x_{j-1} = -2$, since $x_j = 1$.) If $x_{j-1} = -1$, then $x_{j-1}^J = 1$ and Lemma 2.1 for the profile J gives

$$\varepsilon_j^{K_J} u'_j + u_j = p x_{j-1}^J - x_j^J = p - 1.$$

Since $u_j = p$ we have $u'_j = 1$ and $j \in K_J$. Then

$$\mathcal{L}(\sigma', K_J \triangle \{j\}) = \mathcal{L}(\sigma', K_J) - \lambda \cdot p^{-j} = \mathcal{L}(\sigma, J) - \lambda \cdot p^{-j} = \mathcal{L}(\mu_j(\sigma), J)$$

as desired.

If instead $x_{j-1} = 0$, then $x_{j-1}^J = 2$, and Lemma 2.1 for the profile J gives $\varepsilon_j^{K_J} u'_j + u_j = 2p - 1$, so that $j \notin K_J$ and $u'_j = p - 1$. Now consider Lemma 2.1 applied to the profile $J^- := J \triangle \{j\}$. We find that

$$\varepsilon_j^{K_{J^-}} (p - 1) - p = p x_{j-1}^{J^-} - x_j^{J^-}.$$

If $\varepsilon_j^{K_{J^-}} = -1$ then $x_{j-1}^{J^-} = -2$. But since J^- and J either both contain $j-1$ or both do not, this would give $\{x_{j-1}, \varepsilon_{j-1}^{J^-} x_{j-1}^{J^-}, \varepsilon_{j-1}^J x_{j-1}^J\} = \{-2, 0, 2\}$, in contradiction to

Lemma 4.8. It follows that $\varepsilon_j^{K_{J^-}} = 1$, and $j \notin K_{J^-}$. Finally

$$\begin{aligned} \mathcal{L}(\sigma', K_{J^-} \triangle \{j\}) &= \mathcal{L}(\sigma', K_{J^-}) + \lambda \cdot (p-1)p^{-j} \\ &= \mathcal{L}(\sigma, J^-) + \lambda \cdot (p-1)p^{-j} \\ &= \mathcal{L}(\sigma, J) - \lambda \cdot p^{-j} \\ &= \mathcal{L}(\mu_j(\sigma), J) \end{aligned}$$

and this completes the proof. $\square$

Proposition 4.12. *Suppose $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$, equipped with a pinning. Suppose that $x_{j-1} \leq 0$, $u_j = 0$, and that, treating j as an integer, there exists $j' \geq j$ with*

$$x_j, x_{j+1}, \dots, x_{j'} > 0, \quad x_{j'+1} = 0, \quad u_{j+1} = \dots = u_{j'+1} = 0.$$

Then $\mathcal{Z}_{\text{ss}}(\theta_j(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$.

In particular this conclusion holds in case (P.b) if $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ is equipped with a minimal pinning.

Proof. If (P.b) holds and $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ is equipped with a minimal pinning, then case (P) of Theorem 3.3 establishes the existence of j' such that $x_j, x_{j+1}, \dots, x_{j'} > 0$, $x_{j'+1} = 0$, and $u_{j+1} = \dots = u_{j'+1} = 0$. Thus the final sentence follows.

The last sentence of Remark 4.3 shows that σ' is non-critical. In particular $\mathcal{P}(\sigma, \sigma')$ contains every profile, and $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\sigma')$.

Since $x_i \neq 2$ for all i , we have $x_j = x_{j+1} = \dots = x_{j'} = 1$. Now $\varepsilon_j^K u'_j = px_{j-1} - 1$, and since $x_{j-1} \leq 0$, this forces

$$x_{j-1} = 0, \quad j \in K, \quad u'_j = 1.$$

At the other end of the interval we have

$$\varepsilon_{j'+1}^K u'_{j'+1} = px_{j'} - x_{j'+1} = p,$$

and therefore

$$j' + 1 \notin K, \quad u'_{j'+1} = p.$$

By the discussion immediately before Proposition 4.9, it suffices to prove that $\mathcal{L}(\theta_j(\sigma), J) \in \mathcal{L}(\sigma')$ for profiles J such that $j, j+1 \in J$. Fix such a profile, and set $\lambda = 1$ if J has niveau 1 and $\lambda = 1 - q$ if J has niveau 2. Then

$$\mathcal{L}(\theta_j(\sigma), J) = \mathcal{L}(\sigma, J) + \lambda \cdot p^{-j}.$$

Lemma 2.1 at the index j gives $\varepsilon_j^{K_J} = px_{j-1}^J - x_j^J$, so that

$$x_{j-1}^J = 0, \quad x_j^J \in \{-1, 1\}.$$

Similarly at $j' + 1$ we have $\varepsilon_{j'+1}^{K_J} p = px_{j'}^J - x_{j'+1}^J$ and therefore $x_{j'+1}^J = 0$.

