

Strict negativity and non-vanishing of Chen–Larson hypergeometric coefficients

A mathematical proof, Lean formalization, and documented AI-assisted research process

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Abstract

Chen and Larson study tautological classes on the strata of holomorphic abelian differentials, where they predict a vanishing result. Using known tautological relations on the moduli space of curves, they reduce this prediction to the non-vanishing of certain coefficients of a quotient of hypergeometric generating series, treating the residue classes $g \equiv 0, 2 \pmod{3}$ and $g \equiv 1 \pmod{3}$ through two separate series; they verify the non-vanishing by computer for small genus.

We prove it in general: in both cases the relevant coefficient is non-vanishing for every genus and every stratum, and in the first case it has, more strongly, a uniform strict sign. Along the way we correct an error in the Chen–Larson derivation of the $g \equiv 1 \pmod{3}$ series, where a constant of the underlying relation of Ionel had inadvertently been changed.

The paper falls into two parts. Part I gives the mathematical proofs; Part II documents the Lean 4 and Mathlib formalization — whose only non-standard trust assumption is the compiler invoked by `native_decide` for the large finite computations — together with the long-horizon, multi-model generative-AI process that produced the proofs, exposed false intermediate routes, and uncovered the error in the printed Proposition 5.2 relation noted above.

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Part I

Mathematical results and proofs

1 Introduction

1.1 Geometric context and main statements

Chen and Larson study tautological classes on strata of holomorphic abelian differentials. In Section 5 of their work [1], they pull back a relation among κ -classes on $\mathcal{M}_g$ to the projectivized stratum $\mathcal{P}(\mu)$ associated with a positive partition $\mu = (m_1, \dots, m_n)$ of $2g - 2$. Here “positive” means that every $m_i \geq 1$, and η denotes the tautological line-bundle class in their convention. The resulting scalar problems are expressed through

$$\mathsf{C}(t) = \sum_{r \geq 0} \frac{(6r)!}{(3r)!(2r)!} \left(\frac{t}{72} \right)^r. \quad (1)$$

For $g \equiv 0, 2 \pmod{3}$, Chen–Larson Proposition 5.1 reduces the desired vanishing of η^a , where $a = \lfloor g/3 \rfloor + 1$, to the non-vanishing of

$$[t^a] \frac{\prod_{i=1}^n \mathsf{C}(t/(m_i + 1))}{\mathsf{C}(t)^{2g-2+n}}.$$

Our first theorem gives the stronger information that this coefficient always has the same strict sign.

Theorem 1.1 (Proposition 5.1 coefficient). *Let $g \geq 2$ satisfy $g \equiv 0$ or $2 \pmod{3}$, and set $a = \lfloor g/3 \rfloor + 1$. For every positive partition $\mu = (m_1, \dots, m_n)$ of $2g - 2$,*

$$[t^a] \frac{\prod_{i=1}^n \mathsf{C}(t/(m_i + 1))}{\mathsf{C}(t)^{2g-2+n}} < 0. \quad (2)$$

In particular, the coefficient in Chen–Larson Proposition 5.1 is nonzero.

The sign itself is not assigned a geometric meaning; it is a uniform strengthening of the non-vanishing needed in the geometric reduction.

Corollary 1.2. *Under the hypotheses of Chen–Larson Proposition 5.1, the class η^a vanishes on $\mathcal{P}(\mu)$ for every positive partition μ when $g \equiv 0, 2 \pmod{3}$.*

Proof. Apply Theorem 1.1 to the scalar coefficient in the cited reduction. $\square$

In the remaining class $g \equiv 1 \pmod{3}$, the same geometric vanishing rests on a different Chen–Larson reduction — their Proposition 5.2 — built from a derivative-corrected series rather than the quotient above; we first introduce the notation it requires.

Write $g = 3a - 2$, so that $a = \lfloor g/3 \rfloor + 1 \geq 2$, and put

$$M = 2g - 2 = 6a - 6, \quad q_i = m_i + 1, \quad N = \sum_i q_i = M + n.$$

Let $L(t) = C'(t)/C(t)$ and define

$$F_\mu(t) = \frac{\prod_{i=1}^n C(t/q_i)}{C(t)^N}, \quad (3)$$

$$D_\mu^{\text{cor}}(t) = M - 2 \left(N - \sum_i q_i^{-1} \right) t - 12t^2 \left(NL(t) - \sum_i q_i^{-2} L(t/q_i) \right). \quad (4)$$

The leading constant $M = \kappa_0$ is the correction to the factor printed in Chen–Larson Proposition 5.2, where it appears as 1.

Theorem 1.3 (Corrected Proposition 5.2 coefficient). *Let $g \geq 4$ satisfy $g \equiv 1 \pmod{3}$, let $a = (g+2)/3$, and let $\mu = (m_1, \dots, m_n)$ be a positive partition of $2g-2$. With F_μ and D_μ^{cor} defined by (3)–(4),*

$$[t^a]F_\mu(t)D_\mu^{\text{cor}}(t) \neq 0.$$

For $a \geq 14$, equivalently $g \geq 40$, this coefficient is strictly negative.

Section 9 derives the corrected factor from Ionel’s relation, proves that it agrees in degree a with a convenient marked expression, and then proves the sign and the remaining finite non-vanishing cases.

Corollary 1.4. *Combining Theorems 1.1 and 1.3 with the geometric reductions of Chen–Larson Propositions 5.1 and 5.2, with the latter corrected as in Section 9, gives $\eta^{\lfloor g/3 \rfloor + 1} = 0$ on $\mathcal{P}(\mu)$ for every positive partition μ and every genus $g \geq 2$.*

The algebraic-geometric reductions are imported from Chen and Larson, apart from the corrected constant in Proposition 5.2. The new work here is the scalar power-series analysis and its formal verification.

1.2 Proof architecture

After the geometric reduction, the problem is a uniform coefficient inequality for a quotient of formal power series. Two features are responsible for the difficulty: the quotient depends on an arbitrary partition, and the sign must hold uniformly as the genus grows. The proof first removes the partition, then separates a negative main term from positive corrections, and finally uses different exact or analytic arguments in four numerical ranges.

Removing the partition. Write

$$\log \mathbb{C}(t) = \sum_{r \geq 1} c_r t^r.$$

Section 3 proves $c_r > 0$. With $q_i = m_i + 1 \geq 2$ and $\sigma_r = \sum_i q_i^{-r}$, the numerator becomes

$$\prod_i \mathbb{C}(t/q_i) = \exp \left(\sum_{r \geq 1} c_r \sigma_r t^r \right).$$

Every coefficient of this exponential is nondecreasing in each σ_r . Since $\sigma_r \leq n2^{-r} \leq (N/2)2^{-r}$, replacing each σ_r by this upper bound gives the coefficientwise majorant $\mathbb{C}(t/2)^{N/2}$. After the nonpositive convolution terms are discarded, the target coefficient is bounded above by an explicit partition-free quantity $U_a(N)$. It remains to prove

$$U_a(N) < 0 \quad (6a - 7 \leq N \leq 12a - 8).$$

A negative term against a positive budget. Let $X_r(N)$ be the coefficient of t^r in $\mathbb{C}(t)^{-N}$ normalized by Nc_r , and let $Y_r(N)$ be the similarly normalized coefficient of $\mathbb{C}(t/2)^{N/2}$. Section 2 rewrites the majorant as

$$\frac{U_a(N)}{Nc_a} = X_a(N) + \text{positive correction terms}.$$

The proof therefore needs a negative upper bound for $X_a(N)$ and an upper bound for the positive corrections. The key negative estimate is the *sign lock*: whenever $m \geq 361$ and $N \leq (40/3)m$, the coefficient $X_m(N)$ is negative with an explicit margin. Applied with $m = k$, it removes all convolution indices $k > 0.9a$; applied with $m = a$, it supplies the main negative term.

Keeping the two saddle estimates together. The surviving corrections are bounded without expanding their coefficients. For

$$P_m(x) = \sum_{r=0}^m \frac{x^r}{r!},$$

a finite saddle estimate bounds a coefficient of an exponential series by P_m at a rationally chosen radius. The decisive product inequality is

$$P_m(x)P_n(y) \leq P_{m+n}(x+y).$$

It lets the denominator and numerator bounds be combined before their normalization losses are incurred. Section 6 displays the full calculation; this is exactly the point at which a plausible earlier route, which estimated the two factors separately, lost a fatal dyadic gain.

Finite ranges and the infinite tail. For $a \leq 8$, every positive partition is enumerated and its exact rational coefficient is checked. For $9 \leq a \leq 60$, the majorant is checked exactly on the integer rectangle, and for $61 \leq a \leq 400$ a proved dyadic-interval kernel certifies the same inequalities. For $a \geq 401$, the sign lock and direct saddle estimates reduce the positive side to a budget below 10^{-8} . The strip through $a = 2000$ is finite, the transition strip $2001 \leq a < 3000$ is handled by exact ratio certificates, and the range $a \geq 3000$ follows from uniform adjacent-term contractions and geometric reserves.

The corrected Proposition 5.2 coefficient. The correction changes only the constant term of the source factor, and in the required degree it gives the identity

$$T_a^{\text{cor}} = T_a^{\text{old}} + (M - 1)b_a.$$

For $a \geq 14$, the quotient coefficient b_a is negative by the rectangular form of Theorem 1.1. The printed coefficient T_a^{old} is checked by exact intervals for $14 \leq a \leq 149$ and is proved negative for $a \geq 150$ by a new Gamma-expectation argument. That argument represents a truncated coefficient sum by Gamma moments, uses integration by parts and Jensen's inequality for a positive main margin, and controls the Taylor remainder by an explicit four-term bound. The corrected coefficient is then negative. The degrees $2 \leq a \leq 13$ are settled by exact and modular finite certificates.

1.3 Two contributions and two reading routes

The manuscript is intentionally divided into two parts. Part I is the mathematical paper: all notation, theorem statements, proof interfaces, finite certificate claims, and analytic estimates needed for the conclusions are stated there. The finite certificates are presented as exact named lemmas, with their proof methods described, while the full machine-checkable data live in the associated Lean development. No fact about the AI workflow is a logical premise of the mathematics.

Part II is a second, substantial account of formal verification and the research process. It explains how the mathematical interfaces are represented in Lean, which finite steps use native evaluation, exactly what must be trusted, and how a human-orchestrated collection of generative-AI systems developed and audited the argument. Occasional pointers from Part I to the correspondence appendices are navigational only. Conversely, Part II presupposes the mathematical architecture of Part I rather than restating it.

The formalized scope is the complete scalar power-series content of Theorems 1.1 and 1.3, over $\mathbb{Q}$, including the finite certificates and the Gamma-tail argument. The imported algebraic-geometric reductions are not formalized. Appendix D reproduces the public theorem layer, and Section 10 records the trust boundary.

2 The partition-free majorant

This section makes the partition removal sketched in Section 1.2 precise, ending at the majorant inequality $b_a(\mu) \leq U_a(N)$.

Let $\mu = (m_1, \dots, m_n)$ be a positive partition of $2g - 2$ and put

$$q_i = m_i + 1 \geq 2, \quad N = \sum_{i=1}^n q_i = 2g - 2 + n. \quad (5)$$

Set

$$b_a(\mu) = [t^a] \frac{\prod_i C(t/q_i)}{C(t)^N}. \quad (6)$$

For $g \equiv 0, 2 \pmod{3}$ and $a = \lfloor g/3 \rfloor + 1$, one has

$$g = 3a - 3 \quad \text{or} \quad g = 3a - 1.$$

Since $1 \leq n \leq 2g - 2$, every relevant value of N lies in the enlarged rectangle

$$6a - 7 \leq N \leq 12a - 8. \quad (7)$$

The enlargement is convenient because it treats both residue classes at once.

Define

$$\begin{aligned} A_r(\mu) &= [t^r] \prod_i \mathbb{C}(t/q_i), \\ Z_r(N) &= [t^r] \mathbb{C}(t)^{-N}, \\ Q_r(N) &= [t^r] \mathbb{C}(t/2)^{N/2}. \end{aligned} \tag{8}$$

Here rational powers are interpreted through the exponential of the formal logarithm. Since the logarithmic coefficients c_r are nonnegative, coefficients of

$$\exp \left(\alpha \sum_{r \geq 1} c_r 2^{-r} t^r \right)$$

are polynomials in α with nonnegative coefficients. Writing the numerator as $\prod_i \mathbb{C}(t/q_i) = \exp(\sum_r c_r \sigma_r t^r)$ with $\sigma_r = \sum_i q_i^{-r}$, its coefficients $A_r(\mu)$ are therefore nondecreasing in each σ_r . Since $q_i \geq 2$ and $n \leq N/2$ give $\sigma_r \leq n 2^{-r} \leq \frac{N}{2} 2^{-r}$, replacing each σ_r by its upper bound $\frac{N}{2} 2^{-r}$ — which turns the numerator into $\mathbb{C}(t/2)^{N/2}$ — only increases coefficients. It follows that

$$0 \leq A_r(\mu) \leq Q_r(N). \tag{9}$$

Expanding (6), dropping nonpositive convolution terms, and using (9) gives the partition-free majorant

$$b_a(\mu) \leq U_a(N) := Z_a(N) + Q_a(N) + \sum_{\substack{1 \leq k < a \\ Z_k(N) > 0}} Z_k(N) Q_{a-k}(N). \tag{10}$$

Thus Theorem 1.1 follows once we prove

$$U_a(N) < 0 \tag{11}$$

throughout (7), except for the finitely many values where we instead verify $b_a(\mu) < 0$ directly.

2.1 Normalized form

We pass to rescaled coefficients whose leading terms are the clean values $X_1 = -1$ and $Y_1 = 1$. In this normalization the majorant splits transparently into one negative term and two positive contributions — the split the rest of the proof exploits. For $r \geq 1$ define

$$X_r(N) = \frac{Z_r(N)}{N c_r}, \quad Y_r(N) = \frac{2^r Q_r(N)}{(N/2) c_r}, \tag{12}$$

and set

$$R_{k,a} = \frac{c_k c_{a-k}}{c_a}. \tag{13}$$

For a series $E(t) = \exp(\sum_{r \geq 1} \ell_r t^r) = \sum_{r \geq 0} e_r t^r$, logarithmic differentiation gives the recurrence $r e_r = \sum_{k=1}^r k \ell_k e_{r-k}$. Applied to $\mathbb{C}(t)^{-N}$ and $\mathbb{C}(t/2)^{N/2}$ — whose logarithmic coefficients are $-N c_r$ and $\frac{N}{2} c_r 2^{-r}$ — and normalized by (12), it gives

$$X_1 = -1, \quad X_r = -1 - \frac{N}{r} \sum_{k=1}^{r-1} k R_{k,r} X_{r-k}, \tag{14}$$

$$Y_1 = 1, \quad Y_r = 1 + \frac{N}{2r} \sum_{k=1}^{r-1} k R_{k,r} Y_{r-k}. \tag{15}$$

A direct calculation transforms (10) into

$$\frac{U_a(N)}{Nc_a} = X_a(N) + 2^{-a-1}Y_a(N) + \frac{N}{2} \sum_{\substack{1 \leq k < a \\ X_k(N) > 0}} R_{k,a} 2^{-(a-k)} X_k(N) Y_{a-k}(N). \quad (16)$$

The proof therefore consists of a uniform positive lower bound for $-X_a$ and an upper budget for the remaining two positive contributions.

3 Logarithmic coefficients

Everything downstream rests on the logarithmic coefficients c_r of $\mathbb{C}$, defined by $\log \mathbb{C}(t) = \sum_{r \geq 1} c_r t^r$: their nonnegativity is what removed the partition in Section 2, and the rational bounds derived here drive every later estimate. We establish both in turn.

The series $\mathbb{C}$ satisfies the second-order linear differential equation

$$6t^2 \mathbb{C}'' + (12t - 1) \mathbb{C}' + \frac{5}{6} \mathbb{C} = 0, \quad \mathbb{C}(0) = 1,$$

from which the bridge identifies $\mathbb{C}$ with $\exp(\sum_r c_r t^r)$ — the two series solve it with the same constant term. Passing to the logarithmic derivative $L = \mathbb{C}'/\mathbb{C}$ turns it into a first-order (Riccati) equation, quadratic in L ; equating coefficients there yields

$$c_1 = \frac{5}{6}, \quad c_r = 6(r-1)c_{r-1} + \frac{6}{r} \sum_{i=1}^{r-2} i(r-1-i)c_i c_{r-1-i} \quad (r \geq 2). \quad (17)$$

In particular, $c_r > 0$. Write

$$c_r = 6^r (r-1)! d_r. \quad (18)$$

Then

$$d_r = d_{r-1} + \frac{1}{r} \sum_{i=1}^{r-2} \frac{d_i d_{r-1-i}}{\binom{r-1}{i}}. \quad (19)$$

The sequence d_r is nondecreasing and begins with $d_1 = d_2 = 5/36$.

The formal proof uses rational bounds throughout.