Suppose first that $x_j^J = 1$. Put $T = \{j+1, j+2, \dots, j'+1\}$. Lemma 2.1 for J at an index $i \in T$ gives $\varepsilon_i^{K_J} u'_i = px_{i-1}^J - x_i^J$, since $u_i = 0$ for $i \in T$. Multiplying by p^{-i} and summing gives

$$\sum_{i \in T} \varepsilon_i^{K_J} u'_i p^{-i} = x_j^J p^{-j} - x_{j'+1}^J p^{-(j'+1)} = p^{-j}.$$

Consequently

$$\mathcal{L}(\sigma', K_J \triangle T) = \mathcal{L}(\sigma', K_J) + \lambda \cdot p^{-j} = \mathcal{L}(\theta_j(\sigma), J)$$

and the Proposition follows in this case.

If instead $x_j^J = -1$, we have $\varepsilon_j^{K_J} = -x_j^J = 1$ and $j \notin K_J$. Since $u'_j = 1$, we obtain

$$\mathcal{L}(\sigma', K_J \triangle \{j\}) = \mathcal{L}(\sigma', K_J) + \lambda \cdot p^{-j} = \mathcal{L}(\theta_j(\sigma), J).$$

This completes the proof. $\square$

Proposition 4.13. *Suppose $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$, equipped with a pinning. Suppose $j' \geq j$ are integers such that*

$$x_{j-1} = 0, \quad x_j, x_{j+1}, \dots, x_{j'} < 0, \quad x_{j'+1} \leq 0,$$

and

$$u_j = u_{j+1} = \dots = u_{j'} = 0, \quad 0 < u_{j'+1} < p.$$

Then $\mathcal{Z}_{\text{ss}}(\nu_{j'}(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$.

In particular this conclusion holds in case (N.a) if $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ is equipped with a minimal pinning.

Proof. The final sentence follows because Lemma 3.2(ii) rules out $u_{j'+1} = p$ in case (N.a). The last sentence of Remark 4.3 shows that σ' is non-critical. In particular $\mathcal{P}(\sigma, \sigma')$ contains every profile, and $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\sigma')$.

If $u_{i+1} = 0$ and $x_i = -2$ then Lemma 2.1 at $i+1$ gives $u'_{i+1} = |px_i - x_{i+1}| \geq 2p - 2 > p$, a contradiction. Hence $x_i = -1$ for $j \leq i < j'$ while $x_{j'} \in \{-1, -2\}$.

For each $j \leq i \leq j'$, since $u_i = 0$, Lemma 2.14 for any profile J gives either $s_i^J = s_{i-1}^J = 0$ or $d_i^J = d_{i-1}^J = 0$ (but not both, since $x_i \neq 0$). Thus either $s_i^J = 0$ for all $j-1 \leq i \leq j'$ or $d_i^J = 0$ for all $j-1 \leq i \leq j'$. In any case we have $x_i^J = \pm x_i$ for all $j-1 \leq i \leq j'$.

By the discussion preceding Proposition 4.9, it suffices to consider profiles J such that $j', j'+1 \in J$. Fix such a profile, and set

$$J^- = J \triangle \{j'+1\}.$$

Thus $j'+1 \notin J^-$. If $\lambda = 1$ in niveau 1 and $\lambda = 1 - q$ in niveau 2, then a direct calculation gives

$$(4.14) \quad \mathcal{L}(\nu_{j'}(\sigma), J) = \mathcal{L}(\sigma, J^-) + \lambda \cdot p^{-j'}.$$

For each i we have $\varepsilon_i^{K_{J^-}} u'_i - \varepsilon_i^{J^-} u_i = px_{i-1}^{J^-} - x_i^{J^-}$. Multiplying by p^{-i} , summing from j to j' , and using $u_i = 0$ in that range gives

$$\sum_{i=j}^{j'} \varepsilon_i^{K_{J^-}} u'_i p^{-i} = -x_{j'}^{J^-} p^{-j'}.$$

Summing instead from j to $j'+1$ gives

$$\sum_{i=j}^{j'+1} \varepsilon_i^{K_{J^-}} u'_i p^{-i} = (u_{j'+1} - x_{j'+1}^{J^-}) p^{-j'-1}.$$

Thus if $x_{j'}^{J^-} = -1$ then

$$\mathcal{L}(\sigma', K_{J^-} \triangle \{j, \dots, j'\}) = \mathcal{L}(\sigma', K_{J^-}) + \lambda \cdot p^{-j'} = \mathcal{L}(\sigma, J^-) + \lambda \cdot p^{-j'} = \mathcal{L}(\nu_{j'}(\sigma), J),$$

while if $u_{j'+1} - x_{j'+1}^{J^-} = p$ then similarly

$$\mathcal{L}(\sigma', K_{J^-} \triangle \{j, \dots, j'+1\}) = \mathcal{L}(\nu_{j'}(\sigma), J).$$