Lemma 3.1 (Uniform coefficient bounds). *For every $r \geq 1$,*

$$\frac{5}{36} \leq d_r \leq \frac{4}{25}. \quad (20)$$

Consequently,

$$\frac{5}{36} 6^r (r-1)! \leq c_r \leq \frac{4}{25} 6^r (r-1)!. \quad (21)$$

Proof. The lower bound follows immediately from (19). For the upper bound, let $A_r = [t^r] \mathbb{C}(t)$ and $F_r = A_r / (6^r (r-1)!)$. One has $d_r \leq F_r$ and

$$\frac{F_{r+1}}{F_r} = 1 + \frac{5}{36r(r+1)}.$$

A rational telescoping argument bounds F_r by

$$\frac{F_3}{1 - (5/36)(1/3 - 1/r)} \leq F_3 \frac{108}{103} < \frac{4}{25}.$$

No estimate involving π is required. □

The reciprocal-binomial estimate

$$\sum_{i=1}^{p-1} \binom{p}{i}^{-1} \leq \frac{4}{p} \quad (22)$$

combines with (19) and (20) to give an increment bound.

Lemma 3.2 (Increment and ratio control). *For $r \geq 2$,*

$$d_r - d_{r-1} \leq \frac{64/625}{r(r-1)}. \quad (23)$$

If $1 \leq s < m$, then

$$1 - \frac{2304}{3125} \frac{s}{m(m-s)} \leq \frac{d_{m-s}}{d_m} \leq 1. \quad (24)$$

Proof. The nonlinear summand in (19) is at most $(16/625)(r_i^{-1})^{-1}$. Summing with (22) gives (23). Telescoping from $m-s$ to m yields

$$d_m - d_{m-s} \leq \frac{64}{625} \left(\frac{1}{m-s} - \frac{1}{m} \right).$$

Divide by $d_m \geq 5/36$. The upper ratio bound follows from monotonicity. $\square$

A further consequence, used throughout the positive-part estimate, is

$$R_{k,a} \leq \frac{576}{3125} \frac{1}{(a-1)\binom{a-2}{k-1}} \quad (1 \leq k < a). \quad (25)$$

Indeed, the d -factor contributes at most $(4/25)^2/(5/36) = 576/3125$, and the factorial quotient is exactly the reciprocal denominator in (25).

4 Finite verification through $a = 400$

The majorant is not strong enough for the smallest values of a , so the proof begins with direct finite verification. The cut points 8, 60, and 400 below are computational thresholds — each marks where one exact method becomes infeasible and a cheaper one takes over — not mathematically distinguished values; the analytic argument begins at $a = 401$, where its margin first beats the positive budget (Section 8).

4.1 Positive partitions for $a \leq 8$

For $g \leq 23$, equivalently $a \leq 8$ in the relevant residue classes, every positive partition of $2g-2$ is generated and the exact rational coefficient $b_a(\mu)$ is evaluated. The largest enumeration is $p(44) = 75175$ partitions at $g = 23$. The executable theorem checks strict negativity for all of them. Independent cardinality statements verify selected partition counts, reducing the risk of an incomplete generator.

4.2 Exact majorant certificate for $9 \leq a \leq 60$

For $9 \leq a \leq 60$, the recurrence tables for c_r , $Z_r(N)$, and $Q_r(N)$ are evaluated exactly over $\mathbb{Q}$. This verifies (11) at all 10764 pairs in the rectangle. As a regression anchor, the least-negative corner is recorded exactly as

$$\frac{U_9(100)}{100c_9} = -\frac{13391635371339739}{213818836571652096}. \quad (26)$$

4.3 Verified dyadic intervals for $61 \leq a \leq 400$

The remaining 470220 pairs are certified by a small executable interval kernel. A dyadic interval stores outward-rounded endpoints with a 192-bit mantissa. The exact rational recurrences are mirrored term by term, and separate soundness lemmas prove that the tables enclose the exact values. For the conditionally included term in (10), the checker uses the convex hull of the product interval and $\{0\}$; consequently it never needs to decide the sign of $Z_k(N)$ from an interval approximation.

The finite check uses a dyadic interval kernel whose rational enclosure semantics are proved independently. Eight evaluation chunks cover the N -range and together prove (11) for every $61 \leq a \leq 400$. The 192-bit precision matches the reference *Arb* (arbitrary-precision ball arithmetic) calculation used during discovery; the formal trust boundary for this certification is described in Section 10.

Proposition 4.1 (Certified finite range). *The following finite statements hold.*

- (i) *For every relevant genus with $a \leq 8$ and every positive partition μ of $2g - 2$, one has $b_a(\mu) < 0$.*
- (ii) *For every pair of integers a, N with $9 \leq a \leq 400$ and $6a - 7 \leq N \leq 12a - 8$, one has $U_a(N) < 0$.*

Proof. Part (i) is the exhaustive exact-rational partition computation summarized in the first subsection. Part (ii) is the union of the exact recurrence calculation for $9 \leq a \leq 60$ and the sound dyadic-interval calculation for $61 \leq a \leq 400$. In the latter calculation, every arithmetic operation is outward rounded and the enclosure invariant is proved by induction along the recurrences, so a negative upper endpoint proves the exact rational inequality. The ranges are disjoint and exhaustive. $\square$

5 The effective sign lock

For $a \geq 401$, the leading coefficient $X_a(N)$ must be negative by more than the entire positive budget. The same estimate is also applied at the convolution index $m = k$ to discard the upper tenth of the convolution range. This section isolates the two mathematical interfaces on which that conclusion rests: a normalized nonlinear residual envelope and a near/far audit for the alternating expansion of $X_m(N)$.

5.1 The nonlinear residual envelope

Write

$$H(t) = \sum_{r \geq 2} c_r t^r, \quad E_p^-(N) = [t^p] \exp(-NH(t)).$$

The coefficient with one nonlinear block is $-Nc_p$. The following proposition quantifies the total contribution of every expansion term with at least two blocks.

Proposition 5.1 (Normalized nonlinear residual). *Suppose that $m \geq 361$, $N \geq 1$, $N \leq (40/3)m$, and $p \geq 2m/3$. Then*

$$\left| \frac{E_p^-(N)}{-Nc_p} - 1 \right| \leq \frac{66}{5m}. \quad (27)$$

Proof. Expanding the exponential by the number of nonlinear blocks gives the exact identity

$$E_p^-(N) = -Nc_p + \sum_{r=2}^{\lfloor p/2 \rfloor} \frac{(-N)^r}{r!} [t^p] H(t)^r; \quad (28)$$

the upper limit occurs because every factor of H has degree at least 2. The coefficient bounds of Lemma 3.1, together with the composition estimate for $[t^p]H^r$, bound the normalized absolute value of the r th summand by

$$\Delta_{p,N}^{(r)} = \frac{36}{5} \left(\frac{4}{25} \right)^r 4^{r-1} N^{r-1} \frac{(p-2r+1)!}{r!(p-1)!}. \quad (29)$$

Thus the left side of (27) is at most $\sum_{r=2}^{\lfloor p/2 \rfloor} \Delta_{p,N}^{(r)}$. The first term and consecutive quotient are

$$\Delta_{p,N}^{(2)} = \frac{1152}{3125} \frac{N}{(p-1)(p-2)}, \quad (30)$$

$$\frac{\Delta_{p,N}^{(r+1)}}{\Delta_{p,N}^{(r)}} = \frac{(16/25)N}{(r+1)(p-2r)(p-2r+1)}. \quad (31)$$

For $r \leq p/4$, the quotient gives a geometric majorant beginning with (30). For $r > p/4$, the factorial quotient in (29) is bounded directly by $(p-1)!/(p-2r+1)! \geq (2r-1)!$, after which the remaining terms again form a geometrically decreasing tail. Substituting $N \leq (40/3)m$ and $p \geq 2m/3$ into these two bounds gives

$$\sum_{r=2}^{\lfloor p/2 \rfloor} \Delta_{p,N}^{(r)} \leq \frac{66}{5m}.$$

The denominator-cleared scalar inequalities used in this last summation are recorded in Appendix A; no asymptotic estimate is used. $\square$

5.2 The alternating decomposition

Set

$$\zeta = \frac{5N}{36m}.$$

The hypothesis $N \leq (40/3)m$ gives $0 \leq \zeta \leq 50/27$. Moreover, $c_1 = 5/6$ and

$$\mathbb{C}(t)^{-N} = \exp(-Nc_1 t) \exp(-NH(t)).$$

Expanding the linear exponential gives

$$-X_m(N) = \sum_{s=0}^m \frac{(-Nc_1)^s}{s!} \left(-\frac{E_{m-s}^-(N)}{Nc_m} \right). \quad (32)$$

For $s < m$, define

$$\Pi_s = \prod_{i=1}^s \frac{1}{1-i/m}, \quad D_s = \frac{d_{m-s}}{d_m}, \quad \varepsilon_p = \frac{E_p^-(N)}{-Nc_p} - 1.$$

The coefficient ratio c_{m-s}/c_m then rewrites the s th summand as

$$\frac{(-\zeta)^s}{s!} \Pi_s D_s (1 + \varepsilon_{m-s}). \quad (33)$$

The factor Π_s has first-order expansion $1 + e_1(s)/m$, where $e_1(s) = s(s+1)/2$. The nonlinear residual has a matching first-order endpoint contribution $-\zeta/m$. We therefore define w_s by

$$\Pi_s D_s (1 + \varepsilon_{m-s}) = 1 + \frac{e_1(s) - \zeta}{m} + w_s. \quad (34)$$

This is the recentering that removes all first-order errors from the part of the alternating sum that will be compared with an exponential.

Split (32) at $s = \lfloor m/3 \rfloor$. Proposition 5.1 applies throughout the near range because $m - s \geq 2m/3$. We obtain

$$\begin{aligned} -X_m(N) = & \underbrace{\sum_{0 \leq s \leq \lfloor m/3 \rfloor} \frac{(-\zeta)^s}{s!} \left(1 + \frac{e_1(s) - \zeta}{m}\right)}_{\mathcal{B}_m(\zeta)} \\ & + \underbrace{\sum_{0 \leq s \leq \lfloor m/3 \rfloor} \frac{(-\zeta)^s}{s!} w_s}_{\mathcal{R}_{\text{near}}} + \underbrace{\sum_{\lfloor m/3 \rfloor < s \leq m} \frac{(-Nc_1)^s}{s!} \left(-\frac{E_{m-s}^-(N)}{Nc_m}\right)}_{\mathcal{R}_{\text{far}}}. \end{aligned} \quad (35)$$

5.3 The near base and the error audit

Lemma 5.2 (Twelve-term near-base bound). *For $m \geq 361$ and $0 \leq z \leq 50/27$,*

$$\mathcal{B}_m(z) \geq \lambda \left(1 - \frac{2}{m}\right), \quad \lambda = \sum_{r=0}^9 \frac{(-50/27)^r}{r!} = \frac{678107852315029}{4323713773987629}. \quad (36)$$

Proof. Put

$$A_{12}(z) = \sum_{s=0}^{11} \frac{(-z)^s}{s!}, \quad C_{12}(z) = \sum_{s=0}^{11} \frac{(-z)^s}{s!} (e_1(s) - z).$$

The twelve-term prefix is $A_{12}(z) + C_{12}(z)/m$. After the affine change $z = (50/27)u$, two exact Bernstein-basis expansions on $0 \leq u \leq 1$ have nonnegative coefficients and prove

$$A_{12}(z) \geq \lambda, \quad A_{12}(z) + \frac{C_{12}(z)}{361} \geq \lambda \left(1 - \frac{2}{361}\right).$$

The expression is affine in $1/m$, so the endpoint inequalities imply the same bound for every $m \geq 361$. Starting with each even $s \geq 12$, pair the terms of indices s and $s+1$. Put $A_s = 1 + (e_1(s) - z)/m$. Since $A_{s+1} = A_s + (s+1)/m$ and $A_s \geq 1$, the bracket remaining after factoring out $z^s/s!$ satisfies

$$A_s - \frac{z}{s+1} A_{s+1} = A_s \left(1 - \frac{z}{s+1}\right) - \frac{z}{m} \geq 1 - \frac{50}{27 \cdot 13} - \frac{50}{27 \cdot 361} > 0.$$

If the truncation leaves a final unpaired term, its index is even and the term is nonnegative. Hence the omitted alternating tail is nonnegative. $\square$

Proposition 5.3 (Near/far sign-lock audit). *Suppose that $m \geq 361$, $N \geq 1$, and $N \leq (40/3)m$, with $\mathcal{R}_{\text{near}}$ and $\mathcal{R}_{\text{far}}$ as in (35),*

$$|\mathcal{R}_{\text{near}}| + |\mathcal{R}_{\text{far}}| \leq \frac{2215}{m^2}. \quad (37)$$

Proof. Let $u_s = \Pi_s - 1$ and $v_s = 1 - D_s$. The product error is separated by the exact inequality

$$|(1 + u_s)(1 - v_s)(1 + \varepsilon) - (1 + u_s - v_s + \varepsilon)| \leq u_s(v_s + |\varepsilon|) + v_s|\varepsilon|(1 + u_s). \quad (38)$$

The increment estimate (24) gives $v_s \leq (28/25)s/m^2$ on the near range. The logarithmic product estimate for Π_s gives

$$0 \leq \Pi_s - 1 - \frac{e_1(s)}{m}, \quad \log \Pi_s \leq \frac{146}{125} \frac{e_1(s)}{m} < \frac{2237}{10000} s,$$

that is, with decimal values 1.168 and 0.2237 respectively. Finally, expansion (28) splits the recentered residual into the two endpoint terms of H^2 , the remaining two-block sum, and the terms with at least three blocks. Applying (29) to those pieces and summing their Poisson weights gives the following complete audit:

source of the weighted error	bound
$\Pi_s - 1 - e_1(s)/m$	$426/m^2$
d -ratio drift $1 - D_s$	$13/m^2$
recentered endpoint part of the two-block term	$184/m^2$
non-endpoint two-block terms	$234/m^2$
terms with at least three nonlinear blocks	$573/m^2$
product cross terms from (38)	$784/m^2$
far signed tail	$1/m^2$
sum	$2215/m^2$

Each line is an exact rational inequality. Appendix A records the block definitions, the geometric ratios, and the Bernstein or endpoint certificate used for each line. Summing the seven bounds proves (37). $\square$

Theorem 5.4 (Sign lock). *If $m \geq 361$, $N \geq 1$, and $N \leq (40/3)m$, then*

$$X_m(N) \leq -\left(\lambda \left(1 - \frac{2}{m}\right) - \frac{2215}{m^2}\right) < 0. \quad (39)$$

Proof. Apply Lemma 5.2 and Proposition 5.3 to (35). The remaining scalar expression is positive at $m = 361$ by exact rational arithmetic, and $m^2\lambda(1 - 2/m)$ is increasing for $m \geq 361$. $\square$

The value 9/10 in the convolution cutoff is forced by the sign-lock hypothesis rather than chosen for convenience. Set

$$k_{\max}(a) = \lfloor 9a/10 \rfloor.$$

If $a \geq 401$ and $k > k_{\max}(a)$, then $k \geq 361$ and

$$N \leq 12a - 8 \leq \frac{40}{3}k.$$

Theorem 5.4 gives $X_k(N) \leq 0$, so the positive sum in (16) may be restricted to

$$1 \leq k \leq \lfloor 9a/10 \rfloor. \quad (40)$$

For every retained index, $j = a - k \geq a/10$.

6 Finite exponential prefixes and the direct saddle method

The positive corrections in (16) involve a product of two coefficients. Estimating those coefficients separately and only then multiplying loses precisely the dyadic gain needed later. The purpose of this section is therefore not merely to bound each factor, but to retain their joint structure until the two finite exponential prefixes have been combined.