We can therefore suppose for the rest of the argument that neither $x_{j'}^{J^-} = -1$ nor $u_{j'+1} - x_{j'+1}^{J^-} = p$. Lemma 2.14 for J^- at the index $j' + 1$ gives the alternatives

$$(4.15) \quad -u_{j'+1} = ps_{j'}^{J^-} - s_{j'+1}^{J^-} \quad \text{or} \quad d_{j'}^{J^-} = d_{j'+1}^{J^-} = 0.$$

In the second alternative $x_{j'}^{J^-} = x_{j'}$. By hypothesis this is not -1 , so it is -2 . Now Lemma 2.1 gives $\varepsilon_{j'+1}^{K_{J^-}} u'_{j'+1} = -2p + (u_{j'+1} - x_{j'+1})$. Since by hypothesis $u_{j'+1} - x_{j'+1} \neq p$, but also $u_{j'+1} < p$, the only possibility is $u_{j'+1} = u'_{j'+1} = p - 1$ and $x_{j'+1} = -2$. We claim this is impossible. To see this let J' be any profile containing $j' + 2$ but not $j' + 1$. Lemma 2.1 at the index $j' + 1$ gives

$$(\varepsilon_{j'+1}^{K_{J'}} - 1)(p - 1) = px_{j'}^{J'} - x_{j'+1}^{J'}.$$

Using our observation in the third paragraph of the proof that $x_{j'}^{J'} = \pm x_{j'} = \pm 2$, the preceding equation has exactly one solution, with $\varepsilon_{j'+1}^{K_{J'}} = -1$ and $x_{j'}^{J'} = x_{j'+1}^{J'} = -2$. But then Lemma 2.1 at the index $j' + 2$ gives

$$\varepsilon_{j'+2}^{K_{J'}} u'_{j'+2} + u_{j'+2} = -2p - x_{j'+2}^{J'},$$

and this is a contradiction because the left-hand side is at least $-p$ while the right-hand side is at most $-2p + 2$.

So it is the first alternative in (4.15) that must hold. If $s_{j'}^{J^-} = -1$, then by the observation in the third paragraph of the proof we must have $d_{j'}^{J^-} = 0$. But together these give $x_{j'}^{J^-} = -1$, contradicting our running hypothesis. The last remaining possibility is that $s_{j'}^{J^-} = 0$ and $u_{j'+1} = s_{j'+1}^{J^-} = 1$. Since $x_{j'+1} = 0$ we have $x_{j'+1}^{J^-} = 2$. Lemma 2.1 at the index $j' + 1$ is

$$\varepsilon_{j'+1}^{K_{J^-}} u'_{j'+1} - 1 = px_{j'}^{J^-} - 2$$

and so we find that $x_{j'}^{J^-} = 1$, $x_{j'} = -1$, $u'_{j'+1} = p - 1$, and $j' + 1 \notin K_{J^-}$.

If $j' + 2 \notin J$, then $j' + 2 \notin J^-$, and Lemma 2.1 at $j' + 2$ for the profile J^- gives

$$\varepsilon_{j'+2}^{K_{J^-}} u'_{j'+2} - u_{j'+2} + x_{j'+2}^{J^-} = 2p.$$

The left-hand side is at most $p + 2 < 2p$, a contradiction. It follows that $j' + 2 \in J$. Applying the same lemma for the profile J then gives

$$\varepsilon_{j'+2}^{K_J} u'_{j'+2} + u_{j'+2} + x_{j'+2}^J = px_{j'+1}^J.$$

This is impossible if $x_{j'+1}^J = -2$, so $x_{j'+1}^J \in \{0, 2\}$. But in Lemma 4.8 applied at the index $j' + 1 \in J$, the three values for the profiles $\emptyset, J^-, J$ are $0, 2, -x_{j'+1}^J$ respectively. The Lemma thus rules out $x_{j'+1}^J = 2$, and we deduce that $x_{j'+1}^J = 0$. Lemma 2.1 at $j' + 1$ for the profile J then gives $\varepsilon_{j'+1}^{K_J}(p - 1) + 1 = px_{j'}^J$, and we conclude that $j' + 1 \notin K_J$.

Finally, a pair of direct calculations using $u'_{j'+1} = p - 1$ and $u_{j'+1} = 1$ gives

$$\mathcal{L}(\sigma', K_J \triangle \{j' + 1\}) = \mathcal{L}(\sigma, J) + \lambda \cdot (p - 1)p^{-(j'+1)} = \mathcal{L}(\nu_{j'}(\sigma), J)$$

and the proof is complete. $\square$

Before addressing the fourth and final case, we note the following lemma.

Lemma 4.16. *Suppose $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$, equipped with a pinning. Suppose $x_i \leq 0$ for all i , and*

$$x_{j-1} = 0, \quad x_j < 0, \quad u_j > 0, \quad u_{j+1} = 0.$$

Treating j as an integer, let $r > j$ be minimal such that $x_r = 0$. Then

$$u_{j+1} = u_{j+2} = \cdots = u_r = 0.$$

Proof. If σ' is critical then σ is scalar and the result is immediate, so we may assume that σ' is non-critical. Lemma 2.1 at $j+1$ rules out $x_j = -2$, and hence $x_j = -1$. Lemma 2.1 at j then gives

$$j \notin K, \quad u'_j = u_j + 1, \quad 0 < u_j < p.$$

Suppose for the sake of contradiction that some u_i with $j < i \leq r$ is positive, and let m be the least such index. If $m+1 \not\equiv j \pmod{f}$, let $J = \{j\}$. If $m+1 \equiv j \pmod{f}$, take J to be the niveau 2 profile satisfying $J \cap \{j, j+1, \dots, j+f-1\} = \{j\}$. Since σ' is non-critical, in either case $J \in \mathcal{P}(\sigma, \sigma')$.

Lemma 2.1 at the index j for the profile J is

$$\varepsilon_j^{K_J}(u_j + 1) + u_j = px_{j-1}^J - x_j^J.$$

Parity and the bounds $0 < u_j < p$ force $x_j^J = 1$. Hence $s_j^J = 0$ and $d_j^J = 1$.

For $j < i < m$, we have $u_i = 0$ and $i \notin J$. The first alternative of Lemma 2.14 gives $s_{i-1}^J = s_i^J = 0$ while the second gives $d_{i-1}^J = d_i^J = 0$. Furthermore s_i^J, d_i^J are not both 0 for i in this range because $x_i < 0$. Since $d_j^J = 1$, it follows iteratively that the first alternative must be the one that holds for all i in this range, and

$$s_j^J = s_{j+1}^J = \cdots = s_{m-1}^J = 0.$$

Applying Lemma 2.14 at $m \notin J$, its second alternative is again impossible, while the first gives $u_m = s_m^J$.

If $m < r$, then $x_m < 0$, and therefore $s_m^J \leq 0$, a contradiction. Thus $m = r$. Since $x_r = 0$, the equality $u_r = s_r^J > 0$ forces $u_r = s_r^J = 1$ and $x_r^J = 2$. But $r+1 = m+1 \notin J$, so Lemma 2.1 at $r+1$ gives

$$\varepsilon_{r+1}^{K_J} u'_{r+1} - u_{r+1} = 2p - x_{r+1}^J.$$

The left-hand side is at most p , whereas the right-hand side is at least $2p - 2 > p$. This contradiction proves the lemma. $\square$

Proposition 4.17. *Suppose $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$, equipped with a pinning. Suppose $x_i \leq 0$ for all i , and that*

$$x_{j-1} = 0, \quad x_j < 0, \quad u_{j+1} = 0.$$

Suppose moreover that either $u_j > 0$; or $u_i = 0$ for all i ; or else $u_j = 0$, not all u_i are zero, and, treating j as an integer and writing $j' \geq j$ for the least integer such that $u_{j'+1} > 0$, there exists r with

$$j \leq r \leq j', \quad x_r = 0.$$

Then $\mathcal{Z}_{\text{ss}}(\theta_j(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$.

In particular this conclusion holds in case (N.b) if $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ is equipped with a minimal pinning.

Proof. The final sentence follows from Theorem 3.3, which gives $u_{j+1} = 0$ in case (N.b), together with the description of the failure of the additional condition in (N.a). The last sentence of Remark 4.3 shows that σ' is non-critical. In particular $\mathcal{P}(\sigma, \sigma')$ contains every profile, and $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\sigma')$.

We first claim that there exists an integer $r > j$ such that

$$(4.18) \quad x_j, x_{j+1}, \dots, x_{r-1} < 0, \quad x_r = 0, \quad u_{j+1} = \dots = u_r = 0.$$

If $u_j > 0$ or if $u_i = 0$ for all i , take $r > j$ minimal such that $x_r = 0$ (and in the former case, apply Lemma 4.16). In the remaining case take r to be the least index in $[j, j']$ for which $x_r = 0$; the definition of j' gives $u_{j+1} = \dots = u_r = 0$.