For $m \in \mathbb{N}$ and $x \geq 0$, write

$$P_m(x) = \sum_{r=0}^m \frac{x^r}{r!}. \quad (41)$$

6.1 Generic finite saddle lemmas

Lemma 6.1 (Finite-prefix saddle bound). *Let $m \in \mathbb{N}$, let $\rho \geq 0$, and let $\ell_1, \dots, \ell_m \geq 0$. If*

$$\mathcal{E}_m(\ell) = [t^m] \exp \left(\sum_{r=1}^m \ell_r t^r \right),$$

then

$$\rho^m \mathcal{E}_m(\ell) \leq P_m \left(\sum_{r=1}^m \ell_r \rho^r \right). \quad (42)$$

Proof. After the change of variable $t \mapsto \rho t$, the left side is the coefficient of t^m in

$$\exp \left(\sum_{r=1}^m \ell_r \rho^r t^r \right).$$

Expanding by the number q of selected monomials, its q -block contribution is at most

$$\frac{1}{q!} \left(\sum_{r=1}^m \ell_r \rho^r \right)^q.$$

At most m blocks can contribute to degree m , because every block has positive degree. Summing $q = 0, \dots, m$ gives (42). $\square$

Lemma 6.2 (Prefix convolution). *For $m, n \in \mathbb{N}$ and $x, y \geq 0$,*

$$P_m(x)P_n(y) \leq P_{m+n}(x+y). \quad (43)$$

Proof. Expand the product over $0 \leq p \leq m$, $0 \leq q \leq n$, enlarge the rectangle to the triangle $p+q \leq m+n$, and group by $r = p+q$. The binomial identity then gives

$$\sum_{p+q=r} \frac{x^p y^q}{p! q!} = \frac{(x+y)^r}{r!}.$$

Every newly added summand is nonnegative. $\square$

The same argument will be used in the two elementary forms

$$(1+d)^n \leq P_n(nd), \quad \frac{x^r}{r!} \leq P_m(x) \quad (0 \leq r \leq m; d, x \geq 0). \quad (44)$$

6.2 Factorial-weighted exponent masses

Since $c_r = 6^r(r-1)!d_r$, the exponent masses in Lemma 6.1 contain factorial weights. We use the rational saddle parameter

$$\beta = \frac{34}{15}. \quad (45)$$

For $n \geq 40$, put

$$u_{n,r} = \beta^r \frac{(r-1)!}{n^r}.$$

Then

$$\frac{u_{n,r+1}}{u_{n,r}} = \frac{34r}{15n}. \quad (46)$$

The sequence is therefore first decreasing and then increasing, so every term with $4 \leq r \leq n$ is bounded by the sum of the two endpoints of that range. The exact endpoint estimates are

$$n(u_{n,2} + u_{n,3}) \leq \frac{1}{6}, \quad n^2 u_{n,4} \leq \frac{1}{8}, \quad n^2 u_{n,n} \leq \frac{1}{2}. \quad (47)$$

For the last inequality, the quantity $W_n = \beta^n n! / n^{n-1}$ satisfies $W_{40} \leq 1/2$ and

$$\frac{W_{n+1}}{W_n} = \frac{\beta}{(1 + 1/n)^{n-1}} \leq 1 \quad (n \geq 40).$$

Consequently,

$$\sum_{r=2}^n \beta^r \frac{(r-1)!}{n^r} \leq \frac{1}{n}. \quad (48)$$

A second estimate is sharper when the coefficient index is below a square-root scale. If $s \geq 32$ and $2 \leq k \leq s$, the consecutive quotient of $(r-1)!/s^r$ shows that the middle terms are bounded by the low endpoint, and one obtains

$$\sum_{r=2}^k \frac{(r-1)!}{s^r} \leq \frac{1}{s^2} + \frac{8}{s^3} \leq \frac{5}{4s^2}. \quad (49)$$

6.3 Single-factor saddle bounds

Let $Z_k^+(N)$ be the coefficient of t^k in $\exp(N \sum_{r \geq 1} c_r t^r)$. Coefficientwise absolute values give $Z_k(N) \leq Z_k^+(N)$. For $s = \lceil \sqrt{N} \rceil$, Lemma 6.1, Lemma 3.1, and (49) give

$$Z_k^+(N) \leq (6s)^k P_k \left(\frac{5s}{36} + \frac{1}{5} \right) \quad (2 \leq k \leq s). \quad (50)$$

For indices at least 40, use the radii

$$\rho_Z = \frac{17}{45k} = \frac{\beta}{6k}, \quad \rho_Q = \frac{34}{45j} = \frac{\beta}{3j}.$$

The tempered mass estimate (48) yields

$$Z_k^+(N) \leq \left(\frac{45k}{17} \right)^k P_k \left(\frac{17N}{54k} + \frac{4N}{25k} \right), \quad (51)$$

$$Q_j(N) \leq \left(\frac{45j}{34} \right)^j P_j \left(\frac{17N}{108j} + \frac{2N}{25j} \right). \quad (52)$$

The normalization rests on

$$\left(\frac{25r}{68}\right)^r \leq r!, \quad (53)$$

which follows by induction from $(1 + 1/r)^r \leq 68/25$. Together with the lower bound in (21), it implies

$$(6s)^k \leq \frac{36}{5} k c_k \frac{s^k}{k!}, \quad (54)$$

$$2^j \left(\frac{45j}{34}\right)^j = \left(\frac{45j}{17}\right)^j \leq \frac{36}{5} j c_j \left(\frac{6}{5}\right)^j, \quad (55)$$

$$\left(\frac{45r}{17}\right)^r \leq \frac{36}{5} r c_r \left(\frac{6}{5}\right)^r. \quad (56)$$

The identity $(15/34)(68/25) = 6/5$ explains the choice of β .

6.4 The direct product calculation

Theorem 6.3 (Direct product bounds). *Let $a, k, j, N \in \mathbb{N}$ satisfy $k + j = a$, $1 \leq k < a$, and $N \leq 12a$.*

If $s \geq 32$, $2 \leq k \leq s$, $N \leq s^2$, and $j \geq 40$, then

$$\begin{aligned} 2 \cdot 2^j Z_k(N) Q_j(N) &\leq \frac{2592}{25} k j c_k c_j \\ &\times P_{2a} \left(\frac{41}{36} s + \frac{j}{5} + \frac{29}{10} \frac{a}{j} + \frac{1}{5} \right). \end{aligned} \quad (57)$$

If $k, j \geq 40$, then

$$\begin{aligned} 2 \cdot 2^j Z_k(N) Q_j(N) &\leq \frac{2592}{25} k j c_k c_j \\ &\times P_{2a} \left(\frac{a}{5} + \frac{57}{10} \frac{a}{k} + \frac{29}{10} \frac{a}{j} \right). \end{aligned} \quad (58)$$

Proof. Because $Q_j(N) \geq 0$, either $Z_k(N) \leq 0$, in which case both assertions are immediate, or $0 < Z_k(N) \leq Z_k^+(N)$. We treat the latter case.

For the small branch, abbreviate

$$A = \frac{5s}{36} + \frac{1}{5}, \quad B = \frac{17N}{108j} + \frac{2N}{25j}.$$

Equations (50), (52), (54), and (55) give

$$\begin{aligned} 2 \cdot 2^j Z_k(N) Q_j(N) &\leq 2 \cdot 2^j Z_k^+(N) Q_j(N) \\ &\leq \frac{2592}{25} k j c_k c_j \frac{s^k}{k!} \left(\frac{6}{5}\right)^j P_k(A) P_j(B). \end{aligned}$$

Now

$$\frac{s^k}{k!} \leq P_k(s), \quad \left(\frac{6}{5}\right)^j = \left(1 + \frac{1}{5}\right)^j \leq P_j(j/5).$$

Applying Lemma 6.2 three times preserves the joint gain and produces

$$P_k(s) P_k(A) P_j(j/5) P_j(B) \leq P_{2a} \left(s + A + \frac{j}{5} + B \right).$$

The rectangle estimate $N \leq 12a$ gives

$$B \leq \left(\frac{17}{9} + \frac{24}{25} \right) \frac{a}{j} < \frac{29}{10} \frac{a}{j},$$

and $s + A = (41/36)s + 1/5$. This proves (57).

For the tempered branch, set

$$A_k = \frac{17N}{54k} + \frac{4N}{25k}, \quad A_j = \frac{17N}{108j} + \frac{2N}{25j}.$$

Using (51), (52), and (56) for both indices gives

$$\begin{aligned} 2 \cdot 2^j Z_k(N) Q_j(N) &\leq \frac{2592}{25} k j c_k c_j \left(\frac{6}{5} \right)^{k+j} P_k(A_k) P_j(A_j) \\ &= \frac{2592}{25} k j c_k c_j \left(\frac{6}{5} \right)^a P_k(A_k) P_j(A_j). \end{aligned}$$

The residual power satisfies $(6/5)^a \leq P_a(a/5)$, while prefix convolution gives $P_k(A_k) P_j(A_j) \leq P_a(A_k + A_j)$. A final convolution therefore bounds the product by $P_{2a}(a/5 + A_k + A_j)$. Finally,

$$A_k \leq \frac{57}{10} \frac{a}{k}, \quad A_j \leq \frac{29}{10} \frac{a}{j},$$

again by $N \leq 12a$. This proves (58). The factor 2^j has remained inside (55); this is the dyadic gain that disappears if the two normalized factors are bounded independently. $\square$

7 The positive budget for $a \geq 401$

We now insert Theorem 6.3 into the normalized majorant. Let

$$N_- = 6a - 7, \quad N_+ = 12a - 8, \quad s_- = \left\lceil \sqrt{N_-} \right\rceil, \quad s_+ = \left\lceil \sqrt{N_+} \right\rceil, \quad j = a - k.$$

For a rational $x < T$, define the certified exponential upper bound

$$\mathbf{E}_T(x) = \sum_{r=0}^{T-1} \frac{x^r}{r!} + \frac{x^T}{T!} \frac{1}{1 - x/T}. \quad (59)$$

Lemma 7.1 (Finite exponential majorant). *If $T \geq 1$, $m \in \mathbb{N}$, and $0 \leq x < T$, then*

$$P_m(x) \leq \mathbf{E}_T(x). \quad (60)$$

Proof. There is nothing to prove when $m < T$. Otherwise, for $r \geq T$ the quotient of consecutive summands $x^r/r!$ is $x/(r+1) \leq x/T < 1$. Hence the part of $P_m(x)$ beginning at degree T is bounded by the geometric series with first term $x^T/T!$ and ratio x/T , which is exactly the final term in (59). $\square$

For $401 \leq a \leq 2000$, define

$$\Xi_s(a, k) = \frac{1139}{1000} s_+ + \frac{j}{5} + \frac{29}{10} \frac{a}{j} + 1, \quad (61)$$

$$\Xi_t(a, k) = \frac{a}{5} + \frac{57}{10} \frac{a}{k} + \frac{29}{10} \frac{a}{j} + 2. \quad (62)$$

For $401 \leq a \leq 2000$ and every retained k one has $0 \leq \Xi_s(a, k) < 800$. In the tempered regime $s_- < k$ — and hence whenever $k > \lceil \sqrt{N} \rceil$ for an admissible N , which is where Ξ_t is used — one also has $0 \leq \Xi_t(a, k) < 800$. Thus each occurrence of E_{800} in (59) is well defined in the branch where it is used. The direct product theorem and elementary rational absorptions give

$$X_k(N)Y_j(N) \leq \frac{2581}{20} \frac{kj}{NN_+} \mathsf{E}_{800}(\Xi_s(a, k)), \quad k \leq \lceil \sqrt{N} \rceil, \quad (63)$$

$$X_k(N)Y_j(N) \leq \frac{2117}{20} \frac{kj}{N^2} \mathsf{E}_{800}(\Xi_t(a, k)), \quad k > \lceil \sqrt{N} \rceil. \quad (64)$$

For the small branch, the comparison

$$\frac{2592}{25} N_+ \leq \frac{5}{3} \frac{2581}{20} N$$

uses the lower edge $N \geq N_-$; the factor $5/3$ is absorbed as $P_1(2/3)$. The numerical coefficient $1139/1000$ exceeds $41/36$ by $1/9000$. For the tempered branch, $2592/25 < 2117/20$; the extra $+2$ in (62) is the explicit rational reserve used when the remaining prefix factors are absorbed.

Combining (25), (63), and (64), the retained positive convolution is bounded by a sum of explicit terms

$$M_s(a, k) = \frac{65}{N_+} \frac{kj}{(a-1)\binom{a-2}{k-1}} 2^{-j} \mathsf{E}_{800}(\Xi_s(a, k)), \quad (65)$$

$$M_t(a, k) = \frac{96}{N_-} \frac{kj}{(a-1)\binom{a-2}{k-1}} 2^{-j} \mathsf{E}_{800}(\Xi_t(a, k)). \quad (66)$$

The branch union must respect the regime conditions: the small bound (63) applies only for $k \leq \lceil \sqrt{N} \rceil$ and the tempered bound (64) only for $k > \lceil \sqrt{N} \rceil$, and $\lceil \sqrt{N} \rceil$ moves with N across $s_- \leq \lceil \sqrt{N} \rceil \leq s_+$. Outside its regime M_t has no content and can be large, so one may not simply take the unconditional maximum. The theorem-facing edge term is therefore the regime-aware

$$M_{\text{edge}}(a, k) = \begin{cases} M_s(a, k), & k \leq s_-, \\ \max\{M_s(a, k), M_t(a, k)\}, & s_- < k \leq s_+, \\ M_t(a, k), & s_+ < k, \end{cases} \quad (67)$$

which dominates the retained summand for every admissible N : if $k \leq \lceil \sqrt{N} \rceil$ then $k \leq s_+$ and the small bound, maximized at the upper edge $N = N_+$, applies; if $k > \lceil \sqrt{N} \rceil$ then $s_- < k$ and the tempered bound, whose prefactor is maximized at the lower edge $N = N_-$, applies; only in the overlap $s_- < k \leq s_+$ can the applicable branch depend on N , and there the maximum of the two is taken.

For later statements, denote the complete positive side of (16) by

$$\mathcal{V}_a(N) = 2^{-a-1} Y_a(N) + \frac{N}{2} \sum_{\substack{1 \leq k < a \\ X_k(N) > 0}} R_{k,a} 2^{-(a-k)} X_k(N) Y_{a-k}(N). \quad (68)$$

7.1 The finite strip $401 \leq a \leq 2000$

Proposition 7.2 (Finite positive budget). *If $401 \leq a \leq 2000$ and $6a - 7 \leq N \leq 12a - 8$, then*

$$\mathcal{V}_a(N) \leq 10^{-8}.$$

Proof. For every retained index, the regime-aware term $M_{\text{edge}}(a, k)$ dominates the corresponding convolution summand. An exact fixed-point calculation with outward rounding proves

$$\sum_{k \text{ retained}} M_{\text{edge}}(a, k) \leq \frac{1}{200000000} \quad (69)$$

for all $401 \leq a \leq 2000$. The rows are divided into sixteen blocks of one hundred; each block evaluates the rational majorant (59) at scale 10^{20} , and an induction over the arithmetic operations proves that the stored integer is an upper bound for the exact rational value.

The solo contribution is bounded separately by the closed factorial-composition estimate

$$2^{-a-1}Y_a(N) \leq \frac{1}{200000000}. \quad (70)$$

Adding (69) and (70) gives

$$\mathcal{V}_a(N) \leq \frac{1}{200000000} + \frac{1}{200000000} = 10^{-8}. \quad (71)$$

□

7.2 The transition strip $2001 \leq a < 3000$

Proposition 7.3 (Transition-strip positive budget). *If $2001 \leq a < 3000$ and $6a - 7 \leq N \leq 12a - 8$, then*

$$\mathcal{V}_a(N) \leq 10^{-8}.$$

Proof. The direct product inequalities are applied pointwise with truncation cutoffs a and $8a$ in Lemma 7.1. Exact adjacent-quotient certificates cover the lower-tempered transition where a single symbolic monotonicity statement would be inefficient. Each band is bounded by its boundary term times its proved geometric reserve, and the same factorial-composition estimate as in (70) controls the solo term. The denominator-cleared inequalities and the finite list of boundary rows are stated in Appendix C; together they allocate at most one half of 10^{-8} to the convolution and one half to the solo term. □

7.3 The analytic tail $a \geq 3000$

For the infinite range, the direct saddle bounds imply denominator-cleared versions of (63) and (64), with constants 130 and 192. The reciprocal binomial factor is controlled by

$$\binom{n}{k} \geq \left(\frac{n}{k}\right)^k, \quad (72)$$

and by a rational entropy-shadow inequality obtained from the maximal weighted binomial term. The retained range is split into a small branch and lower, middle, and upper tempered bands. Their candidate-majorant quotients satisfy

band	direction	quotient bound ϱ	boundary term
small ($k \leq \lceil \sqrt{N} \rceil$)	forward	$\varrho \leq \frac{1}{2}$	first candidate term
tempered, lower/middle	forward	$\varrho \leq (4a-1)/(4a)$	first term of the band
tempered, upper	reverse	$\varrho \leq (4a-1)/(4a)$	$k = \lfloor 0.9a \rfloor$

The tempered split occurs at $k = \lfloor a/3 \rfloor + 10$. Multiplication by the corresponding reserve $1/(1-\varrho)$ keeps each band within its assigned half-budget. The actual guarded summand at $k = 1$ is zero because $X_1 = -1$; the first small-branch term in the table is only the first term of the positive candidate sequence. The solo term is bounded through a $(10/7)^a$ envelope.

Proposition 7.4 (Infinite positive budget). *If $a \geq 3000$ and $6a - 7 \leq N \leq 12a - 8$, then*

$$\mathcal{V}_a(N) \leq 10^{-8}.$$

Proof. Appendix C proves the candidate domination, the three adjacent-ratio estimates, the endpoint reserves, and the $(10/7)^a$ solo bound as exact rational inequalities. Proposition C.2 there assembles the four bands and the solo term into the displayed budget. $\square$

Theorem 7.5 (Positive envelope). *For every integer $a \geq 401$ and every integer N with $6a - 7 \leq N \leq 12a - 8$,*

$$\mathcal{V}_a(N) \leq 10^{-8}. \quad (73)$$

Proof. Apply Proposition 7.2 when $a \leq 2000$, Proposition 7.3 when $2001 \leq a < 3000$, and Proposition 7.4 when $a \geq 3000$. $\square$

8 Completion of the proof

Proof of Theorem 1.1. For $a \geq 401$, the rectangle gives $N \leq 12a - 8 < (40/3)a$, so Theorem 5.4, applied with $m = a$, yields

$$X_a(N) \leq -\left(\lambda\left(1 - \frac{2}{a}\right) - \frac{2215}{a^2}\right). \quad (74)$$

An exact endpoint comparison at $a = 401$, followed by monotonicity, proves

$$10^{-8} < \lambda\left(1 - \frac{2}{a}\right) - \frac{2215}{a^2}. \quad (75)$$

Combining (16), Theorem 7.5, and (75) gives $U_a(N) < 0$ for every $a \geq 401$.

Proposition 4.1 covers every remaining value $a \leq 400$. $\square$

9 The corrected Proposition 5.2

The residue class $g \equiv 1 \pmod{3}$ requires a second scalar coefficient. We first derive the corrected source factor, then replace it in the required degree by a marked factor whose constant term displays the correction. The resulting identity separates the problem into the already controlled quotient coefficient and a new sign theorem for the series printed by Chen and Larson. The latter is where the Gamma argument enters.

Throughout this section,

$$\begin{aligned}
g &= 3a - 2, & M &= 2g - 2 = 6a - 6, & q_i &= m_i + 1, \\
N &= \sum_i q_i = M + n, & s_r &= \sum_i q_i^{-r}, & F_\mu(t) &= \frac{\prod_i \mathbb{C}(t/q_i)}{\mathbb{C}(t)^N}, \\
b_r(\mu) &= [t^r]F_\mu(t), & L(t) &= \frac{\mathbb{C}'(t)}{\mathbb{C}(t)}.
\end{aligned} \tag{76}$$

9.1 The source correction

Ionel's relation [2, Proposition 1.7, equation (1.13)], with parameter $b = 1$, contributes in the target degree the factor

$$\kappa_0 - 2\kappa_1 t - 12 \sum_{j \geq 1} j c_j \kappa_{j+1} t^{j+1}. \tag{77}$$

On $\mathcal{M}_g$, one has $\kappa_0 = 2g - 2 = M$. Equation (5.4) in the public arXiv v1 of Chen–Larson Proposition 5.2 instead has constant term 1. Dividing Ionel's relation by M would divide every term; changing only the constant term is therefore not a normalization.

Under the stratum pullback used by Chen and Larson, κ_j for $j \geq 1$ contributes $(N - s_j)\eta^j$, whereas κ_0 remains M . Hence (77) becomes $D_\mu^{\text{cor}}(\eta t)$, where

$$D_\mu^{\text{cor}}(t) = M - 2(N - s_1)t - 12t^2 \left(NL(t) - \sum_i q_i^{-2} L(t/q_i) \right). \tag{78}$$

The exponential prefactor is $F_\mu(\eta t)$, so the corrected geometric reduction is

$$\eta^a [t^a] F_\mu(t) D_\mu^{\text{cor}}(t) = 0. \tag{79}$$

The factor printed by Chen and Larson is (78) with the leading M replaced by 1; this is the only change, and it changes the coefficient.

9.2 The marked factor and the correction identity

Define

$$\Phi(t) = 2t + 12t^2 L(t), \quad K_\mu(t) = \sum_i (q_i - 1) \Phi(t/q_i). \tag{80}$$

The marked factors $M - K_\mu(t)$ and $1 - K_\mu(t)$ are not equal to the source factors as formal power series. They may be substituted only after the following degree-sensitive comparison.

Lemma 9.1 (Source–marked bridge). *For a scalar $\varkappa$, let $D_{\mu, \varkappa}$ be the expression in (78) with the leading M replaced by $\varkappa$. Then*

$$F_\mu(D_{\mu, \varkappa} - (\varkappa - K_\mu)) = -2MtF_\mu + 12t^2 F'_\mu. \tag{81}$$

Consequently, for every $r \geq 1$,

$$[t^r]F_\mu D_{\mu, \varkappa} - [t^r]F_\mu(\varkappa - K_\mu) = (12(r - 1) - 2M)b_{r-1}(\mu). \tag{82}$$

Proof. The constants $\varkappa$ cancel. In the t^2 terms, the identity $q_i^{-2} + (q_i - 1)q_i^{-2} = q_i^{-1}$ combines the two sums. The coefficient of t uses $\sum_i (q_i - 1)q_i^{-1} = n - s_1$ and $N - n = M$. Therefore

$$D_{\mu, \varkappa} - (\varkappa - K_\mu) = -2Mt + 12t^2 \left(\sum_i q_i^{-1} L(t/q_i) - NL(t) \right).$$

The parenthesis is F'_μ/F_μ , which proves (81). Taking the coefficient of t^r and using $[t^r](t^2 F'_\mu) = (r-1)b_{r-1}(\mu)$ proves (82). $\square$

At the Proposition 5.2 degree $r = a$, the multiplier in (82) vanishes because $M = 6a - 6$. We may therefore define

$$T_a^{\text{old}}(\mu) := [t^a]F_\mu D_{\mu,1} = [t^a]F_\mu(1 - K_\mu), \quad (83)$$

$$T_a^{\text{cor}}(\mu) := [t^a]F_\mu D_{\mu,M} = [t^a]F_\mu(M - K_\mu). \quad (84)$$

Subtracting the two marked forms gives the central identity

$$\boxed{T_a^{\text{cor}}(\mu) = T_a^{\text{old}}(\mu) + (M-1)b_a(\mu).} \quad (85)$$

9.3 The two large-range sign inputs

For a positive partition of $M = 6a - 6$, the parameter $N = M + n$ satisfies

$$6a - 5 \leq N \leq 12a - 12. \quad (86)$$

This interval lies inside the rectangle used in the Proposition 5.1 proof.

Proposition 9.2 (Rectangular quotient sign). *If $a \geq 9$, $6a - 7 \leq N \leq 12a - 8$, and μ is a positive list with $q_i = m_i + 1$ and $\sum_i q_i = N$, then $b_a(\mu) < 0$.*

Proof. The coefficientwise argument of Section 2 gives $b_a(\mu) \leq U_a(N)$ without using the value of $\sum_i (q_i - 1)$. For $9 \leq a \leq 400$, Proposition 4.1 gives $U_a(N) < 0$; for $a \geq 401$, the proof in Section 8 gives the same inequality throughout the rectangle. $\square$

In particular, (86) gives

$$b_a(\mu) < 0 \quad (a \geq 14). \quad (87)$$

It remains to prove that the printed coefficient is negative as well.

Proposition 9.3 (Printed coefficient in the middle range). *For every $14 \leq a \leq 149$ and every positive partition μ of $6a - 6$,*

$$T_a^{\text{old}}(\mu) < 0.$$

Proof. The coefficients c_r , $b_r(\mu)$, and $[t^r]K_\mu$ satisfy finite rational recurrences. A directed dyadic interval evaluation of these recurrences gives a negative upper endpoint for every partition parameter cell in the stated range. The interval operations are outward rounded, and their enclosure property is proved inductively from the rational recurrences. Thus the finite calculation proves the exact rational inequalities rather than importing a floating-point assumption. The certificate decomposition is listed in Appendix F. $\square$

9.4 The Gamma argument for the printed coefficient

We now prove the genuinely infinite sign input. Define

$$h_r = c_r(N - s_r), \quad \tau_r = \sum_i \frac{m_i}{q_i^r}, \quad k_1 = 2\tau_1, \quad k_r = 12(r-1)c_{r-1}\tau_r \quad (r \geq 2). \quad (88)$$

Then

$$F_\mu(t) = \exp\left(-\sum_{r \geq 1} h_r t^r\right), \quad K_\mu(t) = \sum_{r \geq 1} k_r t^r.$$

Put

$$p = \lfloor a/2 \rfloor, \quad r_0 = a - p - 1, \quad (89)$$

and form the low polynomials and series

$$\begin{aligned} \Lambda(t) &= \sum_{r=1}^p h_r t^r, & J(t) &= \sum_{r=1}^p k_r t^r, \\ E(t) &= \exp(-\Lambda(t)) = \sum_{s \geq 0} e_s t^s, & W(t) &= E(t)(1 - J(t)) = \sum_{s \geq 0} \omega_s t^s. \end{aligned} \quad (90)$$

For $0 \leq s < a$, let

$$\gamma_s = \frac{(a-s-1)!}{6^s(a-1)!}, \quad S_a(\mu) = \sum_{s=0}^{r_0} \gamma_s \omega_s. \quad (91)$$

The weights are Gamma moments: if G has the Gamma distribution of shape a and rate 1, and $T = 1/(6G)$, then $\mathbb{E}_a[T^s] = \gamma_s$.

Lemma 9.4 (Exact normalized split). *Define*

$$H_a = \sum_{s=0}^{r_0} \frac{h_{a-s}\omega_s}{Nc_a}, \quad K_a = \frac{1}{Nc_a} \sum_{s=0}^{r_0} k_{a-s}e_s, \quad \Omega_a = \frac{\omega_a}{Nc_a}. \quad (92)$$

Then

$$-\frac{T_a^{\text{old}}(\mu)}{Nc_a} = H_a + K_a - \Omega_a. \quad (93)$$

Proof. Split both exponent and marked factor at degree p . Since $2(p+1) > a$, a degree- a term contains at most one high coefficient, whether from the exponent or from the marked factor. Every term containing such a high exponent coefficient has complementary low degree at most $r_0 = a - p - 1$. Expanding the product and collecting the high exponent term, the high marked term, and the residual low coefficient gives

$$T_a^{\text{old}} = \omega_a - \sum_{s=0}^{r_0} h_{a-s}\omega_s - \sum_{s=0}^{r_0} k_{a-s}e_s.$$

Division by Nc_a and multiplication by -1 yields (93). $\square$

The next lemma isolates the analytic main term from three coefficient errors.

Lemma 9.5 (Normalized split errors). *For $a \geq 150$ and every positive partition of $6a - 6$,*

$$|H_a - S_a(\mu)| \leq \frac{86069}{3125a^2}, \quad |K_a| \leq \frac{1}{a^2}, \quad |\Omega_a| \leq \frac{1}{a^2}. \quad (94)$$

Proof. For the first estimate, compare $h_{a-s}/(Nc_a)$ with the Gamma weight γ_s . The factorial part is exactly γ_s , while the quotient d_{a-s}/d_a is controlled by Lemma 3.2; the remaining factor $1 - s_{a-s}/N$ is between 0 and 1. Summing the resulting weighted error and the low-degree endpoint correction gives the constant $86069/3125 = 36(2304/3125) + 1$.

For K_a and Ω_a , replace the coefficients of E and W by the positive coefficientwise majorants obtained from $\exp(\sum_{r \leq p} h_r t^r)(1 + \sum_{r \leq p} k_r t^r)$. The bounds (21), the partition inequalities $0 \leq s_r \leq n$ and $0 \leq \tau_r \leq N$, and the factorial-mass estimates of Section 6 reduce both normalized tails to geometric series with sum at most a^{-2} . Appendix B records the exact factorial sums and denominator-cleared endpoint inequalities. $\square$

Lemma 9.6 (Gamma main margin). *For $a \geq 150$ and every positive partition of $6a - 6$,*

$$\mathbb{E}_a \left[W \left(\frac{1}{6G} \right) \right] \geq \frac{9}{40(a-2)}. \quad (95)$$

Proof. Integration by parts against the $\text{Gamma}(a, 1)$ density gives

$$\mathbb{E}_a[W(T)] = \mathbb{E}_a \left[e^{-\Lambda(T)} (MT + 6T^2 \Lambda'(T) - J(T)) \right], \quad T = \frac{1}{6G}. \quad (96)$$

Indeed, differentiating $e^{-\Lambda(1/(6y))}$ produces $6T^2 \Lambda'(T) e^{-\Lambda(T)}$, while the logarithmic derivative of the Gamma density contributes $MT = 6(a-1)T$. The boundary terms vanish at both 0 and $+\infty$, and the displayed integrands are absolutely integrable, so the integration-by-parts identity is justified.

The bracket in (96) has nonnegative coefficients. Its linear coefficient is $M - k_1 \geq 0$; for $1 \leq r < p$, the coefficient of t^{r+1} is

$$6rh_r - k_{r+1} \geq 0,$$

and the terminal coefficient of t^{p+1} is $6ph_p \geq 0$. For $r = 1$, the sharper identity

$$6h_1 - k_2 - 5M = 5 \sum_i \frac{(q_i - 1)(q_i - 2)}{q_i^2} \geq 0$$

gives $6h_1 - k_2 \geq 5M$. Therefore

$$MT + 6T^2 \Lambda'(T) - J(T) \geq 5MT^2 \quad (T \geq 0). \quad (97)$$

Let $G_b \sim \text{Gamma}(b, 1)$ and set $T_b = 1/(6G_b)$. Changing the Gamma shape by two gives

$$5M \mathbb{E}_a[T_a^2 e^{-\Lambda(T_a)}] = \frac{5M}{36(a-1)(a-2)} \mathbb{E}_{a-2}[e^{-\Lambda(T_{a-2})}] \quad (98)$$

$$= \frac{5}{6(a-2)} \mathbb{E}_{a-2}[e^{-\Lambda(T_{a-2})}]. \quad (99)$$

Jensen's inequality now yields $\mathbb{E}_{a-2}[e^{-\Lambda(T_{a-2})}] \geq e^{-\mathbb{E}_{a-2}[\Lambda(T_{a-2})]}$. The coefficient bounds and Gamma moments give the explicit mean estimate

$$\mathbb{E}_{a-2}[\Lambda(T)] \leq \frac{5}{4} + \frac{5}{2(a-3)} + \frac{96a^2}{25(a-5)^3} < \frac{13}{10}. \quad (100)$$

Finally, $e^{-13/10} > 27/100$. Combining (96)–(100) gives

$$\mathbb{E}_a[W(T)] \geq \frac{5}{6(a-2)} \frac{27}{100} = \frac{9}{40(a-2)}.$$

$\square$

The finite sum $S_a(\mu)$ stops at r_0 , so one still has to control a Taylor remainder without interchanging a divergent collection of higher negative Gamma moments.

Lemma 9.7 (Gamma–Taylor truncation). *For $a \geq 150$, let $s_0 = \lfloor a/8 \rfloor$. Then*

$$S_a(\mu) \geq \mathbb{E}_a \left[W \left(\frac{1}{6G} \right) \right] - R_a, \quad (101)$$

where

$$R_a = \frac{920}{2^{r_0+1}} + 2 \left(\frac{5}{6} \right)^a + 9 \left(\frac{9}{10} \right)^{a-s_0} + 920a \left(\frac{3}{10} \right)^{s_0+1} < \frac{1}{a^2}. \quad (102)$$

Proof. Split the Gamma integral into $G \geq a/2$ and $G < a/2$. On the first event, $T \leq x_1 = 1/(3a)$. A positive coefficient majorant for W at $x_2 = 2/(3a) = 2x_1$ is at most 920, so the Taylor terms beyond r_0 contribute at most $920/2^{r_0+1}$.

On the lower event, a Gamma Chernoff estimate gives $\mathbb{P}_a(G < a/2) \leq (5/6)^a$. Since $|W(T)| = e^{-\Lambda(T)}|1 - J(T)| \leq e^{-\Lambda(T)}(1 + J(T)) \leq 2$, the pointwise majorant contributes the second term of R_a . For the remaining Taylor comparison, split the monomial degrees at $s_0 = \lfloor a/8 \rfloor$. The low degrees are bounded by the Gamma-moment ratio and sum to at most $9(9/10)^{a-s_0}$; the high degrees are compared with the x_2 majorant and sum to at most $920a(3/10)^{s_0+1}$. These are exactly the four terms in (102). Direct denominator clearing at the initial values, followed by the monotone ratio bounds of Appendix B, proves $R_a < a^{-2}$ for every $a \geq 150$. $\square$

Proposition 9.8 (Printed coefficient in the Gamma range). *For every $a \geq 150$ and every positive partition μ of $6a - 6$,*

$$-\frac{T_a^{\text{old}}(\mu)}{Nc_a} \geq \frac{9}{40(a-2)} - \frac{95444}{3125a^2} > 0. \quad (103)$$

In particular, $T_a^{\text{old}}(\mu) < 0$.

Proof. Lemmas 9.6 and 9.7 give

$$S_a(\mu) \geq \frac{9}{40(a-2)} - \frac{1}{a^2}.$$

Insert this into the exact split of Lemma 9.4 and use the three estimates in Lemma 9.5. The total a^{-2} coefficient is

$$1 + \frac{86069}{3125} + 1 + 1 = \frac{95444}{3125},$$

which proves the first inequality in (103). After denominators are cleared, positivity of the right side is equivalent to

$$28125a^2 - 3817760a + 7635520 > 0.$$

At $a = 150$ the margin is $3389201/20812500000$; writing the quadratic in powers of $a - 150$ gives positive coefficients, so the margin increases thereafter. $\square$

Propositions 9.3 and 9.8 together give

$$T_a^{\text{old}}(\mu) < 0 \quad (a \geq 14). \quad (104)$$

Combining (85), (87), and (104) proves

$$T_a^{\text{cor}}(\mu) < 0 \quad (a \geq 14), \quad (105)$$

because $M - 1 > 0$.

9.5 The remaining finite degrees

Proposition 9.9 (Corrected coefficient in low degree). *For $2 \leq a \leq 13$ and every positive partition μ of $6a - 6$,*

$$T_a^{\text{cor}}(\mu) \neq 0.$$

Proof. The coefficient is computed from the finite recurrences as

$$T_a^{\text{cor}} = Mb_a - \sum_{r=1}^a k_r b_{a-r}.$$

For $2 \leq a \leq 8$, exhaustive partition enumeration evaluates this rational number exactly. For $9 \leq a \leq 13$, the same recurrence is evaluated modulo the prime 1000000007. The denominator factors are proved invertible modulo that prime, and every resulting residue is nonzero. A rational number whose denominator is invertible and whose reduction is nonzero cannot vanish. The partition-count and modular-certificate inventory appears in Appendix F. $\square$

Proof of Theorem 1.3. Lemma 9.1 identifies the source coefficient $[t^a]F_\mu D_\mu^{\text{cor}}$ with $T_a^{\text{cor}}(\mu)$, because $12(a-1) - 2M = 0$. Proposition 9.9 proves non-vanishing for $2 \leq a \leq 13$, and (105) proves strict negativity for $a \geq 14$. $\square$

Remark 9.10 (The correction changes the answer). For $g = 4$, so $a = 2$ and $M = 6$, and for $\mu = (1^6)$, one has

$$b_2 = -\frac{195}{8}, \quad T_2^{\text{old}} = \frac{45}{8}, \quad T_2^{\text{cor}} = -\frac{465}{4}.$$

Thus the printed and corrected coefficients even have opposite signs. The identity (85) reads $-465/4 - 45/8 = 5(-195/8)$ in this example.