Lemma 2.1 at $j+1$, together with $u_{j+1} = 0$, rules out $x_j = -2$, so $x_j = -1$. Lemma 2.1 at j then gives

$$j \notin K, \quad u'_j = u_j + 1, \quad u_j < p.$$

Lemma 2.1 at r gives $\varepsilon_r^K u'_r = px_{r-1}$, and since $x_{r-1} < 0$ we have

$$x_{r-1} = -1, \quad r \in K, \quad u'_r = p.$$

By the discussion preceding Proposition 4.9, it suffices to consider profiles J such that $j, j+1 \in J$. Fix such a profile, and set $\lambda = 1$ in niveau 1 and $\lambda = 1 - q$ in niveau 2. We have

$$\mathcal{L}(\theta_j(\sigma), J) = \mathcal{L}(\sigma, J) + \lambda \cdot p^{-j}.$$

Lemma 2.1 at j is

$$\varepsilon_j^{K_J}(u_j + 1) + u_j = px_{j-1}^J - x_j^J.$$

If $u_j > 0$ then since $u_j \neq p$ we have $x_j^J = 1$. If $u_j = 0$ then instead $x_{j-1}^J = 0$ and $x_j^J \in \{-1, 1\}$. Note that $x_j^J = -1$ forces $j \notin K_J$. At r we have $\varepsilon_r^{K_J} p = px_{r-1}^J - x_r^J$, and therefore $x_r^J = 0$.

Suppose first that $x_j^J = 1$. Put

$$T = \{j+1, j+2, \dots, r\}.$$

Taking Lemma 2.1 for $i \in T$, multiplying by p^{-i} , and summing gives

$$\sum_{i \in T} \varepsilon_i^{K_J} u'_i p^{-i} = x_j^J p^{-j} - x_r^J p^{-r} = p^{-j},$$

where we use (4.18) to see that $u_i = 0$ in this range. It follows that

$$\mathcal{L}(\sigma', K_J \triangle T) = \mathcal{L}(\sigma, J) + \lambda \cdot p^{-j} = \mathcal{L}(\theta_j(\sigma), J).$$

The other possibility is that $u_j = 0$ and $x_j^J = -1$. In this case $j \notin K_J$ and $u'_j = 1$, so again

$$\mathcal{L}(\sigma', K_J \triangle \{j\}) = \mathcal{L}(\sigma, J) + \lambda \cdot p^{-j} = \mathcal{L}(\theta_j(\sigma), J)$$

and we are done. $\square$

5. NON-INVERTIBLE OPERATIONS

The final ingredient we need for the proof of the main theorem is that non-invertible valid operations f_j give strict inclusions. Note that this is clear for an inclusion of stacks $\mathcal{Z}(\sigma) \subseteq \mathcal{Z}(f_j(\sigma))$, for dimension reasons, but requires justification for sets of semisimple points.

Proposition 5.1. *Suppose that $f_j \in \{\mu_j, \theta_j, \nu_j\}$ is valid and non-invertible on σ . Then*

$$\mathcal{Z}_{\text{ss}}(\sigma) \subsetneq \mathcal{Z}_{\text{ss}}(f_j(\sigma)).$$

More precisely:

- if $f_j = \mu_j$ then $\bar{\rho}(\mu_j(\sigma), \{j+1\}) \notin \mathcal{Z}_{\text{ss}}(\sigma)$;
- if $f_j = \theta_j$ then $\bar{\rho}(\theta_j(\sigma), \emptyset) \notin \mathcal{Z}_{\text{ss}}(\sigma)$;
- if $f_j = \nu_j$ then $\bar{\rho}(\nu_j(\sigma), \emptyset) \notin \mathcal{Z}_{\text{ss}}(\sigma)$.

Proof. Write $\sigma = (u_i; t)$. Since f_j is valid, $u_{j+1} = 0$ if $f_j \in \{\mu_j, \theta_j\}$ and $u_j = 0$ if $f_j = \nu_j$. In all cases σ is non-critical. By Remark 4.7, it is enough to exhibit an element of $\mathcal{L}(f_j(\sigma)) \setminus \mathcal{L}(\sigma)$. Without loss of generality take $t = 0$.

Recall that if $\sum_i p^{-i} y_i = 0$ in $\mathbf{Z}/(q-1)\mathbf{Z}$ then there are unique integers z_i such that $y_i = pz_{i-1} - z_i$. If moreover $y_i \in [-p, p]$ for all i , then the same argument as in Corollary 2.6 gives $|z_i| \leq 1$ for all i .

Suppose first that $f_j = \mu_j$. We have $\mathcal{L}(\mu_j(\sigma), \{j+1\}) = p^{-j}$. If this belonged to $\mathcal{L}(\sigma)$, there would exist a niveau 1 profile I such that $\sum_{i \in I} p^{-i} u_i = p^{-j}$. Apply the observation in the second paragraph of the proof to

$$y_i = \delta_I(i) u_i - \delta_{\{j\}}(i).$$

Since $y_{j+1} = 0$, we obtain $z_j = z_{j+1} = 0$, and therefore $y_j = pz_{j-1} \in \{-p, 0, p\}$. On the other hand since $u_j > 1$ by non-invertibility, we have

$$y_j = \begin{cases} u_j - 1 \in [1, p-1] & \text{if } j \in I, \\ -1 & \text{if } j \notin I, \end{cases}$$

a contradiction.