Remark 9.11 (Scope). The theorem is proved for positive integral partitions $m_i \geq 1$. The corrected pullback factor makes sense in a broader rational-part setting, but no global sign statement is asserted there.

Part II

Formal verification and AI-assisted research process

10 Formalization architecture and trust

Part II treats formal verification and the AI-assisted research process as a second, substantial contribution. It is logically downstream from Part I: the formal development verifies the mathematical statements already formulated there, and the process record explains how those statements and proofs were generated, challenged, and revised. Neither the Lean implementation nor the provenance narrative is needed to follow the mathematical proofs.

The formalization nevertheless contributes more than a terminal correctness check. Its proof interfaces forced the analytic argument into explicit finite-range, sign-lock, saddle, and tail modules; failed interfaces exposed false global strategies; and the source-level formalization of the corrected Proposition 5.2 separated a true coefficient statement from the geometric statement that the printed factor was intended to encode. This section maps the mathematical architecture to Lean and states

the resulting trust boundary. Section 11 then records the AI-assisted workflow that produced and audited both the mathematics and the formal development.

10.1 Public statement layer

The repository now has a concise public facade. Its mathematical content is:

1. the definition of the hypergeometric power series $\mathbb{C}$;
2. the coefficient identity for $\mathbb{C}$;
3. the quotient-series identity

$$\mathbb{C}(t)^N F_\mu(t) = \prod_i \mathbb{C}(t/(m_i + 1));$$

4. strict negativity of the coefficient in Theorem 1.1;
5. the non-vanishing corollary.

The bridge from the recurrence-defined c_r to the formal power series is proved by showing that both $\mathbb{C}$ and $\exp(\sum_{r \geq 1} c_r t^r)$ solve the same formal differential equation with constant term one.

For the corrected Proposition 5.2 the public facade is `Prop52/Theorem.lean`. It imports `Prop52/Source.lean`, where the source coefficient `sourceCoeff`, the coefficient-level product pinning, the source-marked bridge `sourceCorrectedCoeff_eq`, and the $g = 4$ executable checks live. The facade itself exposes only two genus-indexed source theorems: one for non-vanishing and one for strict negativity. The exact Lean identifiers are listed in Appendix D. The marked theorem names `correctedCoeff_nonvanishing` and `correctedCoeff_neg` are retained behind the facade in `Prop52/Assembly.lean`; the bridge turns them into the source-shaped statement. Both public theorems carry the same axiom report as the Proposition 5.1 facade.

10.2 Proof layers

The production proof has the following conceptual dependency chain:

layer	role
series bridge	identifies the recurrence-defined coefficients with the Chen–Larson generating series
majorant	replaces the partition-dependent numerator by $Q_r(N)$ and proves $b_a(\mu) \leq U_a(N)$
finite certificates	handles $a \leq 400$ by enumeration, exact rationals, and verified dyadic intervals
sign lock	eliminates $k > \lfloor 0.9a \rfloor$ and supplies the negative margin
direct saddle	proves the finite-prefix product inequalities (57) and (58)
positive assembly	checks the bounded edge budget and closes the prefix and infinite tail by ratio/reserve estimates
completion	combines all ranges into the unconditional public theorem

10.3 Trust boundary

Almost all of the argument is checked by Lean’s small logical kernel, the same trusted core that underlies every Mathlib proof. The exception is the family of large finite computations — the enumerations and the exact and interval certificates — which are run by `native_decide`: Lean compiles the finite Boolean check to machine code, executes it, and accepts the result. This is faster by orders of magnitude than kernel reduction and is what makes the millions of certificate cases feasible, but it widens the trusted base for those steps to the Lean compiler and its runtime, together with any relevant native implementations they invoke. Following the Lean reference manual [3], each such use is recorded by a named axiom, and for the present development the two additional native-evaluation axioms are `Lean.ofReduceBool` and `Lean.trustCompiler`, alongside the standard `propext`, `Classical.choice`, and `Quot.sound`. Nothing else — in particular no `sorry`, no project-specific axiom, and no floating-point assumption — enters the final theorems.

This is verifiable in one command. The repository ships a short audit script that prints, for the series bridge, the majorant inequality, the Proposition 5.1 negativity theorem and non-vanishing corollary, and the corrected Proposition 5.2 theorems, the full list of axioms each depends on; a reader can reproduce it from a clean checkout with

```
lake exe cache get    # download the prebuilt Mathlib
lake build            # check every proof, including the certificates
lake env lean scripts/PublicAxiomsReport.lean # print the axiom lists
```

The completed proof and the compact public facade are published as a tagged repository release. As an additional audit, the external checker `lean4checker` can replay the ordinary (non-native) proof objects and can detect certain bugs involving import state or declaration processing. It does not, however, remove the compiler trust attached to the native-evaluation certificates: proofs that depend on `Lean.trustCompiler` cannot at present be validated by an external proof checker without first replacing those computations by kernel-checkable certificates [3].

11 AI-assisted research process and provenance

The AI-assisted research process is a substantive subject of this manuscript, not merely a disclosure attached to the proof. The record studies a concrete case in which general-purpose generative systems proposed the central mathematical architecture, generated and repaired a large Lean development, performed adversarial audits, and identified an error in the printed source relation. The mathematical conclusions remain those of Part I and stand independently of this provenance; the purpose here is instead to make the research method inspectable and to distinguish precisely among mathematical generation, formal certification, and human orchestration.

The project began after Hannah Larson’s talk *Tautological classes on the strata of differentials* in the Algebraic Geometry and Moduli seminar at ETH Zurich on 10 June 2026, which described the conjecture of Chen and Larson. The author selected the problem and then worked almost entirely through general-purpose AI tools. The human role was orchestration: designing and relaying prompts, maintaining the computational and Lean environment, deciding from high-level progress signals when an independent review should be solicited, curating the provenance record, and taking responsibility for the final manuscript. The author did not originate or modify a mathematical lemma, estimate, proof strategy, or Lean proof. The systems generated the mathematical content and most of the formal development, but no theorem-level claim was accepted merely because a model asserted it: acceptance rested on a Lean proof or, for expository reformulations of already

formalized statements, on an independently checked mathematical derivation.

We describe the workflow through chronological *human–AI interaction cards*, following the format proposed by Feng et al. [5]. Each card records the task, model, essential prompt, reply summary, and resulting human action; its title links to the public conversation when one exists. The prompts are reproduced verbatim or in close paraphrase because prompt design and the routing of work among independent systems are the most transferable parts of the method. The replies are summarized, with the most relevant exchanges archived alongside the source.

In the content-based terminology of Feng et al., the mathematical generation is classified as **Level A**: no human supplied or edited the mathematical lemmas, estimates, proof strategies, or Lean constructions. This label does not mean that the project was unattended. The author selected the target, organized the agents, interpreted only high-level signs of progress or stalling, and relayed independent diagnoses. In particular, the interventions in Cards 6 and 7 were mathematical diagnoses produced by separate AI models and forwarded to the worker without human alteration. The classification is therefore about the source of the mathematical content, not about the absence of human project management. We make no process-based claim about the mathematical significance of the theorem: that assessment belongs to peer review, and correctness rests on the proof rather than on the provenance.

The workflow had two phases. During mathematical development, one ChatGPT 5.5 Pro conversation was the central strand; independent conversations alternated between adversarial audits and strategic guidance, and their findings were fed back into that strand over ten drafts. During formalization, the work moved to Lean under a persistent autonomous goal. When that run stalled, two fresh-model reviews diagnosed defects in the current architecture and supplied pivots that were relayed to the worker. The cards below make this routing, including the points of failure and correction, explicit.

Card 1: Central investigation strand

ChatGPT 5.5 Pro

Task. Attack the Chen–Larson coefficient problem from scratch and develop a candidate proof.

Prompt.

Can you have a look at Conjecture 1.4 of arXiv:2603.23850 (and its explicit reformulation Proposition 5.1) and make a serious, sustained, high-effort research attack at trying to crack it? This is an open problem, but in similar conversations you already managed to crack several such problems, and you might have a perspective on this power series that is not available easily to the authors. Would be really curious if you can make progress here!

Reply (summary). Proposed the coefficientwise majorization that removes the partition, the normalized coefficients X_r, Y_r , and the strategy of a negative leading term against small positive corrections.

Continuation. This single conversation became the central strand: in later turns it produced the $\text{\LaTeX}$ write-up and Python verification scripts, and carried the mathematics through ten successive drafts, fed by the audits and guidance of Cards 2 and 3.

Card 2: Adversarial mathematical audit**ChatGPT 5.5 Pro**

Task. Surface every defect in a draft package (notes and code) before trusting it.

Prompt.

Consider the appended zip file. Please read the contained mathematical notes in detail and inspect all relevant code files, then proceed with a full mathematical audit, going paragraph by paragraph to identify any mathematical errors, gaps in arguments, hand-waving, misapplications of known results, or mis-cited theorems — anything that could affect the validity of the presented arguments. Then give me a full report.

Reply (summary). Returned concrete defects — a far-tail constant valid only in a smaller range, a positive-part bound vacuous at $a = 401$, a single-edge scan that missed the worst value of N — each then repaired.

Human action. Fed each reported defect back into the central strand for repair, and re-ran the audit on the next draft.

Technique. A fresh conversation used purely adversarially: an exhaustive, paragraph-level audit of notes and code, repeated each revision.

Card 3: Strategic guidance**Claude Fable 5**

Task. Get higher-level direction, not line edits, when a draft stalled.

Prompt.

I wanted to ask for advice about the appended pure mathematics research note. For the theorem it tries to prove it shows some partial progress, but gaps remain. Please look through it carefully, think hard, do your own explorations or search for sources, and then give me a report focused on the most helpful advice possible for a second AI which will be tasked to continue pushing for a solution — concrete pointers to results, obstructions, a higher-level pivot, anything that could help crack the problem.

Reply (summary). Returned strategic guidance — which effective-constant program to pursue, the worst-case regimes to target, and higher-level pivots — phrased as instructions the authoring model could act on.

Human action. Forwarded the guidance into the central strand to shape the next revision.

Technique. A second model used not to find errors but to suggest direction, with the prompt explicitly asking for advice aimed at the next AI in the loop.

Card 4: First formalization push**Claude Fable 5 (Claude Code)**

Task. Begin the Lean development: the statement layer, the series bridge, the partition-free majorant, the finite certificates, and the sign lock.

Prompt (excerpt).

Following your advice, the author put a tenth revision package together... this could be the basis of a first push towards a formalization by you. Take the time to review any changes, then if you think the basis is solid enough please go ahead and start developing the Lean code here... The goal would be to isolate the desired end theorem in a file with as few imports, notation etc. as possible, with a clear certificate that the proof is sorry- and axiom-free.

Reply (summary). Produced the bridge identifying $\mathbb{C}$ with $\exp(\sum c_r t^r)$, the majorant inequality, the exact and dyadic-interval finite certificates, and the rational toolkit underlying the sign lock.

Human action. Reviewed and integrated the layers, then handed the remaining analytic tail to a dedicated completion run.

Card 5: Long-horizon completion drive**OpenAI Codex CLI**

Task. Drive the formalization to completion with minimal supervision.

Prompt.

/goal Please continue working on the Lean formalization of the argument, until it is either finished or until you hit a serious obstacle which requires either mathematical input or technical assistance beyond the standard difficulties and problem solving of the formalization process.

Reply (summary). Over many cycles of inspection, coding, compilation, benchmarking, and repair, the worker proposed proof decompositions, searched Mathlib, refactored, generated finite-certificate checkers, and profiled slow computations.

Human action. Monitored progress, spot-checked the Lean state, and intervened only at the obstacle the goal itself anticipated (Card 6).

Technique. A persistent goal that doubles as a completion condition attached to the thread [6]; before this, continuation had to be re-prompted every twenty to thirty minutes.

Card 6: “Is the worker stuck?” — a review of the process**ChatGPT 5.5 Pro; Claude Opus 4.8**

Task. Decide whether the multi-day Codex run was healthy exploration or a dead end, and how to finish.

Prompt (excerpt).

Please review both the proof and the formalization files in detail. (i) Do you see any mathematical errors or obstructions in the strategy, and are they fixable? (ii) The formalization has run with a Codex worker for about two days; is this the expected process of investigation and optimization, or is the worker stuck in a local minimum or dead end? (iii) Any advice on how to finish, which could be forwarded to the worker.

Reply (summary). Run in parallel in both systems; the reviews found no fatal obstruction in the main strategy but flagged unresolved analytic estimates and a proliferation of proof interfaces, with the worker reshaping one product-tail reduction rather than proving it, and advised freezing the completion route around a single direct combined-product inequality.

Human action. Forwarded the advice to the worker and gathered the worker’s own specific blocking questions for the deeper review of Card 7.

Technique. Reviewing the process, not only the mathematics, and running the same review in two independent systems.

Card 7: The correction that unstuck the proof**ChatGPT 5.5 Pro; Claude Opus 4.8**

Task. Resolve the worker’s blocking questions about the large-tail product estimate.

Prompt (relayed, excerpt).

[Relaying the worker’s own questions:] What is the most direct analytic proof of the sharp large-tail product target, preferably by modifying the existing split rather than introducing a large generated table?

Reply (summary). The forwarded answer showed that the attempted route — bounding the two factors separately and multiplying — is not merely expensive but false at the first large-tail cell $(a, k) = (3000, 4)$, off by about $3.26 \cdot 10^{557}$ because the split discards a factor 2^{a-k} ; it prescribed the fix used in Section 6: keep the product together, bound it with finite exponential prefixes, prove the prefix-convolution inequality, and normalize only at the last step.

Human action. Relayed the corrected route to the worker, which implemented it as the direct-saddle method; the proof closed shortly afterward.

Technique. Escalating the worker’s own questions to a fresh, stronger review, and letting a provably false target redirect the architecture.

What the two interventions show. The episode in Cards 6 and 7 is worth keeping in the record. The autonomous worker had made genuine local progress around an interface that was globally false: separating the two factors and multiplying them loses the dyadic cancellation, and no

amount of finite refinement could have repaired it. What exposed this was not a sharper estimate but the formal target itself, which refused to close. Formal verification thus did more than confirm a known proof: it prevented a plausible but invalid analytic architecture from being mistaken for a theorem, and it marked exactly the moment at which the autonomous worker needed an outside perspective. That perspective, too, came from an AI model; the author’s part was the orchestration — reading a high-level signal that the run had stalled and routing the question to a fresh review — not any judgement about the mathematics. The mathematical content was AI-generated within a human-orchestrated multi-agent process, and the human decisions that mattered were about *when* the worker was stuck, not how to fix it.

Scope of this record. The seven cards are a curated, not exhaustive, account: they record the interactions that materially shaped the proof and omit many routine continuation prompts and unproductive sessions.

The write-up itself was likewise AI-produced, in three stages. The original technical research note — the tenth revision, focused purely on the mathematics — was authored by ChatGPT 5.5 Pro in the initial research strand of Card 1. After the formalization was complete, a fresh expository draft of the present paper was produced by ChatGPT 5.5 Pro in a new conversation, following the author’s high-level instructions on organization and intended audience. The subsequent revisions — including those prompted by an external review of the release candidate — were carried out in Claude Code under the author’s direction, together with further ChatGPT 5.5 Pro review. As with the mathematics, the author’s contribution to the write-up was direction and editorial judgement rather than composition; and acceptance of each formalized claim rested on its Lean proof rather than on any model’s assurance.

11.1 Addendum: the corrected Proposition 5.2

The same methodology produced the companion result, with two features worth recording: an AI audit caught a genuine error in the *printed* (arXiv v1) relation, and the formalization then ran to completion without further mathematical or technical intervention.

After the Proposition 5.1 result was shared with the authors — who were open to the approach and asked whether the remaining case $g \equiv 1 \pmod{3}$ (Proposition 5.2) could now be attacked — the author posed that problem to a ChatGPT 5.5 Pro conversation,¹ which produced a candidate non-vanishing proof for the *printed* Proposition 5.2 within three rounds. A separate ChatGPT 5.5 Pro audit of that candidate² flagged that the derivation of the series in Proposition 5.2 was itself in error: the constant $\kappa_0 = 2g - 2$ of Ionel’s relation had been replaced by 1. An independent AI audit of the Chen–Larson paper alone³ confirmed the error and supplied the corrected derivation of Section 9.1; asked to validate the point, the authors agreed, in private correspondence (June 2026), that the printed relation was in error. (As of June 2026, the public arXiv v1 still prints the leading constant 1.)

The corrected target was fed back into the original conversation, which — in two further rounds with intermediate feedback⁴ — assembled a complete bundle: an informal proof of the corrected statement, an independent audit, the pinned Proposition 5.1 dependency, and a detailed Lean formalization plan with milestones. That bundle was handed to a fresh OpenAI Codex CLI worker, which completed the entire Lean formalization — the rectangle export, the correction identity, the

¹<https://chatgpt.com/share/6a3bc8c9-e06c-83eb-b3e5-d098bb04573f>

²<https://chatgpt.com/share/6a3bc91c-0460-83eb-857d-d821168e38b7>

³<https://chatgpt.com/share/6a393ddd-bbc0-83eb-aa30-44806c9edc58>

⁴<https://chatgpt.com/share/6a3bc93b-3c68-83eb-8735-cb143fe10b12>

finite and modular certificates, the in-Lean re-proof of the printed-coefficient sign, and the two public theorems (sorry-free, with no project-specific axioms) — *autonomously, with no further mathematical or technical intervention*, in roughly twenty hours. In the classification of [5] this is again Level A, and the human role was smaller still than for Proposition 5.1, reducing to relaying the problem, the audit verdict, and the finished bundle.

Two points deserve emphasis. First, the error was in the *printed manuscript*, not in any formalization: the series printed in Proposition 5.2 is a valid power series, but it is not the one Ionel’s relation produces, so a formal proof about the printed coefficient would have certified a true statement that does not carry the intended geometric meaning. The defect was surfaced by adversarial AI review of the paper itself — exactly the kind of error formal verification does not catch, because verification certifies *what one states*, while deciding what to state still needs independent mathematical judgement. Second, the contrast with Proposition 5.1 is instructive: given a sufficiently detailed, AI-authored formalization plan, the autonomous worker needed none of the mid-run redirection (Cards 6 and 7) that the first development had required.

Quantitative notes. The Proposition 5.1 completion drive ran as a single Codex `/goal` thread, for which the goal tracker recorded about sixty-four hours of elapsed time; the separate Proposition 5.2 drive took roughly twenty hours by the same measure. These are elapsed wall-clock figures as logged, not active compute time: they include waiting and compilation and are not summed over parallel workers. The calendar span of the Proposition 5.1 thread was longer still — it stayed open from 15 to 20 June 2026 — because the goal was paused between work sessions. Human active time was not separately measured, and token counts and monetary cost were not retained; the figures are given only to convey scale. The formal proof was checked under Lean 4.27.0 and a Mathlib revision pinned in the package manifest.

On the persistent goal. The `/goal` prompt of Card 5 is reproduced as it was actually used. By later best-practice standards [6] it is deliberately minimal: it states the desired outcome and the condition under which to stop and ask for help, but not the verification command, the permitted files, or an iteration policy. In practice the evolving Lean goal state, successful compilation, and the axiom reports served as the implicit evidence surface; a reproduction attempt should make those conditions explicit.

Limitations. Formal verification establishes the formal theorem relative to the trust boundary of Section 10. It does not by itself establish novelty or significance, the correctness of bibliographic claims, or the faithfulness of every sentence of the informal paper to the formal definitions; those remain matters of human and community judgement. The development also illustrates characteristic failure modes of AI-assisted research, which this record deliberately preserves: plausible but false intermediate estimates, a proliferation of proof interfaces, and convincing local progress on a globally false target. Curated, redacted records of the command-line-agent sessions — every on-topic human instruction from the selected sessions, with the AI side summarized — together with the most relevant review exchanges are archived under `docs/ai-correspondence/`, and the card titles link to the primary public conversations, so that the provenance can be inspected directly.

12 Reproducibility and repository organization

The repository is organized so that a reader can audit the public statements without first traversing the development history. Its four entry points are

`Prop51/Theorem.lean`, `Prop52/Theorem.lean`, `Prop51.lean`, `Prop52.lean`.

The two `Theorem.lean` files state the hypergeometric series, the source-shaped quotient identities, and the public sign or non-vanishing theorems. The root files import the complete corresponding libraries. The paper-to-Lean map in Appendix E identifies the backend declaration for every mathematical interface used in Part I; Appendix F records the inputs and outputs of the finite certificates.

The archived release fixes the Lean toolchain at Lean 4.27.0 and pins the Mathlib revision in the package manifest. From a clean checkout the complete verification and the public axiom audit are run by

```
lake exe cache get
lake build
lake env lean scripts/PublicAxiomsReport.lean
```

The first command obtains the matching prebuilt Mathlib cache, the second checks both proposition libraries and all finite certificates, and the third prints the axiom dependencies of the public theorem layer. The release also contains the certificate-generation provenance and the curated AI correspondence under `docs/ai-correspondence/`. Thus the mathematical proof, formal certificate boundary, and process record can be inspected separately while remaining tied to the same versioned source tree.

A Residual-envelope and sign-lock audit details

This appendix records the scalar estimates used in Propositions 5.1 and 5.3. The purpose is not to reproduce the Lean implementation, but to state the two analytic interfaces and the ordinary inequalities used to certify them. All constants below are rational, and every finite polynomial assertion is certified after positive denominators have been cleared.

A.1 Summing the nonlinear residual blocks

Recall the block majorant

$$\Delta_{p,N}^{(r)} = \frac{36}{5} \left(\frac{4}{25} \right)^r 4^{r-1} N^{r-1} \frac{(p-2r+1)!}{r!(p-1)!} \quad (2 \leq r \leq p/2).$$

Write

$$\Delta_{p,N}^{\text{near}} = \sum_{2 \leq r \leq p/4} \Delta_{p,N}^{(r)}, \quad \Delta_{p,N}^{\text{far}} = \sum_{p/4 < r \leq p/2} \Delta_{p,N}^{(r)}.$$

Under the sign-lock hypotheses, $m \geq 361$, $2m \leq 3p$, and $N \leq (40/3)m$. In particular,

$$p \geq 241, \quad N \leq 20p. \tag{106}$$

For $2 \leq r < p/4$, the exact quotient from (31) is bounded by

$$\frac{\Delta_{p,N}^{(r+1)}}{\Delta_{p,N}^{(r)}} \leq \frac{64R}{75p}, \quad R = 20.$$

The right side is $256/(15p) < 1$; in fact it is at most $1/11$ for $p \geq 241$. Combining this with the first block gives

$$\Delta_{p,N}^{\text{near}} \leq \frac{1152}{3125} \frac{20p}{(p-1)(p-2)} \frac{1}{1-256/(15p)} \quad (107)$$

$$\leq \frac{196416}{15625m}. \quad (108)$$

For the last inequality, use $m \leq 3p/2$, first to obtain

$$20mp \leq 31(p-1)(p-2),$$

and then use $(1 - 256/(15p))^{-1} \leq 11/10$. These are polynomial inequalities on $p \geq 241$.

For the far blocks, the factorial quotient satisfies

$$\frac{(p-2r+1)!}{(p-1)!} \leq \frac{1}{(2r-1)!} \quad (p/4 < r \leq p/2).$$

Together with $N \leq 20p < 80r$, this gives

$$\Delta_{p,N}^{(r)} \leq \hat{\Delta}_r := \frac{9}{5} \left(\frac{16}{25} \right)^r \frac{(80r)^{r-1}}{r!(2r-1)!}. \quad (109)$$

The consecutive quotient is

$$\frac{\hat{\Delta}_{r+1}}{\hat{\Delta}_r} = \frac{256}{5} \frac{(1+1/r)^{r-1}}{(2r)(2r+1)} \leq \frac{1}{2} \quad (r \geq 34), \quad (110)$$

where $(1+1/r)^r < e < 68/25$ is sufficient for the last inequality. The first far index is $r_* = \lfloor p/4 \rfloor + 1 \geq 61$. Applying the factorial lower bounds at this index and then summing the geometric tail gives

$$\Delta_{p,N}^{\text{far}} \leq \frac{1}{1000m}. \quad (111)$$

The endpoint inequality behind (111) is stronger than needed: after multiplication by m , its left side decreases with p once $p \geq 241$, so it is enough to clear denominators at the first admissible value.

Adding (108) and (111) yields

$$\sum_{r=2}^{\lfloor p/2 \rfloor} \Delta_{p,N}^{(r)} \leq \left(\frac{196416}{15625} + \frac{1}{1000} \right) \frac{1}{m} < \frac{66}{5m},$$

which is the last step in Proposition 5.1.

A.2 The seven sign-lock budgets

We next make explicit how the error w_s in (34) is divided. For $p = m - s$, write the normalized nonlinear expansion as

$$\varepsilon_p = \sum_{r=2}^{\lfloor p/2 \rfloor} \varepsilon_p^{(r)}, \quad \varepsilon_p^{(r)} = -\frac{(-N)^r}{r!Nc_p} [t^p] H(t)^r.$$

The two-block term is separated into its endpoint compositions $(2, p-2)$ and $(p-2, 2)$ and the remaining compositions. Thus

$$\varepsilon_p = \varepsilon_p^{(2,\text{end})} + \varepsilon_p^{(2,\text{mid})} + \varepsilon_p^{(\geq 3)}, \quad \varepsilon_p^{(2,\text{end})} = -N \frac{c_2 c_{p-2}}{c_p}.$$

The endpoint term is the source of the first-order correction $-\zeta/m$ in (34). Let

$$u_s = \Pi_s - 1, \quad v_s = 1 - D_s, \quad z_* = \frac{50}{27}.$$

After adding and subtracting $e_1(s)/m$ and $-\zeta/m$, the absolute value of w_s is bounded by six nonnegative contributions: the second-order residual of Π_s , the drift v_s , the recentered endpoint two-block term, the middle two-block term, the terms with at least three blocks, and the product cross terms in (38). With the Poisson weight $z_*^s/s!$, their complete sums satisfy

$$\sum_{s \leq m/3} \frac{z_*^s}{s!} \left| \Pi_s - 1 - \frac{e_1(s)}{m} \right| \leq \frac{426}{m^2}, \quad (112)$$

$$\sum_{s \leq m/3} \frac{z_*^s}{s!} (1 - D_s) \leq \frac{13}{m^2}, \quad (113)$$

$$\sum_{s \leq m/3} \frac{z_*^s}{s!} \left| \varepsilon_{m-s}^{(2,\text{end})} + \frac{\zeta}{m} \right| \leq \frac{184}{m^2}, \quad (114)$$

$$\sum_{s \leq m/3} \frac{z_*^s}{s!} \left| \varepsilon_{m-s}^{(2,\text{mid})} \right| \leq \frac{234}{m^2}, \quad (115)$$

$$\sum_{s \leq m/3} \frac{z_*^s}{s!} \left| \varepsilon_{m-s}^{(\geq 3)} \right| \leq \frac{573}{m^2}, \quad (116)$$

$$\sum_{s \leq m/3} \frac{z_*^s}{s!} \mathcal{X}_s \leq \frac{784}{m^2}. \quad (117)$$

Here $\mathcal{X}_s$ is exactly the right-hand side of (38), after the three pieces of ε_{m-s} have been substituted. The inequalities use only the following elementary inputs:

- $1 - D_s \leq (28/25)s/m^2$ on $3s \leq m$;
- $\log \Pi_s \leq (146/125)e_1(s)/m < (2237/10000)s$;
- the block estimate (29), with the endpoint two-block compositions removed where appropriate;
- finite Poisson-moment bounds for $\sum z_*^s/s!$, $\sum s z_*^s/s!$, and the corresponding quadratic and quartic moments.

For (112), the exponential remainder inequality $0 \leq e^x - 1 - x \leq x^2 e^x / 2$ is applied to $x = \log \Pi_s$; the tilt $e^{(2237/10000)s}$ is absorbed by replacing z_* with a fixed larger rational parameter. For (114), the exact endpoint ratio $N c_2 c_{p-2} / c_p$ is compared with $5N / (36m^2)$ before the Poisson sum is taken. The two remaining block estimates follow from the same composition bounds used in the residual envelope. Finally, (117) is obtained by multiplying the already stated pointwise bounds, rather than introducing a new analytic estimate.

The numerical Poisson bounds are finite rational certificates. To verify one of them, substitute $z_* = 50/27$, truncate after the last degree that occurs, clear the positive factorial denominators, and

transform the resulting polynomial in the auxiliary endpoint parameter to the Bernstein basis on $[0, 1]$. Every coefficient is nonnegative. This procedure also certifies the two polynomial inequalities in Lemma 5.2; it is a finite arithmetic check, not an appeal to numerical sampling.

It remains to account for the far signed tail. Taking absolute values in the last sum of (35), applying the rational saddle bound to $E_{m-s}^-(N)$, and summing the resulting Poisson tail from $s = \lfloor m/3 \rfloor + 1$ gives

$$|\mathcal{R}_{\text{far}}| \leq \frac{1}{m^2}. \quad (118)$$

The first omitted term dominates the tail because its adjacent quotient is less than one; after clearing factorials, the endpoint inequality is checked at $m = 361$ and is monotone thereafter. Equations (112)–(118) sum to

$$\frac{426 + 13 + 184 + 234 + 573 + 784 + 1}{m^2} = \frac{2215}{m^2},$$

which proves Proposition 5.3.

B Scalar estimates in the Proposition 5.2 Gamma tail

This appendix supplies the quantitative interfaces used in Section 9.4. Its organization mirrors the proof: first the three coefficient errors in the exact split, then the Gamma moment entering Jensen's inequality, and finally the four Taylor-remainder terms.

B.1 The three split errors

Put

$$\alpha_{a,s} = \frac{h_{a-s}}{Nc_a} \quad (0 \leq s \leq r_0).$$

The factorial part of c_{a-s}/c_a is the Gamma weight γ_s from (91); the remaining factor is $D_s = d_{a-s}/d_a$, together with $1 - s_{a-s}/N$. The coefficientwise positive majorant for W therefore gives the exact aggregate estimate

$$\sum_{s=0}^{r_0} |\alpha_{a,s} - \gamma_s| |\omega_s| \leq \frac{86069}{3125a^2}. \quad (119)$$

The constant has a useful decomposition

$$\frac{86069}{3125} = 36 \left(\frac{2304}{3125} \right) + 1.$$

Here $2304/3125$ is the increment constant in the d -ratio estimate, the factor 36 is the closed factorial-moment sum, and the final 1 covers the partition-dependent factor s_{a-s}/N . Thus (119) is exactly the first inequality of (94), not a rounded asymptotic estimate.

For the other two terms, let $\widehat{E}$ and $\widehat{W}$ be obtained from E and W by taking absolute values coefficientwise. Positivity gives

$$|e_s| \leq [t^s] \exp(\Lambda_+(t)), \quad |\omega_s| \leq [t^s] \exp(\Lambda_+(t))(1 + J(t)),$$

where $\Lambda_+(t) = \sum_{r \leq p} h_r t^r$. Evaluating these majorants at the rational saddle radii used in Section 6, and then applying the factorial-mass bounds there, yields

$$\frac{1}{Nc_a} \sum_{s=0}^{r_0} k_{a-s} |e_s| \leq \frac{1}{a^2}, \quad (120)$$

$$\frac{|\omega_a|}{Nc_a} \leq \frac{1}{a^2}. \quad (121)$$

The endpoint inequalities in both cases are polynomial after multiplication by the positive factorial denominators. Their shifted forms have nonnegative coefficients in $a - 150$, so the check at $a = 150$ propagates to all larger a .

B.2 The exponent moment in the Jensen step

Let G_{a-2} have Gamma shape $a - 2$ and rate one and put $T = 1/(6G_{a-2})$. Separating the $r = 1$ term from the remaining terms in Λ gives

$$\mathbb{E}_{a-2}[\Lambda(T)] \leq \frac{5M}{24(a-3)} \quad (122)$$

$$+ \frac{8M}{25} \left(\frac{1}{(a-3)(a-4)} + \frac{p-2}{3 \binom{a-3}{3}} \right). \quad (123)$$

The first line uses $h_1 \leq 5M/4$ and the first negative Gamma moment. For $r \geq 2$, the coefficient bound $c_r \leq (4/25)6^r(r-1)!$ cancels the factor 6^{-r} in the Gamma moment; the remaining reciprocal falling factorials are bounded by their first term and a binomial tail. Since $p \leq a/2$ and

$$\binom{a-3}{3} = \frac{(a-3)(a-4)(a-5)}{6},$$

the right side of (123) is at most

$$\frac{5}{4} + \frac{5}{2(a-3)} + \frac{96a^2}{25(a-5)^3}. \quad (124)$$

After positive denominators are cleared, the assertion that (124) is smaller than $13/10$ becomes

$$100a^4 - 14480a^3 + 110040a^2 - 410000a + 662500 > 0.$$

Writing this polynomial in powers of $a - 150$ gives

$$100(a-150)^4 + 45520(a-150)^3 + 7094040(a-150)^2 + 405202000(a-150) + 4170062500,$$

so it is positive for every $a \geq 150$. This proves (100) with no numerical approximation.

The other scalar endpoint in the Jensen step is $e^{-13/10} > 27/100$. Equivalently, $e^{13/10} < 100/27$; a finite positive Taylor bound for $e^{3/10}$, multiplied by the standard rational upper bound for e , proves this strict inequality. Together with the Gamma shape shift in (99), this yields the margin $9/(40(a-2))$.

B.3 The four-term truncation residue

Set $s_0 = \lfloor a/8 \rfloor$ and let R_a be the quantity in (102). Each term has a distinct origin:

term	source
$920/2^{r_0+1}$	Taylor degrees above r_0 on $G \geq a/2$, where the evaluation radius is half the radius used for the coefficient majorant
$2(5/6)^a$	the Gamma lower-tail event $G < a/2$, using $ e^{-\Lambda(T)}(1 - J(T)) \leq 2$
$9(9/10)^{a-s_0}$	retained Taylor degrees $s \leq s_0$ on the lower event, controlled by the negative-Gamma-moment ratio
$920a(3/10)^{s_0+1}$	degrees $s > s_0$, controlled by the same coefficient majorant at the larger radius

After the coefficient bounds for Λ and J are inserted, the exact endpoint certificate for the majorant is

$$\widehat{W}\left(\frac{2}{3a}\right) \leq 920.$$

The Gamma Chernoff estimate gives $\mathbb{P}(G < a/2) \leq (5/6)^a$. The low-degree moment comparison contributes the ratio $9/10$, while the radius comparison in the high degrees contributes $3/10$.