Suppose next that $f_j = \theta_j$. We have $\mathcal{L}(\theta_j(\sigma), \emptyset) = -p^{-j}$. If this belonged to $\mathcal{L}(\sigma)$, there would be a niveau 1 profile I such that $\sum_{i \in I} p^{-i} u_i = -p^{-j}$. Apply the observation in the second paragraph to

$$y_i = \delta_I(i) u_i + \delta_{\{j\}}(i).$$

Validity of θ_j gives $0 \leq y_i \leq p$ for all i . Again $y_{j+1} = 0$, so $z_j = z_{j+1} = 0$. Since $y_j > 0$, the equation $y_j = pz_{j-1} - z_j$ gives $z_{j-1} = 1$. But whenever $z_i = 1$, the inequality $y_i \geq 0$ forces $z_{i-1} = 1$. Iterating cyclically eventually gives $z_{j+1} = 1$, a contradiction.

Finally suppose that $f_j = \nu_j$. We have $\mathcal{L}(\nu_j(\sigma), \emptyset) = -p^{-j} + u_{j+1} p^{-(j+1)}$. If this belonged to $\mathcal{L}(\sigma)$, there would be a niveau 1 profile I such that

$$\sum_{i \in I} p^{-i} u_i = -p^{-j} + u_{j+1} p^{-(j+1)}.$$

Apply the observation in the second paragraph to

$$y_i = \delta_I(i) u_i + \delta_{\{j\}}(i) - u_{j+1} \delta_{\{j+1\}}(i).$$

Here $y_j = 1$, $y_{j+1} \geq -(p-1)$ because $u_{j+1} \neq p$ by non-invertibility, and $y_i \geq 0$ for $i \neq j+1$. The equation $y_j = pz_{j-1} - z_j$ forces $z_{j-1} = 0$ and $z_j = -1$. The equation $y_{j+1} = -p - z_{j+1}$, together with $y_{j+1} \geq -(p-1)$, forces $z_{j+1} = -1$. But now $y_{j+2} = -p - z_{j+2} < 0$, contrary to $y_{j+2} \geq 0$. This completes the proof. $\square$

6. THE MAIN THEOREMS

We are now ready to prove Theorem 1.14, thus also establishing Theorems 1.9, 1.10, and 1.12 as a consequence of Proposition 1.15. For convenience we restate the theorem here, in the language of phantom Serre weights.

Theorem 6.1. *We have the following.*

- (i) *We have $\mathcal{Z}_{\text{ss}}(\sigma) = \mathcal{Z}_{\text{ss}}(\sigma')$ if and only if either $\sigma \sim \sigma'$ or else there exists t such that either $\sigma = (\text{BT}; t)$ and $\sigma' = (\text{St}; t)$ or vice-versa.*
- (ii) *Suppose $\mathcal{Z}_{\text{ss}} \subsetneq \mathcal{Z}'_{\text{ss}}$ is a simple inclusion in $\mathcal{Z}_{\text{ss}}$. Then either:*
 - *there exists σ and a valid operation $f_j \in \{\mu_j, \theta_j, \nu_j\}$ such that $\mathcal{Z}_{\text{ss}} = \mathcal{Z}_{\text{ss}}(\sigma)$ and $\mathcal{Z}'_{\text{ss}} = \mathcal{Z}_{\text{ss}}(f_j(\sigma))$, or*
 - *we have $\mathcal{Z}_{\text{ss}} = \mathcal{Z}_{\text{ss}}(\text{ur}; t)$ and $\mathcal{Z}'_{\text{ss}} = \mathcal{Z}_{\text{ss}}(\text{cr}; t)$ for some t .*
- (iii) *Each of the inclusions listed in (ii) is simple except for the ones of the form $\mathcal{Z}_{\text{ss}}(\sigma) \subsetneq \mathcal{Z}_{\text{ss}}(f_j(\sigma))$ with $f_j \in \{\theta_j, \nu_j\}$, $(u_j, u_{j+1}) = (0, 0)$, and either $u_{j-1} = 1$ or $u_{j+2} = p$.*

Proof. (i) The ‘if’ direction is clear. To check the ‘only-if’ direction, suppose that $\mathcal{Z}_{\text{ss}}(\sigma) = \mathcal{Z}_{\text{ss}}(\sigma')$. In particular $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$. Choose $\tilde{\sigma} \sim \sigma$ such that $\mathcal{Z}_{\text{ss}}(\tilde{\sigma}) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ can be equipped with a minimal pinning. Consider what Theorem 4.1 says about this inclusion. All of the six bullet points listed in the theorem are proper inclusions, thanks to Remark 4.4 and Proposition 5.1. Since the inclusion $\mathcal{Z}_{\text{ss}}(\tilde{\sigma}) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$ is not proper, the conclusion of the theorem cannot hold, and so either $\tilde{\sigma} = \sigma'$ or $\{\tilde{\sigma}, \sigma'\} = \{(\text{BT}; t), (\text{St}; t)\}$. In either case we are done. In the first case we have $\sigma \sim \tilde{\sigma} = \sigma'$; in the second case, since neither $(\text{BT}; t)$ nor $(\text{St}; t)$ is the source or target of any of the invertible operations, we have $\sigma = \tilde{\sigma}$, and $\{\sigma, \sigma'\} = \{(\text{BT}; t), (\text{St}; t)\}$.