For completeness, the inequality $R_a < a^{-2}$ is propagated in steps of eight, which preserves $s_0 = \lfloor a/8 \rfloor$ modulo its initial class. After multiplication by a^2 , each of the four summands is nonincreasing under $a \mapsto a + 8$ for $a \geq 150$. Exact rational evaluation at $a = 150, 151, \dots, 157$ gives a value below 1 in each residue class. Hence

$$a^2 R_a < 1 \quad (a \geq 150),$$

which is the final inequality in (102).

C Exact ratio and reserve estimates for the large positive tail

This appendix supplies the scalar inequalities behind the ratio-reserve argument of the large positive tail. Its purpose is twofold: to record the exact inequalities and certificate interfaces used by the Lean proof of Section 7, so that the large positive tail can be audited together with the formal certificate files, and to record exactly where the finite prefix certificates enter. All quantities below are rational. We use the convention $0^0 = 1$.

C.1 Candidate summands

Fix $a > 2000$. Put

$$N_- = 6a - 7, \quad N_+ = 12a - 8, \quad s_- = \lceil \sqrt{N_-} \rceil, \quad s_+ = \lceil \sqrt{N_+} \rceil, \quad (125)$$

and

$$K = \left\lfloor \frac{9a}{10} \right\rfloor, \quad \ell = \max\{1, s_- + 1\}, \quad m = \left\lfloor \frac{a}{3} \right\rfloor + 10, \quad j_k = a - k. \quad (126)$$

For $a > 2000$ one has

$$1 \leq \ell \leq m < K < a. \quad (127)$$

The entropy-shadow factor is

$$\mathcal{H}_a(k) = \frac{(k-1)^{k-1}(j_k-1)^{j_k-1}}{(a-2)^{a-2}}. \quad (128)$$

The maximal-term estimate for the binomial expansion gives

$$\frac{1}{(a-1)\binom{a-2}{k-1}} \leq \mathcal{H}_a(k) \quad (1 \leq k < a-1). \quad (129)$$

For $k = 1$ this is simply $1/(a-1) \leq 1$. For $k \geq 2$, put $n = a-2$ and $r = k-1$. In the binomial expansion

$$n^n = \sum_{i=0}^n \binom{n}{i} r^i (n-r)^{n-i},$$

the summand at $i = r$ is maximal. Hence

$$n^n \leq (n+1) \binom{n}{r} r^r (n-r)^{n-r},$$

which rearranges exactly to (129).

Recall the rational exponential upper bound

$$\mathbb{E}_T(x) = \sum_{q=0}^{T-1} \frac{x^q}{q!} + \frac{x^T}{T!} \frac{1}{1-x/T}, \quad 0 \leq x < T. \quad (130)$$

Write

$$\Xi_s(a, k) = \frac{1139}{1000} s_+ + \frac{j_k}{5} + \frac{29}{10} \frac{a}{j_k} + 1, \quad (131)$$

$$\Xi_t(a, k) = \frac{a}{5} + \frac{57}{10} \frac{a}{k} + \frac{29}{10} \frac{a}{j_k} + 2, \quad (132)$$

and set

$$L_s(a, k) = \mathbb{E}_a(\Xi_s(a, k)), \quad L_t(a, k) = \mathbb{E}_{8a}(\Xi_t(a, k)). \quad (133)$$

The two candidate summands used in the formal tail proof are

$$S_a(k) = \frac{65}{N_+} k j_k \mathcal{H}_a(k) 2^{-j_k} L_s(a, k), \quad (134)$$

$$T_a(k) = \frac{96}{N_-} k j_k \mathcal{H}_a(k) 2^{-j_k} L_t(a, k). \quad (135)$$

They are not heuristic asymptotic expressions: they are the exact rational majorants summed by the Lean proof.

To relate them to the normalized convolution in (16), define

$$\mathcal{C}_a(N, k) = \frac{N}{2} R_{k,a} 2^{-j_k} \max\{X_k(N), 0\} Y_{j_k}(N). \quad (136)$$

The direct saddle estimates, the bound (25), and (129) imply

$$\mathcal{C}_a(N, k) \leq S_a(k) \quad \text{if } k \leq \lceil \sqrt{N} \rceil, \quad (137)$$

$$\mathcal{C}_a(N, k) \leq T_a(k) \quad \text{if } \lceil \sqrt{N} \rceil < k. \quad (138)$$

The sign lock removes all $k > K$. Consequently, for every $N_- \leq N \leq N_+$,

$$\sum_{1 \leq k < a} \mathcal{C}_a(N, k) \leq \Sigma_s(a) + \Sigma_t(a), \quad (139)$$

where

$$\Sigma_s(a) = \sum_{k=1}^{\min\{K, s_+\}} S_a(k), \quad \Sigma_t(a) = \sum_{k=\ell}^K T_a(k). \quad (140)$$

The overlap is deliberately counted twice. This loses a negligible amount and allows the two regimes to be summed independently.

C.2 The common adjacent quotient

Let $1 \leq r < K$ and write $j = a - r$. Removing the constants and exponential shells from either (134) or (135) gives the same adjacent quotient

$$\mathfrak{q}_a(r) = 2 \frac{(r+1)(j-1)r^r(j-2)^{j-2}}{rj(r-1)^{r-1}(j-1)^{j-1}}. \quad (141)$$

Thus

$$\frac{S_a(r+1)}{S_a(r)} = \mathfrak{q}_a(r) \frac{L_s(a, r+1)}{L_s(a, r)}, \quad \frac{T_a(r+1)}{T_a(r)} = \mathfrak{q}_a(r) \frac{L_t(a, r+1)}{L_t(a, r)}. \quad (142)$$

For $r \geq 2$ (and hence, in the retained range, $j \geq 3$), it is useful to factor

$$\mathfrak{q}_a(r) = 2 \frac{r+1}{j} \mathcal{P}_a(r), \quad (143)$$

where

$$\begin{aligned} \mathcal{P}_a(r) &:= \frac{r^r}{r(r-1)^{r-1}} \frac{(j-1)(j-2)^{j-2}}{(j-1)^{j-1}} \\ &= \left(1 + \frac{1}{r-1}\right)^{r-1} \left(1 + \frac{1}{j-2}\right)^{-(j-2)}. \end{aligned} \quad (144)$$

Since $n \mapsto (1 + 1/n)^n$ is increasing, one obtains

$$\mathcal{P}_a(r) \leq 1 \quad \text{if } r-1 \leq j-2, \quad \mathcal{P}_a(r) \geq 1 \quad \text{if } j-2 \leq r-1. \quad (145)$$

We repeatedly use the elementary geometric estimates

$$F_{r+1} \leq qF_r \quad (u \leq r < v) \implies \sum_{r=u}^v F_r \leq \frac{F_u}{1-q}, \quad (146)$$

$$F_{r-1} \leq qF_r \quad (u < r \leq v) \implies \sum_{r=u}^v F_r \leq \frac{F_v}{1-q}, \quad (147)$$

valid for nonnegative F_r and $0 \leq q < 1$.

C.3 The small branch

Lemma C.1 (Small adjacent ratio). *For $a > 2000$ and $1 \leq r < \min\{K, s_+\}$, one has*

$$S_a(r+1) \leq \frac{1}{2}S_a(r). \quad (148)$$

Proof. Since $s_+ = \lceil \sqrt{12a-8} \rceil$, an elementary integer estimate gives

$$12s_+ \leq a \quad (a > 2000), \quad (149)$$

because $\lceil \sqrt{12a-8} \rceil < \sqrt{12a-8} + 1$ and

$$(a-12)^2 - 144(12a-8) = a^2 - 1752a + 1296 > 0 \quad (a \geq 2001)$$

(positive at $a = 2001$ and increasing for $a > 876$), so $12(\sqrt{12a-8} + 1) \leq a$. As $r < s_+$, hence $r+1 \leq s_+$, this gives $12(r+1) \leq a$; writing $j = a - r$ then yields $j \geq 11(r+1) + 1$, and therefore

$$272(r+1) \leq 25j. \quad (150)$$

For $r \geq 2$ the factorization (143)–(144) bounds the first power ratio by $e < \frac{68}{25}$ (by the classical bound $(1+1/n)^n < e$) and the reciprocal second ratio by 1, so $\mathcal{P}_a(r) \leq \frac{68}{25}$ and

$$\mathfrak{q}_a(r) = 2 \frac{r+1}{j} \mathcal{P}_a(r) \leq \frac{136(r+1)}{25j} \leq \frac{1}{2} \quad (151)$$

by (150). For $r = 1$ one has directly

$$\mathfrak{q}_a(1) = \frac{4}{j} \left(\frac{j-2}{j-1} \right)^{j-2} \leq \frac{4}{j} \leq \frac{1}{2},$$

since $j = a - 1 > 1999$, so (151) holds throughout $1 \leq r < \min\{K, s_+\}$. The small exponent is nonincreasing: with $j = a - r$, (131) gives

$$\Xi_s(a, r+1) - \Xi_s(a, r) = -\frac{1}{5} + \frac{29a}{10j(j-1)} \leq 0,$$

since $12(r+1) \leq a$ yields $j \geq \frac{11a}{12} + 1$ and hence $29a \leq 2j(j-1)$, so

$$\Xi_s(a, r+1) \leq \Xi_s(a, r), \quad (152)$$

and monotonicity of (130) gives $L_s(a, r+1) \leq L_s(a, r)$. Substituting (151) and this bound into (142) proves (148). $\square$

The first candidate has the exact form

$$S_a(1) = \frac{65(a-1)}{12a-8} 2^{-(a-1)} L_s(a, 1), \quad (153)$$

because $\mathcal{H}_a(1) = 1$. The exponential shell satisfies

$$L_s(a, 1) \leq \left(\frac{3}{2} \right)^a. \quad (154)$$

Indeed $\Xi_s(a, 1) \leq 3a/10$, and the finite rational exponential bound at $3a/10$ is dominated by the negative-binomial envelope $(3/2)^a$. Therefore

$$800000000 S_a(1) \leq 800000000 \frac{65(a-1)}{12a-8} 2^{-(a-1)} \left(\frac{3}{2}\right)^a. \quad (155)$$

The right-hand side is nonincreasing for $a \geq 3$; its value at $a = 2001$ is at most one by an exact rational computation. Hence

$$800000000 S_a(1) \leq 1 \quad (a > 2000). \quad (156)$$

Combining Lemma C.1, (146), and (156) gives

$$\Sigma_s(a) \leq 2S_a(1) \leq \frac{1}{400000000}. \quad (157)$$

The actual guarded convolution term at $k = 1$ vanishes because $X_1 = -1$. The nonzero value $S_a(1)$ is retained only as a convenient first-term reserve for the candidate sequence.

C.4 The lower tempered band

Put

$$q_a = \frac{4a-1}{4a} = 1 - \frac{1}{4a}. \quad (158)$$

The lower tempered band is $\ell \leq k \leq m$. Its desired step estimate is

$$T_a(r+1) \leq q_a T_a(r) \quad (\ell \leq r < m). \quad (159)$$

On this range $r-1 \leq j-2$, so $\mathcal{P}_a(r) \leq 1$ by (145), and therefore

$$\mathfrak{q}_a(r) \leq 2 \frac{r+1}{j}. \quad (160)$$

Define the sharp exponential quotient target

$$U_a^\sharp(r) = \frac{1}{2} \frac{j}{r+1} q_a. \quad (161)$$

Then (160) gives the exact budget identity

$$\mathfrak{q}_a(r) U_a^\sharp(r) \leq q_a. \quad (162)$$

Thus it remains only to prove

$$\frac{L_t(a, r+1)}{L_t(a, r)} \leq U_a^\sharp(r). \quad (163)$$

There are two subranges. If

$$3(r+1) \leq a, \quad (164)$$

then

$$\Xi_t(a, r+1) \leq \Xi_t(a, r), \quad U_a^\sharp(r) \geq 1. \quad (165)$$

Hence (163) follows immediately from monotonicity of $\mathbf{E}_{8a}$.

The only remaining indices are the ten offsets

$$r = \left\lfloor \frac{a}{3} \right\rfloor + t, \quad 0 \leq t < 10. \quad (166)$$

For $a \geq 3000$ the following two rational inequalities hold uniformly in t :

$$\Xi_t(a, r+1) + \frac{175}{4a} \leq \Xi_t(a, r), \quad (167)$$

$$1 \leq U_a^\sharp(r) \left(1 + \frac{175}{12a}\right)^3. \quad (168)$$

The second estimate is reduced to the nonnegativity of

$$550656a^3 - 76358800a^2 - 559763750a + 144703125. \quad (169)$$

After writing $a = 3000 + x$, this polynomial becomes

$$550656x^3 + 4879545200x^2 + 14408999436250x + 14178803653453125, \quad (170)$$

which is manifestly nonnegative for $x \geq 0$.

The finite-exponential three-step shift inequality says that, whenever $y + d \leq z$,

$$\mathsf{E}_T(y) \leq \left(1 + \frac{d}{3}\right)^{-3} \mathsf{E}_T(z) \quad (171)$$

provided the displayed arguments lie in the certified range. Applying it with $d = 175/(4a)$ and using (167)–(168) proves (163) for every $a \geq 3000$.

For the finite prefix $2001 \leq a < 3000$, define

$$\theta_a = 1 - \frac{213}{5a} = \frac{5a - 213}{5a}. \quad (172)$$

Exact rational certificates verify, for every guarded pair (a, t) with (166),

$$\frac{L_t(a, r+1)}{L_t(a, r)} \leq \theta_a, \quad (173)$$

$$(4a) \mathfrak{q}_a(r) \theta_a \leq 4a - 1. \quad (174)$$

The first computation is split into twenty blocks of fifty consecutive values of a , each containing the ten offsets $t = 0, \dots, 9$. The second is a single 999×10 raw-rational block. These are exact Boolean checks over $\mathbb{Q}$ executed by Lean’s native evaluator; no floating-point comparison is used. Equations (173) and (174) imply (159) in the prefix strip. Together with the symbolic argument above, they prove (159) for every $a > 2000$.

C.5 The upper tempered band

The upper band is $m + 1 \leq k \leq K$ and is summed in reverse. The desired step is

$$T_a(r-1) \leq q_a T_a(r) \quad (m+1 < r \leq K). \quad (175)$$

Set

$$V_a^\sharp(r) = \mathfrak{q}_a(r-1) q_a. \quad (176)$$

By the exact quotient identity (142), it is enough to prove

$$L_t(a, r-1) \leq V_a^\sharp(r) L_t(a, r). \quad (177)$$

In the far upper band

$$3a \leq 5r, \quad (178)$$

one has

$$\Xi_t(a, r-1) \leq \Xi_t(a, r), \quad V_a^\sharp(r) \geq 1. \quad (179)$$

The second inequality follows from the lower form of (145), applied to the raw quotient at $r-1$. Thus (177) is immediate in this range.

It remains to treat the middle band

$$m+1 < r \leq K, \quad 5r < 3a. \quad (180)$$

The exponent shift is bounded by

$$\Xi_t(a, r-1) \leq \Xi_t(a, r) + \delta_a, \quad \delta_a = \frac{45}{a}. \quad (181)$$

Put

$$\Theta_a = \left(1 - \frac{79}{75}\delta_a\right)^{-1}. \quad (182)$$

The rational exponential estimate used in Lean is

$$\mathbf{E}_{8a}(E + \delta_a) - \mathbf{E}_{8a}(E) \leq \frac{79}{75}\delta_a \mathbf{E}_{8a}(E + \delta_a), \quad (183)$$

valid here because

$$E + \delta_a \leq \frac{8a-1}{24}. \quad (184)$$

Rearranging (183) and using (181) gives

$$L_t(a, r-1) \leq \Theta_a L_t(a, r). \quad (185)$$

It remains to show $\Theta_a \leq V_a^\sharp(r)$.

For this purpose write the raw quotient at $r-1$ as

$$\mathfrak{q}_a(r-1) = \frac{2r}{a-r+1} \mathcal{P}_a(r-1). \quad (186)$$

Define

$$W_a(r) = \Theta_a \frac{4a}{4a-1} \frac{a-r+1}{2r}. \quad (187)$$

Then $\Theta_a \leq V_a^\sharp(r)$ is equivalent to

$$W_a(r) \leq \mathcal{P}_a(r-1). \quad (188)$$

There are two elementary subcases.

If $a+1 \leq 2r$, then direct rational simplification gives

$$W_a(r) \leq 1, \quad (189)$$

and (145) gives $1 \leq \mathcal{P}_a(r-1)$.

If $2r < a + 1$, then

$$W_a(r) \leq 1 - \frac{1}{a}. \quad (190)$$

The required matching lower bound for the power product is obtained from the telescoping inequality

$$\mathcal{P}_a(s) \geq 1 - \frac{(j_s - 2) - (s - 1)}{2(s - 1)(j_s - 2)} \quad (s - 1 \leq j_s - 2). \quad (191)$$

For $s = r - 1$ and the present range, the fraction on the right is at most $1/a$; hence

$$1 - \frac{1}{a} \leq \mathcal{P}_a(r - 1). \quad (192)$$

This proves (188), then (177), and finally (175) throughout the upper band.

C.6 Endpoint reserves and the tempered budget

The two endpoint exponential shells admit the common envelope

$$L_t(a, \ell) \leq \left(\frac{10}{7}\right)^a, \quad L_t(a, K) \leq \left(\frac{10}{7}\right)^a. \quad (193)$$

The theorem-facing endpoint estimates are

$$800000000(4a)T_a(\ell) \leq 1, \quad 800000000(4a)T_a(K) \leq 1. \quad (194)$$

We record the coarse envelopes used to prove them.

For the lower endpoint, one has $\ell \leq a/4$, discards $\mathcal{H}_a(\ell) \leq 1$, and keeps only the dyadic decay. This yields

$$800000000(4a)T_a(\ell) \leq G_-(a) := 3200000000 a^3 2^{-(a - \lfloor a/4 \rfloor)} \left(\frac{10}{7}\right)^a. \quad (195)$$

The four-step comparison

$$G_-(a + 4) \leq G_-(a) \quad (196)$$

follows from

$$a + 4 \leq \frac{401}{400}a, \quad 10000 \left(\frac{401}{400}\right)^3 \leq 19208. \quad (197)$$

Exact rational checks at $a = 2001, 2002, 2003, 2004$ initialize the four residue classes and prove the first inequality in (194).

For the upper endpoint, the entropy shadow is retained. The elementary bounds

$$\mathcal{H}_a(K) \leq \left(\frac{901}{1000}\right)^{\lfloor 89a/100 \rfloor} \left(\frac{101}{1000}\right)^{\lfloor a/10 \rfloor - 1} \quad (198)$$

give

$$\begin{aligned} 800000000(4a)T_a(K) &\leq G_+(a) \\ &:= 3200000000 a^3 \left(\frac{901}{1000}\right)^{\lfloor 89a/100 \rfloor} \left(\frac{101}{1000}\right)^{\lfloor a/10 \rfloor - 1} 2^{-\lfloor a/10 \rfloor} \left(\frac{10}{7}\right)^a. \end{aligned} \quad (199)$$

The one-hundred-step comparison

$$G_+(a + 100) \leq G_+(a) \quad (200)$$

follows from

$$\left(\frac{21}{20}\right)^3 \left(\frac{901}{1000}\right)^{89} \left(\frac{101}{1000}\right)^{10} \left(\frac{10}{7}\right)^{100} \leq 2^{10}. \quad (201)$$

Exact rational checks for $2001 \leq a \leq 2100$ initialize the one hundred residue classes and prove the second inequality in (194).

Since $1 - q_a = 1/(4a)$, the lower and upper geometric reserves satisfy

$$\sum_{k=\ell}^m T_a(k) \leq \frac{T_a(\ell)}{1 - q_a} = 4aT_a(\ell) \leq \frac{1}{800000000}, \quad (202)$$

$$\sum_{k=m+1}^K T_a(k) \leq \frac{T_a(K)}{1 - q_a} = 4aT_a(K) \leq \frac{1}{800000000}. \quad (203)$$

Therefore

$$\Sigma_t(a) \leq \frac{1}{400000000}. \quad (204)$$

Together with (157), this gives the product part of the positive budget:

$$\Sigma_s(a) + \Sigma_t(a) \leq \frac{1}{200000000} \quad (a > 2000). \quad (205)$$

C.7 The solo term

The remaining term in (16) is $2^{-a-1}Y_a(N)$. The sharp solo estimate is proved first at the upper edge $N = N_+$ by splitting the closed factorial-composition expansion into low-degree and remainder blocks. In the notation of the formal development, the closed block sum is

$$\begin{aligned} \Delta_a(N) &= \sum_{s=0}^{a-1} \frac{(Nc_1/4)^s}{s!} \sum_{\substack{r \geq 1 \\ 2r \leq a-s}} \frac{(2N/25)^r 3^{a-s} 4^{r-1} (a-s-2r+1)!}{r!} \\ &\quad + \frac{(Nc_1/4)^a}{a!}, \end{aligned} \quad (206)$$

which is `positiveLargeTailSoloSharpGcompClosedFactorialSplitBlockSum` in Lean. The denominator-cleared estimate at the upper edge is

$$4 \cdot 2^a \Delta_a(N_+) \leq 29 a c_a \left(\frac{10}{7}\right)^a. \quad (207)$$

Monotonicity in N then yields the normalized bound

$$Y_a(N) \leq \frac{29}{2} \frac{a}{N} \left(\frac{10}{7}\right)^a \quad (N_- \leq N \leq N_+). \quad (208)$$

Since $N \geq 6a - 7$ and $a \geq 401$, one has $a/N \leq 1/5$. Therefore

$$\begin{aligned} 2^{-a-1}Y_a(N) &\leq \frac{29}{4} \frac{a}{N} \left(\frac{5}{7}\right)^a \\ &\leq \frac{29}{20} \left(\frac{5}{7}\right)^a. \end{aligned} \quad (209)$$

Writing

$$H_{\text{solo}}(a) = 2900000000 \left(\frac{5}{7}\right)^a,$$

one has the exact adjacent- a relation

$$H_{\text{solo}}(a+1) = \frac{5}{7} H_{\text{solo}}(a). \quad (210)$$

Thus the sequence is decreasing, and the exact rational check at $a = 401$ gives

$$2900000000 \left(\frac{5}{7}\right)^a \leq 1. \quad (211)$$

Consequently

$$2^{-a-1} Y_a(N) \leq \frac{1}{2000000000}. \quad (212)$$

Proposition C.2 (Large positive-tail budget). *For every $a > 2000$ and every N with $6a - 7 \leq N \leq 12a - 8$,*

$$2^{-a-1} Y_a(N) + \frac{N}{2} \sum_{\substack{1 \leq k < a \\ X_k(N) > 0}} R_{k,a} 2^{-(a-k)} X_k(N) Y_{a-k}(N) \leq 10^{-8}. \quad (213)$$

Proof. The sign lock removes $k > K$. The remaining product sum is at most (205) by (137)–(139). The solo term is at most (212). Their sum is $1/2000000000 + 1/2000000000 = 10^{-8}$. $\square$

C.8 Correspondence with the formal proof

For audit purposes, the principal Lean declarations corresponding to this appendix are listed below. The declaration names are stable in the release version of the repository.

paper item	Lean declaration or file
L_s, L_t	<code>positiveSmallLargeExp</code> ; <code>positiveTemperedLargeExp</code>
S_a, T_a	<code>positiveSmallEntropyShadowExpMajorantTerm</code> ; <code>positiveTemperedEntropyShadowExpMajorantTerm</code>
raw quotient q_a	<code>positiveEntropyShadowBaseStepRawQuotient</code>
power product $\mathcal{P}_a$	<code>positiveEntropyShadowBaseStepRawPowerProduct</code>
small ratio $1/2$	<code>positiveSaddleLargeTailCandidateSmallRawBaseHalfCertificate</code>
lower split $m = \lfloor a/3 \rfloor + 10$	<code>positiveLargeExpTemperedSplit</code>
lower sharp target $U_a^\sharp$	<code>positiveTemperedLowerSharpExpQuotientTarget</code>
finite prefix target θ_a	<code>positiveTemperedLowerPrefixTopOffsetExpRatioTarget</code>
finite prefix certificates	<code>PositiveSaddlePrefixExpRatioCertificate.lean</code> ; <code>PositiveSaddlePrefixRawBudget.lean</code>
upper-middle shift $45/a$	<code>positiveTemperedUpperMiddleExpShiftBudget</code>
upper-middle factor Θ_a	<code>positiveTemperedUpperMiddleExpShiftFactor</code>
small first reserve	<code>positiveSaddleLargeTailCandidateSmallFirstReserveCertificate_closed</code>
tempered endpoint reserves	<code>positiveSaddleLargeTailCandidateUnitReserveCertificate_temperedTenSevenths_closed</code>
assembled hybrid ratio certificate	<code>positiveSaddleLargeTailCandidateRefinedAtomicCertificate_hybridClosed</code>
solo block sum Δ_a	<code>positiveLargeTailSoloSharpGcompClosedFactorialSplitBlockSum</code>
solo block-sum cleared	<code>positiveLargeTailSoloSharpGcompClosedFactorialSplitBlockSumTenSeventhsCleared</code>
solo saddle cleared	<code>positiveLargeTailSoloSharpGcompSaddleTenSeventhsCleared</code>
solo scalar budget	<code>positiveLargeTailSoloTenSeventhsScalarBudget_of_ge_401</code>

D Public Lean statement

The following is a Lean-style rendering of the public facade, lightly edited for readability (e.g. $\mathbb{Q}[[X]]$ for the rational power-series type and ASCII $\leq$, $\neq$ for the order and disequality); it is not literal source. The exact, machine-checked file is `Prop51/Theorem.lean` in the repository. It is reproduced to make the correspondence between the paper and the formal theorem immediate.

```
import Prop51.BCoeffSeries
import Prop51.Completion

namespace Prop51

open PowerSeries

noncomputable abbrev chenLarsonC :  $\mathbb{Q}[[X]]$  := Cseries

theorem coeff_chenLarsonC (k : Nat) :
  coeff k chenLarsonC =
    (Nat.factorial (6*k) :  $\mathbb{Q}$ ) /
    ((Nat.factorial (3*k) :  $\mathbb{Q}$ ) *
     (Nat.factorial (2*k) :  $\mathbb{Q}$ ) *  $72^k$ )

noncomputable abbrev chenLarsonSeries (mu : List Nat) :  $\mathbb{Q}[[X]]$  :=
  bSeries mu

theorem chenLarsonSeries_spec (mu : List Nat) :
  chenLarsonC ^ ((mu.map (fun m => m + 1)).sum) *
  chenLarsonSeries mu
  = (mu.map (fun m =>
    rescale ((m + 1 : Nat) :  $\mathbb{Q}$ )(-1)) chenLarsonC)).prod

theorem chenLarsonCoefficient_neg
  {g : Nat} (hg : 2 ≤ g) (hmod : g % 3 ≠ 1)
  {mu : List Nat} (hmu : IsPartitionOf mu (2 * g - 2)) :
  coeff (g / 3 + 1) (chenLarsonSeries mu) < 0

theorem chenLarsonCoefficient_ne_zero
  {g : Nat} (hg : 2 ≤ g) (hmod : g % 3 ≠ 1)
  {mu : List Nat} (hmu : IsPartitionOf mu (2 * g - 2)) :
  coeff (g / 3 + 1) (chenLarsonSeries mu) ≠ 0

end Prop51
```

For the corrected Proposition 5.2 the analogous public facade is `Prop52/Theorem.lean`. Besides re-exporting the series of Proposition 5.1, it *defines* the corrected source factor D_μ^{cor} coefficientwise, pins its coefficients, and states the two final source theorems (again lightly edited for readability).

```
import Prop51.Theorem
import Prop52.Source

namespace Prop52

-- B_mu(X) is re-exported from Prop 5.1.
noncomputable abbrev chenLarsonSeries (mu : List Nat) : Q[[X]] :=
  Prop51.chenLarsonSeries mu

-- The corrected factor D^cor, defined coefficientwise.
-- Geometrically lead = M(g/3+1) = 2g-2; printed factor uses 1.
noncomputable abbrev chenLarsonProp52SourceFactor
  (lead : Q) (mu : List Nat) : Q[[X]] :=
  mk (sourceFactorCoeff lead mu)

-- Its coefficients pin D^cor exactly:
theorem coeff_sourceFactor_zero (lead : Q) (mu : List Nat) :
  coeff 0 (chenLarsonProp52SourceFactor lead mu) = lead

theorem coeff_sourceFactor_one (lead : Q) (mu : List Nat) :
  coeff 1 (chenLarsonProp52SourceFactor lead mu)
    = -2 * ((N mu : Q) - sPower mu 1)

theorem coeff_sourceFactor_succ_succ
  (lead : Q) (mu : List Nat) (r : Nat) :
  coeff (r + 2) (chenLarsonProp52SourceFactor lead mu)
    = -12 * (r + 1) * Prop51.c (r + 1)
      * ((N mu : Q) - sPower mu (r + 2))

-- Non-vanishing for g >= 2, g = 1 (mod 3):
theorem chenLarsonProp52Coefficient_nonvanishing
  {g : Nat} (hg : 2 <= g) (hmod : g % 3 = 1)
  {mu : List Nat} (hmu : Prop51.IsPartitionOf mu (2 * g - 2)) :
  coeff (g / 3 + 1)
    (chenLarsonSeries mu
      * chenLarsonProp52SourceFactor (M (g / 3 + 1)) mu) != 0

-- ... and strict negativity for g >= 40:
theorem chenLarsonProp52Coefficient_neg
  {g : Nat} (hg : 40 <= g) (hmod : g % 3 = 1)
  {mu : List Nat} (hmu : Prop51.IsPartitionOf mu (2 * g - 2)) :
  coeff (g / 3 + 1)
    (chenLarsonSeries mu
      * chenLarsonProp52SourceFactor (M (g / 3 + 1)) mu) < 0

end Prop52
```

Here the non-vanishing facade carries $2 \leq g$ where Theorem 1.3 states $g \geq 4$; the two select the same genera, since for $g \equiv 1 \pmod{3}$ the condition $g \geq 2$ already forces $g \geq 4$. The strict-negativity

facade carries $40 \leq g$, matching $g \geq 40$ directly.

E Paper-to-Lean correspondence

paper item	principal Lean declaration or file
hypergeometric series C	<code>Aseq, Cseries</code> in <code>ExpSeries.lean</code>
$C = \exp(\sum c_r X^r)$	<code>Cseries_eq_expSeries_c</code> in <code>Bridge.lean</code>
quotient-series identity	<code>bSeries_official</code> in <code>BCoeffSeries.lean</code>
partition-free majorant	<code>bCoeff_le_U</code> in <code>Majorant.lean</code>
coefficient bounds	<code>d_lb, d_ub, d_increment, d_ratio_lb</code> in <code>DNorm.lean</code>
sign lock	<code>Xnorm_le_neg_final_margin</code> in <code>SignLock.lean</code>
finite saddle and prefix convolution	<code>expCoeff_saddle, expPrefix_mul_le</code> in <code>SaddleDirect.lean</code>
direct product estimates	<code>Bq_Qq_small_core, Bq_Qq_tempered_core</code> in <code>SaddleDirect.lean</code>
finite edge budget	<code>positiveSaddleFiniteScaledEdgeBudget</code> in <code>Prop51/Generated/PositiveSaddleFiniteScaledEdge.lean</code>
closed proof	<code>completion_of_directSaddle_closed</code> and <code>coefficientNegativity</code>
public theorem	<code>chenLarsonCoefficient_neg, chenLarsonCoefficient_ne_zero</code> in <code>Prop51/Theorem.lean</code>
<i>Corrected Proposition 5.2</i>	
corrected source coefficient T_a^{cor}	<code>sourceCoeff</code> in <code>Prop52/Source.lean</code>
source-marked bridge	<code>sourceCorrectedCoeff_eq</code> in <code>Prop52/Source.lean</code>
central identity (85)	<code>correctedCoeff_eq_printedCoeff_add</code> in <code>Prop52/Assembly.lean</code>
rectangle quotient sign	<code>Prop51.bCoeff_neg_of_rectangle</code> in <code>Prop51/Rectangle.lean</code>
finite range $2 \leq a \leq 13$	<code>correctedCoeff_finite_nonvanishing</code> in <code>Prop52/Finite.lean</code>
printed sign, $14 \leq a \leq 149$	<code>printedCoeffNegativityMid_closed</code> in <code>Prop52/MidBridge.lean</code>
printed sign, Gamma tail	<code>printedTailERealSeriesHasSum</code> in <code>Prop52/GammaCompletion.lean</code>
public theorems (Prop. 5.2)	<code>chenLarsonProp52Coefficient_nonvanishing, chenLarsonProp52Coefficient_neg</code> in <code>Prop52/Theorem.lean</code>

F Computational certificate inventory

range	method	verification surface
$a \leq 8$	exact partition enumeration	exact rational coefficient for every positive partition; selected partition-cardinality checks
$9 \leq a \leq 60$	exact rational majorant	all 10764 rectangle pairs
$61 \leq a \leq 400$	192-bit outward dyadic intervals	proved enclosure of recurrence tables; eight native chunks
$401 \leq a \leq 2000$	direct saddle plus fixed-point edge certificate	sixteen 100-row shards, scale 10^{20} ; analytic solo theorem
$2001 \leq a < 3000$	direct saddle plus generated prefix ratios	pointwise product and solo bounds; finite quotient/raw-budget chunks
$a \geq 3000$	symbolic rational tail	adjacent quotients, reverse upper band, unit reserves, and sharp solo envelope

For the corrected Proposition 5.2 the ranges are as follows.

range	method	verification surface
$2 \leq a \leq 8$	exact rational partition enumeration	exact corrected coefficient for every positive partition (with permutation/generator completeness)
$9 \leq a \leq 13$	modular certificate at $p = 1,000,000,007$	nonzero residue, plus the rational-to- $\mathbb{F}_p$ reduction bridge
$14 \leq a \leq 149$	exact dyadic interval certificate	strict negativity of the printed marked coefficient T_a^{old}
$a \geq 150$	Gamma-integral, moment, and Taylor-truncation estimates	real exponential power-series identity for the upper-event integrand
$a \geq 14$ (assembly)	central identity (85) with the rectangle	$T_a^{\text{cor}} < 0$ from $T_a^{\text{old}} < 0$ and $b_a < 0$

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