(ii) This part is essentially immediate from Theorem 4.1 and Proposition 5.1. Write $\mathcal{Z}_{\text{ss}} \subsetneq \mathcal{Z}'_{\text{ss}}$ as $\mathcal{Z}_{\text{ss}}(\sigma) \subsetneq \mathcal{Z}_{\text{ss}}(\sigma')$ equipped with a minimal pinning. Since the inclusion is strict we have $\sigma \neq \sigma'$ and $\{\sigma, \sigma'\} \neq \{(\text{BT}; t), (\text{St}; t)\}$. Theorem 4.1 tells us that either $\mathcal{Z}_{\text{ss}} = \mathcal{Z}_{\text{ss}}(\text{ur}; t)$ and $\mathcal{Z}'_{\text{ss}} = \mathcal{Z}_{\text{ss}}(\text{cr}; t)$ for some t , or else one of the four middle bullet points holds and we have

$$\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(f_j(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(\sigma')$$

for some valid and non-invertible f_j . Proposition 5.1 tells us that the left-hand inclusion is strict, and the hypothesis that $\mathcal{Z}_{\text{ss}} \subsetneq \mathcal{Z}'_{\text{ss}}$ is simple implies $\mathcal{Z}_{\text{ss}}(f_j(\sigma)) = \mathcal{Z}_{\text{ss}}(\sigma')$.

(iii) We already saw in the introduction, in the discussion following the statement of Theorem 1.12, that each of the listed exceptions is non-simple on the level of the stacks $\mathcal{Z}$. For each listed exception, that discussion exhibits $f_j(\sigma) = f(g(h(\sigma)))$ where f, g, h are valid operations, two of which are non-invertible and one of which is invertible. It follows from Proposition 5.1 that two of the inclusions in

$$\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(h(\sigma)) \subseteq \mathcal{Z}_{\text{ss}}(g(h(\sigma))) \subseteq \mathcal{Z}_{\text{ss}}(f(g(h(\sigma)))) = \mathcal{Z}_{\text{ss}}(f_j(\sigma))$$

are strict, and so $\mathcal{Z}_{\text{ss}}(\sigma) \subseteq \mathcal{Z}_{\text{ss}}(f_j(\sigma))$ cannot be simple.

It remains to show that each of the inclusions that we claim to be simple is actually simple. The simplicity of $\mathcal{Z}_{\text{ss}}(\text{ur}; t) \subseteq \mathcal{Z}_{\text{ss}}(\text{cr}; t)$ was already established by Proposition 2.9. It remains to consider inclusions $\mathcal{Z}_{\text{ss}}(\sigma) \subsetneq \mathcal{Z}_{\text{ss}}(f_j(\sigma))$ for valid f_j not of the exceptional type.

Since parts (i) and (ii) have been proved, observe that by Proposition 1.15, we already have Proposition 1.15(i), Theorem 1.9, and Theorem 1.10 at our disposal. If

there is an intermediate inclusion $\mathcal{Z}_{\text{ss}}(\sigma) \subsetneq \mathcal{Z}_{\text{ss}}(\sigma') \subsetneq \mathcal{Z}_{\text{ss}}(\sigma'') = \mathcal{Z}_{\text{ss}}(f_j(\sigma))$, where we take $\sigma'' = f_j(\sigma)$ unless $f_j(\sigma) = (\text{St}; t)$, in which case we take $\sigma'' = (\text{BT}; t)$, then there is also a chain

$$\mathcal{Z}(\sigma) \subsetneq \mathcal{Z}(\sigma') \subsetneq \mathcal{Z}(\sigma'').$$

Since each stack in this chain is irreducible,

$$\dim \mathcal{Z}(f_j(\sigma)) = \dim \mathcal{Z}(\sigma'') \geq \dim \mathcal{Z}(\sigma) + 2.$$

Write $\sigma = (u_i; t)$. Since the codimension of $\mathcal{Z}(\sigma)$ is given by $\#\{i : u_i = 0\}$, the operation must reduce the cardinality of this set by 2, i.e., we are in the case where the operation is either θ_j or ν_j with $(u_{j-1}, \dots, u_{j+2}) = (u_{j-1}, 0, 0, u_{j+2})$ and $u_{j-1} \neq 1$, $u_{j+2} \neq p$. Since in this situation $\theta_j = \nu_j$, it suffices to consider θ_j .¹

Suppose, then, that we have a chain

$$\mathcal{Z}_{\text{ss}}(\sigma) \subsetneq \mathcal{Z}_{\text{ss}}(\sigma') \subsetneq \mathcal{Z}_{\text{ss}}(\theta_j(\sigma)).$$

We can and do assume that $\mathcal{Z}_{\text{ss}}(\sigma) \subsetneq \mathcal{Z}_{\text{ss}}(\sigma')$ is simple. Since σ' cannot be regular, because otherwise $\mathcal{Z}_{\text{ss}}(\sigma')$ would be maximal in $\mathcal{Z}_{\text{ss}}$, we have $\sigma' \sim f_k(\tilde{\sigma})$ for some $\tilde{\sigma} \sim \sigma$, some index k , and a valid non-invertible operation f_k . We want to derive a contradiction.

Writing $\tilde{\sigma} = (\tilde{u}_r; \tilde{t})$, we claim that we still have

$$\tilde{u}_j = \tilde{u}_{j+1} = 0, \quad \tilde{u}_{j-1} \neq 1, \quad \tilde{u}_{j+2} \neq p,$$

and in addition we have

$$(6.2) \quad \theta_j(\tilde{\sigma}) \sim \theta_j(\sigma).$$

Indeed, any invertible operation that alters one of $u_{j-1}, \dots, u_{j+2}$ can only be μ_{j-2}/ν_{j-2} , changing u_{j-1} from 0 to p or vice-versa, or μ_{j+2}/ν_{j+2} , changing u_{j+2} from 1 to 0 or vice-versa. Thus any invertible operation on σ preserves the given conditions on $(u_{j-1}, \dots, u_{j+2})$ and has support disjoint from that of θ_j , hence commutes with θ_j . The claim follows. Consequently, we may replace σ with $\tilde{\sigma}$, and σ' with $f_k(\tilde{\sigma})$, so that our chain becomes

$$\mathcal{Z}_{\text{ss}}(\sigma) \subsetneq \mathcal{Z}_{\text{ss}}(f_k(\sigma)) \subsetneq \mathcal{Z}_{\text{ss}}(\theta_j(\sigma)).$$

We will show in this situation that $f_k = \theta_j$ or ν_j , which provides the necessary contradiction.

A direct calculation gives

$$(6.3) \quad \{\mathcal{L}(\theta_j(\sigma), J) : J \subseteq \mathbf{Z}/f\mathbf{Z}\} = \{\mathcal{L}(\sigma, J) + \gamma \cdot p^{-j} : J \subseteq \mathbf{Z}/f\mathbf{Z}, \gamma \in \{-1, 0, 1\}\}.$$

Take $J' = \{k+1\}$ if $f_k = \mu_k$ and $J' = \emptyset$ if $f_k = \theta_k$ or ν_k . Then

$$\mathcal{L}(f_k(\sigma), J') = t + \eta \quad \text{with} \quad \eta := \begin{cases} p^{-k} & f_k = \mu_k \\ -p^{-k} & f_k = \theta_k \\ -p^{-k} + u_{k+1}p^{-(k+1)} & f_k = \nu_k \end{cases}$$

and since J' has niveau 1, this value must be in the set in (6.3). We can then write

$$\eta = \sum_{i \in J} u_i p^{-i} + \gamma \cdot p^{-j}$$

¹In the Lean formalization available at <https://github.com/davidsavitt/KLSposet>, this argument that appeals to the Emerton–Gee stacks is replaced by a combinatorial argument.

for some $J \subseteq \mathbf{Z}/f\mathbf{Z}$. Define y_i by the formulas

$$\begin{aligned} y_i &= \delta_J(i)u_i + \gamma \cdot \delta_{\{j\}}(i) - \delta_{\{k\}}(i), \\ y_i &= \delta_J(i)u_i + \gamma \cdot \delta_{\{j\}}(i) + \delta_{\{k\}}(i), \\ y_i &= \delta_J(i)u_i + \gamma \cdot \delta_{\{j\}}(i) + \delta_{\{k\}}(i) - u_{k+1}\delta_{\{k+1\}}(i) \end{aligned}$$

in the three cases $f_k = \mu_k, \theta_k, \nu_k$ respectively. As usual we find $y_i = pz_{i-1} - z_i$ with $|z_i| \leq 1$ for all i .

Suppose $k \neq j$ in the θ_k and ν_k cases. Note also that $k \neq j, j+1$ in the μ_k case, because μ_j and μ_{j+1} are not valid.

If $f_k = \mu_k$ then we have $y_{j+1} = 0$, while in the other two cases $y_{j+1} \in \{0, 1\}$. Either way we have $z_j = 0$. Moreover, in all three cases $y_j = \gamma$, so the equation $y_j = pz_{j-1} - z_j$ gives $\gamma = pz_{j-1}$. It follows that $\gamma = 0$. Thus in fact $\mathcal{L}(f_k(\sigma), J') = \mathcal{L}(\sigma, J) \in \mathcal{L}(\sigma)$. Looking at the choice of J' and comparing with the list in Proposition 5.1, we reach a contradiction. $\square$

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